Fundamental Solution via Invariant Approach for a Brain Tumor Model and its Extensions

Andrew G. Johnpillai[a], Fazal M. Mahomed[b], and Saeid Abbasbandy[c]

[a] Department of Mathematics, Eastern University, Sri Lanka
[b] School of Computational and Applied Mathematics, DST-NRF Centre of Excellence in Mathematical and Statistical Sciences; Differential Equations, Continuum Mechanics and Applications, University of the Witwatersrand, Johannesburg, Wits 2050, South Africa and School of Mathematics and Statistics, University of New South Wales, Sydney NSW 2052 Australia
[c] Department of Mathematics, Imam Khomeini International University Ghazvin, 34149-16818, Iran

Reprint requests to A. G. J; Tel.: +94 65 2240753, E-mail: andrewgratienj@yahoo.com

Z. Naturforsch. **69a**, 725 – 732 (2014) / DOI: 10.5560/ZNA.2014-0064
Received April 17, 2014 / revised August 30, 2014 / published online November 5, 2014

We firstly show how one can use the invariant criteria for a scalar linear $(1+1)$ parabolic partial differential equations to perform reduction under equivalence transformations to the first Lie canonical form for a class of brain tumor models. Fundamental solution for the underlying class of models via these transformations is thereby found by making use of the well-known fundamental solution of the classical heat equation. The closed-form solution of the Cauchy initial value problem of the model equations is then obtained as well. We also demonstrate the utility of the invariant method for the extended form of the class of brain tumor models and find in a simple and elegant way the possible forms of the arbitrary functions appearing in the extended class of partial differential equations. We also derive the equivalence transformations which completely classify the underlying extended class of partial differential equations into the Lie canonical forms. Examples are provided as illustration of the results.

Key words: Linear $(1+1)$ Parabolic Partial Differential Equations; Lie Canonical Forms; One-Parameter Groups of Transformations; Cauchy Problem for the Brain Tumor Model; Fundamental Solution.

1. Introduction

The applications of Lie symmetries and operator approaches to differential equations have attracted considerable attention ever since the initial seminal works of Lie on this subject. As a result, over the years a large number of contributions have been reported by many authors in the literature.

Lie in [1] studied properties of the one-parameter groups of transformations of the linear $(1+1)$ parabolic partial differential equation (PDE) of one space and one time variable,

$$u_t = a(t,x)u_{xx} + b(t,x)u_x + c(t,x)u, \qquad (1)$$

where a, b, and c are smooth functions of t and x. Moreover, he gave the complete group classification of the parabolic PDE (1). He provided all the canonical forms of the PDE (1) for which (1) admits nontrivial point symmetry algebras of dimensions 1, 2, 4, and 6 (apart from the infinite dimensional algebra of trivial Lie point symmetries of the superposition operators).

Ovsiannikov in his work [2] further studied the parabolic PDE (1) by using the reduction to the fourth Lie canonical form which is the special case of (1) with $a=1$ and $b=0$. Bluman in [3] investigated the symmetry properties of the parabolic PDE (1) and developed a mapping algorithm. Equation (1) was first reduced to the fourth Lie canonical form for this purpose.

In Johnpillai and Mahomed [4], the practical criterion in terms of the coefficients of the equation, for equivalence of the parabolic PDE (1) under point transformations for the autonomous case to the classical heat equation, was given. A brief account of the re-

© 2014 Verlag der Zeitschrift für Naturforschung, Tübingen · http://znaturforsch.com

ducibility of the PDE (1) to the heat and second Lie canonical form was reported in [5].

In a more recent work, Mahomed [6] obtained a complete invariant characterization of the scalar linear $(1+1)$ parabolic PDE (1). He derived refined invariant criteria for reductions to the four Lie canonical forms thus giving practical conditions in terms of the coefficients of the parabolic PDE (1), for reduction to simpler forms. In [7], the invariant method is applied to a bond pricing PDE taken from [8].

Here in this paper, we show how one can use these invariant criteria for a class of brain tumor models to find the equivalence transformations which reduce the PDE to the first Lie canonical form, i.e. the classical heat PDE. We also construct the fundamental solution for the model via these transformations by utilizing the fundamental solution of the one-dimensional heat equation. The solution of the Cauchy problem of the model equation is also obtained using its fundamental solution. We also apply the invariant method to the extended form of the class of brain tumor models which is a subclass of the PDEs of the form (1). We demonstrate the utility of the invariant method to find in simple and elegant manner the possible forms of the arbitrary functions appearing in the extended class of PDEs. We also derive the equivalence transformations which completely classify the underlying class of PDEs so that it belongs to the Lie canonical forms.

