Optimal Campaign Strategies in Fractional-Order Smoking Dynamics

Anwar Zeb[a,b], Gul Zaman[b], Il Hyo Jung[c], and Madad Khan[a]

[a] Department of Mathematics, COMSATS Institute of Information Technology Abbottabad, K.P.K, Pakistan

[b] Department of Mathematics, University of Malakand, Chakdara, Dir (L), K.P.K, Pakistan

[c] Department of Mathematics, Pusan National University, San 30, Geumjeong-Gu, Busan 609-735, South Korea

Reprint requests to M. K.; E-mail: madadmath@yahoo.com

Z. Naturforsch. **69a**, 225 – 231 (2014) / DOI: 10.5560/ZNA.2014-0020
Received December 23, 2013 / revised March 10, 2014 / published online May 21, 2014

This paper deals with the optimal control problem in the giving up smoking model of fractional order. For the eradication of smoking in a community, we introduce three control variables in the form of education campaign, anti-smoking gum, and anti-nicotive drugs/medicine in the proposed fractional order model. We discuss the necessary conditions for the optimality of a general fractional optimal control problem whose fractional derivative is described in the Caputo sense. In order to do this, we minimize the number of potential and occasional smokers and maximize the number of ex-smokers. We use Pontryagin's maximum principle to characterize the optimal levels of the three controls. The resulting optimality system is solved numerically by MATLAB.

Key words: Giving up Smoking Model; Fractional Order Derivatives; Optimal Control; Numerical Analysis.

Mathematics Subject Classification 2010: 92D25, 49J15, 93D20

1. Introduction

Most infectious diseases could be driven towards eradication if adequate and timely steps like vaccination, treatment, educational and enlightenment campaign, etc are taken in the course of the epidemic. However, many of these diseases eventually become endemic in our society due to lack of adequate policies and timely interventions to mitigate the spread of the diseases. Consequently, there is the need for proactive steps towards controlling the spread of infectious diseases, particularly those ones for which both vaccine and cure are available. Moreover, it is often cheaper to prevent the occurrence of a disease than to cure it. Optimal control and ramifications have found applications in many different fields, including aerospace, process control, robotics, bio-engineering, economics, finance and management sciences. Before the arrival of digital computers in the 1950s, only fairly simple optimal control problems could be solved. The arrival of digital computers has enabled the application of optimal control theory and methods to many complex problems. The study of epidemic models is strongly related to the possibility of evaluation of different control strategies: screening and educational campaigns [1], the vaccination campaign [2], and the resource allocation [3].

Smoking is the biggest cause of both preventable and premature deaths throughout the world and therefore special attention is required to control it. Smoking-related diseases are cause of over 440 000 deaths in the US annually and for the UK this figure stands over 105 000 annually. The life expectancy of the smoker is cut short by 10 – 12 years and more than half of all smokers die from smoking-related diseases [4]. Comparative smoking facts show that the risk of heart attack is 70% higher among smokers than non-smokers. The incidence of lung cancer is ten times greater in smokers than non-smokers and one out of ten of people that smoke will die from this disease. Some 80% of smokers will at one time be diagnosed with heart disease, emphysema or chronic bronchitis of the diseases attributable to the tobacco habit, 29% are from lung cancer, and 24% are caused by heart disease [5]. Without these health diseases cancers have also been linked to smoking, including cancer of the throat, mouth, stomach, cervix, breast, and pancreas. All this is small wonder as cigarette smoke has been found to contain over 4000 chemical compounds and toxins, all with very harmful components to human health. Moreover, the financial burden imposed by cigarette smoking is enormous. Smoking-related illness in the United States costs 96 billion each year in medical costs and 97 billion in lost productivity due to premature mortality [6] and the human toll on survivors and caregivers of individuals affected by tobacco-related illness is incalculable. In addition, there is a growing concern about new tobacco products being marketed to smokers and non-smokers as alternatives for use in smoke-free environments. Dual use of cigarettes and smokeless tobacco can sustain tobacco addiction, encouraging continued tobacco use among smokers who might quit. Consumer miss perceptions regarding the safety of the use of these products, independently and concurrently with smoking pose an ongoing challenge to tobacco prevention and control efforts.

Several authors did a lot of work in order to understand the dynamics of smoking. In 2000, Castillo–Garsow et al. [7] for the first time proposed a simple giving up smoking mathematical model. In that work they considered a system with a total constant population which is divided into three subclasses. Zaman [8] extended the work of Castillo–Garsow et al. [7] and developed a model taking into account the occasional smokers compartment in the giving up smoking model and presented its qualitative behaviour. But all these work has been done for the integer-order differential equations.