The outline of the paper is as follows. In Section 2 we review the invariant criteria for the scalar linear parabolic PDE (1) as given in [6]. In Section 3 we apply this invariant method to a brain tumor model taken from [9, 10]. We present the equivalence transformations which reduce the underlying model to the linear heat equation. Moreover, we also construct the fundamental solution of the model by means of these transformation formulae. The closed-form solution of the Cauchy problem of the underlying equation is then constructed making use of its fundamental solution. Section 4 deals with the application of the invariant method to the extended form of the class of brain tumor models which is a subclass of the PDEs of the form (1). Concluding remarks are made in Section 5.

2. Preliminaries

It is well-known from the work of Lie [1] that the equivalence transformations of the linear parabolic equation (1) is an infinite group that consists of the linear change of the dependent variable given by

$$\bar{u} = \sigma(t,x)u, \ \ \sigma \neq 0, \tag{2}$$

as well as invertible transformations of the independent variables

$$\bar{t} = \phi(t), \ \bar{x} = \psi(t,x), \ \dot{\phi} \neq 0, \ \psi_x \neq 0, \tag{3}$$

where ϕ, ψ, and σ are arbitrary functions and the over dot indicates differentiation with respect to t.

Two linear parabolic PDEs of the form (1) are called equivalent if one can be transformed to the other by the equivalence transformations (2) and (3).

It is shown in Lie [1] that a scalar linear parabolic PDE (1) has the undermentioned four canonical forms:

$$\begin{aligned} &u_t = u_{xx}, \\ &u_t = u_{xx} + \frac{A}{x^2}u, \ \ A \neq 0, \\ &u_t = u_{xx} + c(x)u, \ \ c \neq 0, \ \ A/x^2, \\ &u_t = u_{xx} + c(t,x)u, \ \ A \neq 0. \end{aligned} \tag{4}$$

The heat equation which is the first Lie canonical form has six nontrivial point symmetries in addition to the infinite number of trivial superposition symmetries. The second, third, and fourth Lie canonical forms in (4) have nontrivial symmetries 4, 2, and 1, respectively.

Theorem 1 ([6]). *The necessary and sufficient conditions for the reduction of the scalar linear $(1+1)$ parabolic PDE (1) to*

(a) the heat equation

$$\bar{u}_{\bar{t}} = \bar{u}_{\bar{x}\bar{x}} \tag{5}$$

via the transformations

$$\begin{aligned} \bar{t} &= \phi(t), \\ \bar{x} &= \pm\int[\dot{\phi}\,a^{-1}]^{1/2}\mathrm{d}x + \beta(t), \\ \bar{u} &= \nu(t)|a|^{-1/4}u\exp\bigg[\int\frac{b}{2a}\mathrm{d}x \\ &\quad -\frac{1}{8}\frac{\ddot{\phi}}{\dot{\phi}}\bigg(\int\frac{\mathrm{d}x}{|a|^{1/2}}\bigg)^2 \\ &\quad -\frac{1}{2}\int\frac{1}{|a|^{1/2}}\partial_t\bigg(\int\frac{\mathrm{d}x}{|a|^{1/2}}\bigg)\mathrm{d}x \\ &\quad \mp\frac{1}{2}\frac{\dot{\beta}}{|\dot{\phi}|^{1/2}}\int\frac{\mathrm{d}x}{|a|^{1/2}}\bigg], \end{aligned} \tag{6}$$

where $\dot{\phi}$ and a have the same sign, are that the coefficients of the PDE (1) with ϕ, β, and ν of the transformations (6) defined by $f(t), g(t)$, and $h(t)$ given by

$$f(t) = \frac{1}{16}\frac{\ddot{\phi}^2}{\dot{\phi}^2} - \frac{1}{8}\left(\frac{\ddot{\phi}}{\dot{\phi}}\right)_t ,$$
$$g(t) = \pm\frac{1}{4}\frac{\ddot{\phi}}{\dot{\phi}}\frac{\dot{\beta}}{\dot{\phi}^{1/2}} \mp \frac{1}{2}\left(\frac{\dot{\beta}}{\dot{\phi}^{1/2}}\right)_t ,$$
$$h(t) = \frac{1}{4}\frac{\ddot{\phi}}{\dot{\phi}} + \frac{1}{4}\frac{\dot{\beta}^2}{\dot{\phi}} + \frac{\dot{\nu}}{\nu}, \tag{7}$$

satisfy the invariant condition

$$2L_x + 2M_x - N_x = 0, \tag{8}$$

where

$$L = |a|^{1/2}(|a|^{1/2}J_x)_x ,$$
$$M = |a|^{1/2}[|a|^{1/2}\partial_t(b/2a)]_x ,$$
$$N = |a|^{1/2}\partial_t^2(1/|a|^{1/2}), \tag{9}$$

and J is

$$J = c - \frac{b_x}{2} + \frac{ba_x}{2a} + \frac{a_{xx}}{4} - \frac{3}{16}\frac{a_x^2}{a} - \frac{a_t}{2a} - \frac{b^2}{4a}, \tag{10}$$

as well as the constraining relation

$$J + \partial_t\int\frac{b}{2a}\mathrm{d}x - \frac{1}{2}\int\frac{1}{|a|^{1/2}}\partial_t^2\left(\int\frac{\mathrm{d}x}{|a|^{1/2}}\right)\mathrm{d}x + f(t)\left(\int\frac{\mathrm{d}x}{|a|^{1/2}}\right)^2 + g(t)\int\frac{\mathrm{d}x}{|a|^{1/2}} + h(t) = 0 \tag{11}$$

holds;

(b) the second Lie canonical equation

$$\bar{u}_{\bar{t}} = \bar{u}_{\bar{x}\bar{x}} + \frac{A}{\bar{x}^2}, \tag{12}$$

where $A \neq 0$ is a constant, via transformations (6) with $\beta = 0$, are that the coefficients of the PDE (1) with ϕ and ν in the transformations (6) defined by $f(t)$ and $h(t)$ in (7), satisfy the invariant condition, provided that condition (8) does not hold,

$$20L_x + 20M_x - 10N_x + 10(|a|^{1/2}M_x)_x\int\frac{\mathrm{d}x}{|a|^{1/2}} - 5(|a|^{1/2}N_x)_x\int\frac{\mathrm{d}x}{|a|^{1/2}} + 10(|a|^{1/2}L_x)_x\int\frac{\mathrm{d}x}{|a|^{1/2}} + [|a|^{1/2}(|a|^{1/2}L_x)_x]_x\left(\int\frac{\mathrm{d}x}{|a|^{1/2}}\right)^2 + [|a|^{1/2}(|a|^{1/2}M_x)_x]_x\left(\int\frac{\mathrm{d}x}{|a|^{1/2}}\right)^2 - \frac{1}{2}[|a|^{1/2}(|a|^{1/2}L_x)_x]_x\left(\int\frac{\mathrm{d}x}{|a|^{1/2}}\right)^2 = 0, \tag{13}$$

where L, M, N, and J are as given in (9) and (10), together with the constraining relation

$$A = \left(\int\frac{\mathrm{d}x}{|a|^{1/2}}\right)^2\Bigg[J + \partial_t\int\frac{b}{2a}\mathrm{d}x - \frac{1}{2}\int\frac{1}{|a|^{1/2}}\partial_t^2\left(\int\frac{\mathrm{d}x}{|a|^{1/2}}\right)\mathrm{d}x + f(t)\left(\int\frac{\mathrm{d}x}{|a|^{1/2}}\right)^2 + h(t)\Bigg] \tag{14}$$

being satisfied;

(c) the third Lie canonical equation

$$\bar{u}_{\bar{t}} = \bar{u}_{\bar{x}\bar{x}} + \bar{c}(\bar{x})\bar{u}, \tag{15}$$

are that the coefficients of the PDE (1) satisfy the invariant criterion, provided that the conditions (8) and (13) do not hold,

$$\frac{\partial}{\partial t}\left[J + \partial_t\left(\int\frac{b}{2a}\mathrm{d}x\right) - \frac{1}{2}\int\frac{1}{|a|^{1/2}}\partial_t^2\left(\int\frac{\mathrm{d}x}{|a|^{1/2}}\right)\mathrm{d}x\right] = 0; \tag{16}$$

moreover, $\bar{c}$ in (15) satisfy

$$\varepsilon\bar{c} = J + \partial_t\left(\int\frac{b}{2a}\mathrm{d}x\right) - \frac{1}{2}\int\frac{1}{|a|^{1/2}}\partial_t^2\left(\int\frac{\mathrm{d}x}{|a|^{1/2}}\right)\mathrm{d}x. \tag{17}$$

It must be noted that if $a > 0$, then $\varepsilon = 1$, otherwise $\varepsilon = -1$;

(d) the fourth Lie canonical form

$$\bar{u}_{\bar{t}} = \bar{u}_{\bar{x}\bar{x}} + \bar{c}(\bar{t},\bar{x})\bar{u}, \tag{18}$$

are that the coefficients of the parabolic PDE (1) satisfy the condition, provided that the conditions (8), (13), and (16) do not hold,

$$\frac{\partial}{\partial t}\left[J + \partial_t\left(\int\frac{b}{2a}\mathrm{d}x\right) - \frac{1}{2}\int\frac{1}{|a|^{1/2}}\partial_t^2\left(\int\frac{\mathrm{d}x}{|a|^{1/2}}\right)\mathrm{d}x\right] \neq 0. \tag{19}$$

3. Application of the Invariant Method to Brain Tumor Model and Fundamental Solution

In this section, we consider the following linear parabolic PDE which is a particular class of (1) given by

$$u_t = \Omega u_{xx} + \frac{2\Omega}{x} u_x + F(t,x)u\,, \tag{20}$$

where Ω is an arbitrary constant and $F(t,x)$ is an arbitrary function of its variables. In the brain tumor model, (20) takes the form

$$u_t = \Omega u_{xx} + \frac{2\Omega}{x} u_x + [p - k(t)]u\,, \tag{21}$$

where $u(t,x)$ denotes the concentration of tumor cells at location x at time t, the arbitrary constant Ω is known as the diffusion coefficient which captures the invasiveness of the cells of the tumor called glioblastoma multiforme, p is the proliferation rate of the tumor, $k(t)$ is an arbitrary function which is the (therapy-dependent) killing rate at time t. Moreover, the tumor spread is assumed to be spherically symmetric in this class of models, and x measures the distance from the centre (i.e. the origin of glioblastoma multiforme), (see [9, 10] and the references therein).

3.1. Equivalence Transformation Formulae and Reduction to Lie Canonical Forms

Here we use the theorem of invariant approach discussed in Section 2 to find the equivalence transformations which reduce the model (21) to the Lie canonical forms.

From (21), we deduce $a(t,x) = \Omega$, $b(t,x) = 2\Omega/x$ and $c(t,x) = p - k(t)$. By using (10), we evaluate J as

$$J = p - k(t)\,. \tag{22}$$

Firstly we look at the reduction of (21) to the first canonical form, i.e. the linear heat equation (5). Invoking (9), we obtain $L = 0, M = 0$, and $N = 0$ for the PDE (21). Thus obviously the invariant condition (8) is satisfied and we get reduction to the heat equation (5). The relation (11) gives $f = 0, g = 0$, and $h = k(t) - p$.

There now arise two cases depending on the sign of Ω.

Case 1 ($\Omega > 0$).
In this case, using (6) and (7), we find the following equivalence transformation formulae:

$$\begin{aligned}&\bar{t} = t + a_1\,, \quad \bar{x} = \Omega^{-1/2}x + b_1\,,\\ &\bar{u} = b_2\Omega^{-1/4}x \exp\left[\int_0^t k(r)\mathrm{d}r - pt\right]u\,,\end{aligned} \tag{23}$$

where a_1 and b_i, $i = 1, 2$, are constants, which reduces (21) to the linear classical heat equation (5).

Case 2 ($\Omega < 0$).
Here the equivalence transformation which reduces (21) to the linear one-dimensional heat equation (5) is

$$\begin{aligned}&\bar{t} = -t + a_1\,, \quad \bar{x} = (-\Omega)^{-1/2}x + b_1\,,\\ &\bar{u} = b_2(-\Omega)^{-1/4}x \exp\left[\int_0^t k(r)\mathrm{d}r - pt\right]u\,,\end{aligned} \tag{24}$$

here again a_1 and b_i, $i = 1, 2$, are constants.

Since $L = M = N = 0$ for the PDE (21), the invariant conditions for reduction to the second, third, and fourth Lie canonical forms are not satisfied, thus we do not obtain equivalence transformations formulae which enable reduction for these cases.

3.2. The Fundamental Solution

In this section, we construct the fundamental solution of the Cauchy problem for the brain tumor model [10]

$$\begin{aligned}&u_t - \Omega u_{xx} - \frac{2\Omega}{x}u_x + [k(t) - p]u = 0\,,\\ &0 < x < x_0\,, \ \ 0 < t < t_0\,,\\ &u(t_0,x) = \delta(x - x_0)\,.\end{aligned} \tag{25}$$

Here the function $\delta(x)$ is the Dirac delta measure. Moreover, we use the approach as given in the work [11].

Now we utilize the equivalence transformations (23) and (24), which reduce the brain tumor model (21) to the classical heat equation (5), in the construction of the fundamental solutions of the underlying model.

The fundamental solution of the Cauchy problem for the heat equation (5) is well-known [12], and we give it in bar coordinates as follows:

$$\bar{u}(\bar{t},\bar{x}) = \frac{1}{\sqrt{4\pi\bar{t}}}\exp\left(-\frac{\bar{x}^2}{4\bar{t}}\right). \tag{26}$$

Case 1 ($\Omega > 0$).

For the above-mentioned case, we look for the fundamental solution for the Cauchy problem (25). The solution (26) is transformed using the transformation formulae (23) so that we derive the following solution for (21):

$$u(t,x) = \frac{\Omega^{1/4}}{2b_2 x\sqrt{\pi(t+a_1)}} \cdot \exp\left[pt - \int_0^t k(r)\mathrm{d}r - \frac{\left(\frac{x}{\Omega^{1/2}}+b_1\right)^2}{4(t+a_1)}\right]. \tag{27}$$

The solution in (27) is a fundamental solution whenever

$$\lim_{t\to t_0} u(t,x) = \delta(x-x_0) \tag{28}$$

is satisfied. Evidently, the functional form of $u(t,x)$ in (27) suggests that it is not possible to obtain a solution that satisfies the condition (28).

Case 2 ($\Omega < 0$).

In this case, we construct the fundamental solution for the Cauchy problem (25) using the equivalence transformation (24). Thus by applying the transformation formulae (24) to the solution (26), we obtain the solution

$$u(t,x) = \frac{(-\Omega)^{1/4}}{2b_2 x\sqrt{\pi(a_1-t)}} \cdot \exp\left\{pt - \int_0^t k(r)\mathrm{d}r - \frac{\left[\frac{x}{(-\Omega)^{1/2}}+b_1\right]^2}{4(a_1-t)}\right\}. \tag{29}$$

Thus the Cauchy problem (25) has the fundamental solution (29) provided the limit (28) holds.

Let $t_0 := a_1$ and $z(x) - z_0 := \frac{x}{(-\Omega)^{1/2}} + b_1$, where the constant z_0 is defined by $z_0 = z(x_0)$. By using the well-known limit

$$\lim_{p\to 0}\frac{1}{\sqrt{4p\pi}}\exp\left[-\frac{(x-x_0)^2}{4p}\right] = \delta(x-x_0), \tag{30}$$

we obtain from (29) that

$$\lim_{t\to t_0} u(t,x) = \frac{(-\Omega)^{1/4}}{b_2 x} \cdot \exp\left[pt_0 - \int_0^{t_0} k(r)\mathrm{d}r\right]\delta(z-z_0). \tag{31}$$

To find the Dirac function in the new variable $z = z(x)$, we use the formula as given in [13], that is,

$$\delta(x-x_0) = \left|\frac{\partial z(x)}{\partial x}\right|_{x=x_0}\delta(z-z_0). \tag{32}$$

Now using the condition (28), we deduce that

$$b_2 = (-\Omega)^{3/4} x_0^{-1}\exp\left[pt_0 - \int_0^{t_0} k(r)\mathrm{d}r\right]. \tag{33}$$

Thus by substituting (33) for b_2 into (29), we obtain the following fundamental solution of the Cauchy problem (25):

$$u(t,x) = \frac{x_0}{\sqrt{4(-\Omega)\pi(t_0-t)}}x^{-1} \cdot \exp\left[-p(t_0-t) + \int_t^{t_0} k(r)\mathrm{d}r - \frac{(x-x_0)^2}{4(-\Omega)(t_0-t)}\right]. \tag{34}$$

Remark 1. The concentration of tumor cells denoted by $u(t,x)$ given by the solutions (27), (29), and (34) of (21) can be explained by taking into account the critical nature of the value of the expression $p-k(t)$. In the case when $p-k(t) > 0$, the number of cells increases exponentially with time which is one of the characteristics of glioblastoma multiforme and the cause for the fatal outcome. The case in which $p-k(t) < 0$, one can see that the solutions imply a rapid drop in the number of cells. The two- and three-dimensional sketches for the solution (34) are given in Figures 1 and 2, respectively.

We now find another form of solution for the PDE (21) with boundary conditions during the period of treatment for the growth of the tumor as follows. Clearly the concentration of the tumor cells $u(t,0) = 0$. During the chemotherapy treatment the killing rate $k(t) = k$ is a constant. Therefore,

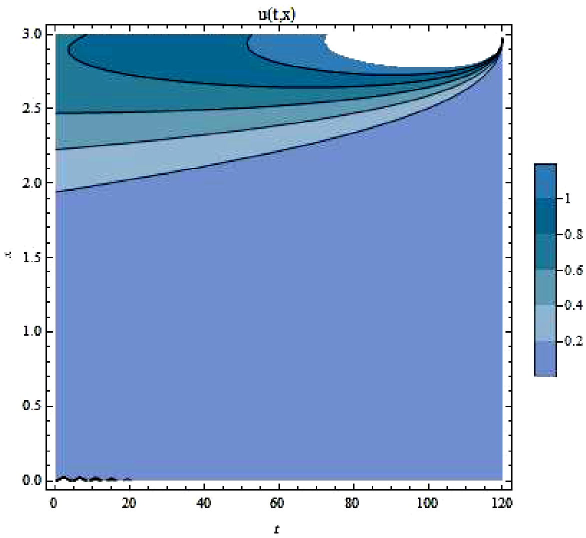

Fig. 1 (colour online). Two-dimensional sketch for the solution (34) for $0 \le x \le 3$ and $0 \le t \le 120$. The concentration of tumor cells is denoted by $u(t,x)$. From [10], it is known that for the glioblastoma multiforme cells, the diffusion coefficient is $-\Omega = 0.0013\,\text{cm}^2$ per day and the proliferation rate of the tumor is $p = 0.0393$ per day. The tumor is diagnosed when the diameter of the tumor that is visually detectable reaches 3 cm.

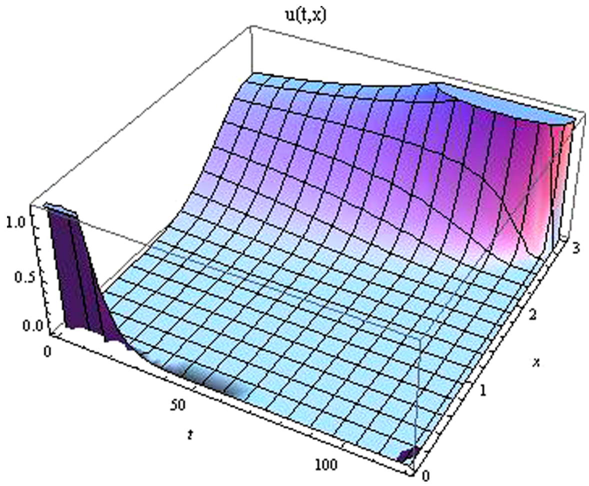

Fig. 2 (colour online). Three-dimensional sketch for the solution (34) for $0 \le x \le 3$ and $0 \le t \le 120$. When $p - k(t) > 0$, the number of cells increases exponentially with time. For $p - k(t) < 0$, the solutions imply a rapid drop in the number of cells. During the chemotherapy treatment the killing rate $k(t) = k$ is a constant. If the tumor is decreasing in size during the chemotherapy treatment, it implies that $k > p$. Therefore, one can assume that the killing rate is $k = 0.04$ per day.

$\exp\left[\int_0^t k(r)\mathrm{d}r - pt\right] = \exp[(k-p)t]$. If $k - p > 0$, then the concentration of tumor cells is decreased rapidly, so we can assume $u(t,x_0) = 0$. The transformed boundary conditions are then $\bar{u}(t,0) = 0$ and $\bar{u}(t,x_0) = 0$. Now another form of solution in series of the heat equation (5) for these boundary conditions is [14]

$$\bar{u} = \sum_{n=1}^{\infty} A_n \exp\left[-\left(\frac{n\pi}{x_0}\right)^2 \bar{t}\right] \sin\left(\frac{n\pi\bar{x}}{x_0}\right), \quad (35)$$

where A_n are constants, and (35) converges uniformly for $0 \le x \le x_0$, $t \ge 0$. Thus by using the transformation (24), the solution of (21) is given by

$$\begin{aligned} u(t,x) = &\sum_{n=1}^{\infty} \frac{A_n(-\Omega)^{1/2}}{b_2 x} \\ &\cdot \exp\left[pt - \int_0^t k(r)\mathrm{d}r - \left(\frac{n^2\pi^2}{x_0^2}\right)(-t + a_1)\right] \\ &\cdot \sin\left(\frac{n\pi}{x_0}\left[(-\Omega)^{-1/2}x + b_1\right]\right). \end{aligned} \quad (36)$$

3.3. Cauchy Initial Value Problem and Closed-Form Solution of (25)

Here in this section, we derive the closed-form analytical solution of the Cauchy problem (25). We make use of the appropriate results on the evolution of the Cauchy problem from [11, 15], with necessary adaptation, to obtain the solution of our problem.

Let $S(t,x;\xi,\tau)$ be the fundamental solution of the Cauchy problem

$$\begin{aligned} u_t - \Omega u_{xx} - \frac{2\Omega}{x} u_x + [k(t) - p]u = 0\,, \\ u(0,x) = \varphi(x)\,. \end{aligned} \quad (37)$$

If we suppose the existence and uniqueness of the solution for the Cauchy problem (37), then it is given by [11, 15]

$$u(t,x) = \int_{\mathbb{R}^n} S(t,x;\xi,0)\varphi(\xi)d\xi\,. \quad (38)$$

Now let $s := x_0$, $t_0 := T$ in the fundamental solution (34). If we note that $\varphi(x) = 1$, then the solution to the Cauchy problem (25) using (38) results in

$$u(t,x) = \frac{x^{-1}}{\sqrt{4(-\Omega)\pi(T-t)}}$$

$$\cdot \exp\left[-p(T-t)+\int_t^T k(r)\mathrm{d}r\right] \cdot \int_0^\infty s\exp\left[-\frac{(x-s)^2}{4(-\Omega)(T-t)}\right]\mathrm{d}s\,. \tag{39}$$

The value of the integral in (39) can be obtained in terms of the Kummer confluent hypergeometric functions as we have $T > t$.

4. The Invariant Approach to the Extended Linear Parabolic PDE (20)

In this section, we show how one can use the invariant method to find in a simple manner the possible forms of $F(t,x)$ for which the linear parabolic PDE (20) can be transformed to the Lie canonical forms. To this end, we again use the theorem given in Section 2.

We first consider the reduction to the heat equation. The parabolic PDE (20) has

$$L = |\Omega| F_{xx}\,,\; M = N = 0\,, \text{and } J = F(t,x)\,. \tag{40}$$

The invariant criterion (8) for reduction to the heat equation yields

$$L_x = 0\,. \tag{41}$$

Since $|\Omega| \neq 0$, we obtain

$$F_{xxx} = 0\,. \tag{42}$$

Thus from (42), we infer that

$$F(t,x) = B(t)x^2 + C(t)x + D(t)\,, \tag{43}$$

where $B(t)$, $C(t)$, and $D(t)$ are arbitrary functions. From the constraining relation (11), we obtain, $f(t) = -|\Omega|B(t)$, $g(t) = -|\Omega|^{1/2}C(t)$, and $h(t) = -D(t)$ which imply that the transformations (6) depends on the solution of a Riccati equation for the transformation in t. The others are then derived from this.

Now we look at the possibility when the parabolic PDE (20) is reducible to the second Lie canonical form (12) by making use of the invariant condition (13) which simplifies to

$$x^2 L_{xxx} + 10xL_{xx} + 20L_x = 0\,, \tag{44}$$

where $L_x \neq 0$, so from (40) and (44), we derive

$$x^2 F_{xxxxx} + 10xF_{xxxx} + 20F_{xxx} = 0\,. \tag{45}$$

Thus from (45), we have

$$F(t,x) = B(t)x^{-1} + C(t)x^{-2} + D(t)x^2 + E(t)x + G(t)\,, \tag{46}$$

where $B(t)$, $C(t)$, $D(t)$, $E(t)$, and $G(t)$ are arbitrary functions. Moreover, by invoking the constraining relation (14), one can determine a simplified form for the function $F(t,x)$ given by

$$F(t,x) = A\Omega x^{-2} + D(t)x^2 + G(t)\,, \tag{47}$$

where $D(t)$ and $G(t)$ are arbitrary functions. Construction of the transformations (6) again requires the solution of a Riccati equation.

Next we consider the instance when the parabolic PDE (20) is equivalent to the third Lie canonical form (15). One can readily find from the invariant condition (16) of Theorem 1 that

$$\frac{\partial J}{\partial t} = 0\,, \tag{48}$$

provided that the invariant conditions (8) and (13) do not hold. Thus from (40) and (48), we have $F_t = 0$, and hence $F(t,x) = F(x)$ and $\bar{c} = F(x)$ from (17). Therefore, for the functional forms of $F(x)$ which are not of the forms of (43) and (47), the PDE (20) can be reduced to the third canonical form. A transformation which does the reduction is $\bar{t} = t$, $\bar{x} = x/\sqrt{\Omega}$, $\bar{u} = xu$. For example, the parabolic PDE $u_t = \Omega u_{xx} + \frac{2\Omega}{x}u_x + x^3 u$ is reducible to the third Lie canonical PDE of the form $\bar{u}_{\bar{t}} = \bar{u}_{\bar{x}\bar{x}} + \bar{c}(\bar{x})\bar{u}$, where $\bar{c}(\bar{x}) = x^3$, by means of the above equivalence transformation.

We now look at the situation when the parabolic PDE (20) is reducible to the fourth Lie canonical form (18). The invariant condition (19) reduces into

$$\frac{\partial J}{\partial t} \neq 0\,, \tag{49}$$

provided that (8) and (13) and (16) do not hold. So we have $F_t \neq 0$ and then one gets reduction to the fourth fourth Lie canonical form. Moreover, the transformation which reduces to the form $\bar{u}_{\bar{t}} = \bar{u}_{\bar{x}\bar{x}} + F(\bar{t},\bar{x})\bar{u}$ is again $\bar{t} = t$, $\bar{x} = x/\sqrt{\Omega}$, $\bar{u} = xu$. An example for this case is the parabolic equation $u_t = \Omega u_{xx} + \frac{2\Omega}{x}u_x + \gamma(t)x^4 u$, which is reducible to the fourth Lie canonical PDE of the form $\bar{u}_{\bar{t}} = \bar{u}_{\bar{x}\bar{x}} + \bar{c}(\bar{t},\bar{x})\bar{u}$, where $\bar{c}(\bar{t},\bar{x}) = \gamma(t)x^4$, under the above equivalence transformation.

5. Concluding Remarks

We have studied a class of scalar linear $(1+1)$ parabolic PDEs which models brain tumors. These models are used to weave the insights gained from experimental observations of individual biochemical and biomechanical processes into a coherent description of tumor growth. We applied the invariant method [6] to find an equivalence transformation which reduced the underlying class of PDEs to the first Lie canonical form, i.e. the classical heat PDE. We constructed a fundamental solution for the brain tumor PDE using the equivalence transformation and the well-known fundamental solution of the heat equation. These transformation formulae and the fundamental solution for (21) were not reported previously in the literature. Using this approach, for the first time the closed-form solution of the Cauchy problem (25) for the brain tumor model is derived. We also obtained another solution for the model under discussion with possible boundary conditions, using the transformation formulae and a known solution of the heat equation with boundary conditions. We also utilized the invariant approach to the extended form of the class of brain tumor models to perform classification via equivalence transformations to reduce the class of PDEs in a simple way to the four Lie canonical forms.

It is noteworthy that the authors (see [16, 17]) introduced the group theoretical approach called invariance principle for boundary value problems to find fundamental solution of the Cauchy problem of linear PDEs. The above-mentioned method is a combination of application of Lie symmetries and the use of the theory of generalized functions. That is, if the boundary value problem is invariant under group G, then the fundamental solution is found among the functions invariant under G.

In the invariant approach to obtain fundamental solutions of parabolic linear PDEs [6], one utilizes the equivalence transformation formulae which map the given PDEs to the Lie canonical forms, their fundamental solutions, and the properties of the Dirac function.

Acknowledgements

F. M. is Visiting Professorial Fellow at UNSW for 2014. He thanks the NRF of South Africa for a research grant. He is also thankful to the School of Mathematics and Statistics at UNSW where this work was completed for enabling research facilities.

We are thankful to the referees and the editor for their useful comments which improved the presentation of this paper.

[1] S. Lie, Archiv for Mathematik og Naturvidenskab **VI**, 328 (1881) [in German]. Reprinted in S. Lie, Gesammelte Abhandlungen, 3 paper XXXV.
[2] L. V. Ovsiannikov, Group Properties of Differential Equations, Academic Press, New York 1982.
[3] G. W. Bluman, SIAM J. Appl. Math. **43**, 1259 (1983).
[4] I. K. Johnpillai and F. M. Mahomed, J. Phys. A: Math. Gen. **34**, 11033 (2001).
[5] F. M. Mahomed and C. A. Pooe, in: Proceedings of the International Conference Modern Group Analysis 9 (Mogran 9), O. V. Rudenko and O. A. Sapozhnikov (Eds.), p. 525, 16th ISNA, Moscow State University, Moscow 2003.
[6] F. M. Mahomed, J. Nonlin. Math. Phys. **15**, 112 (2008).
[7] F. M. Mahomed, K. S. Mahomed, R. Naz, and E. Momoniat, Math. Comput. Appl. **18**, 244 (2013).
[8] W. Sinkala, P. G. L. Leach, and J. G. O'Hara, J. Differ. Equations **244**, 2820 (2008).
[9] L. M. Wein, J. T. Wu, A. G. Ianculescu, and R. K. Puri, Cell Proliferat. **35**, 343 (2002).
[10] L. M. Wein and D. Koplow, D Sloan School of Management, Preprint, MIT (1999).
[11] C. A. Pooe, F. M. Mahomed, and C. Wafo Soh, Nonlin. Dynam. **36**, 69 (2004).
[12] P. J. Olver, Applications of Lie Groups to Differential Equations, 2nd edn., Springer, New York 1993.
[13] R. Courant and D. Hilbert, Methods of Mathematical Physics, Interscience, New York 1962.
[14] J. D. Logan, Applied Partial Differential Equations, Springer-Verlag, New York 1998.
[15] J. Hadamard, Lectures on Cauchy's Problem in Linear Partial Differential Equations, Yale University Press, New Haven, Connecticut 1923.
[16] V. A. Baikov, R. K. Gazizov, and N. H. Ibragimov, in: CRC Handbook of Lie Group Analysis of Differential Equations, N. H. Ibragimov (Ed.) vol. 3, CRC Press, Boca Raton, Florida 1996.
[17] R. K. Gazizov and N. H. Ibragimov, Nonlin. Dynam. **17**, 387 (1998).