In the present days researchers try to work on fractional-order differential equations because of best presentation and applications of many phenomena. Differential equations of fractional order have been the focus of many studies due to their frequent appearance in different applications ie fluid mechanics, biology, physics, Epidemiology, and engineering. Recently, a large amount of literatures are developed concerning the application of fractional differential equations in nonlinear dynamics (see for example [9 – 15]). The differential equations with fractional order have recently proved to be valuable tools to the modelling of many real world problems. This is because of the fact that the realistic modelling of a physical phenomenon does not depend only on the instant time, but also on the history of the previous time which can also be successfully achieved by using fractional calculus. The reason of using fractional-order differential equations is that fractional-order derivatives are naturally related to systems with memory which exist in most biological systems. They are also closely related to fractals which are abundant in biological systems. For this purpose Erturk et al. [16] introduced fractional order to the giving up smoking model and find the analytic approximate solution using a multi-step differential transform method. The model presented by Erturk et al. [16] is given by

$$\begin{aligned}
D_t^\alpha P(t) &= \alpha - \beta_1 P(t)L(t) - (d_1+\mu)P(t) + \tau Q(t)\,,\\
D_t^\alpha L(t) &= \beta_1 P(t)L(t) - \beta_2 L(t)S(t) - (d_2+\mu)L(t)\,,\\
D_t^\alpha S(t) &= \beta_2 L(t)S(t) - (\gamma + d_3 + \mu)S(t)\,,\\
D_t^\alpha Q(t) &= \gamma S(t) - (\mu + d_4 + \tau)Q(t)\,.
\end{aligned} \tag{1}$$

Here μ is the natural death rate, γ is the recover rate from infection, β is the transmission coefficient, δ is the quit rate of smoking, d_1, d_2, d_3, and d_4 represent the death rate for potential smokers, occasional smoker, smoker, and ex-smoker related to smoking disease, respectively. Additionally, the rate at which the ex-smoker in the population becomes potential smoker again is presented by τ.

The total population is given by

$$N(t) = P(t) + L(t) + S(t) + Q(t)\,. \tag{2}$$

By the conservation law, we have

$$D_t^\alpha N(t) = \alpha - (\mu + d_1 + d_2 + d_3 + d_4)N(t)\,, \tag{3}$$

with

$$N(0) = P(0) + L(0) + S(0) + Q(0)\,.$$

Most programs that attempt to reduce smoking in young people focus on prevention of smoking initiation. However, effective smoking cessation programs designed to address the unique needs of young people who are already using tobacco are rare. It has been recommended that cessation programs educate youth about tobacco as well as help them become and stay motivated to quit.

In order to control the spread of smoking in a community, we develop an optimal control strategy in this paper. We introduce three functions in the form of education campaign, anti-smoking gum, and anti-nicotine drug/medicine. Our goal in this work is to minimize the number of potential and smoking individuals and maximize the number of ex-smoker by applying these three control variables. To do this, first we adjust the control functions in the fractional-order smoking model. Then we derived the optimality conditions for the fractional-order model. Finally, we show parameters estimation and represent numerical simulations.

The paper is organized as follows: In Section 2, we give some basic definitions and formulation of the model. In Section 3, we derive an optimal control problem. Parameters estimation and numerical results are discussed in Section 4. Finally, we give a conclusion in Section 5.

2. Fractional Control Theory with Preliminaries

In this section, we present the fractional optimal control theory and give some basic definitions which are necessary for the subsequent sections. In 2004, a general formulation was presented for fractional optimal control problem (FOCP) by Agrawal [13]:

$$J\left[u(t)\right] = \theta\left[X(t_\mathrm{f}),t_\mathrm{f}\right] + \int_0^{t_\mathrm{f}} \phi\left[X(t),u(t),t\right]\mathrm{d}t\,.$$

Definition 1. Left-sided Caputo fractional derivative of $f(t)$ is defined as

$$\begin{aligned}{}_a^{\mathrm{C}}D_t^{\alpha} f(t) = {} & \frac{1}{\Gamma(n-\alpha)} \\ & \cdot \int_a^t (t-\tau)^{n-\alpha-1}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^n f(\tau)\,\mathrm{d}\tau\,,\end{aligned} \tag{4}$$

and right-sided Caputo fractional derivative of $f(t)$ is defined as

$$\begin{aligned}{}_t^{\mathrm{C}}D_b^{\alpha} f(t) = {} & \frac{1}{\Gamma(n-\alpha)} \\ & \cdot \int_t^b (\tau-t)^{n-\alpha-1}\left(-\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^n f(\tau)\,\mathrm{d}\tau\,,\end{aligned} \tag{5}$$

where α is the order of the derivative such that $n-1 < \alpha < n$.

Definition 2. Left-sided Riemann–Liouville fractional derivative of $f(t)$ is defined as

$$\begin{aligned}{}_aD_t^{\alpha} f(t) = {} & \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \\ & \cdot \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\,\mathrm{d}\tau\,,\end{aligned} \tag{6}$$

and right-sided Riemann–Liouville fractional derivative of $f(t)$ is defined as

$$\begin{aligned}{}_tD_b^{\alpha} f(t) = {} & \frac{1}{\Gamma(n-\alpha)}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \\ & \cdot \int_t^b (\tau-t)^{n-\alpha-1} f(\tau)\,\mathrm{d}\tau\,,\end{aligned} \tag{7}$$

where α is the order of the derivative such that $n-1 < \alpha < n$.

Consider the following fractional differential system (FDS):

$$\begin{aligned}&{}_0^{\mathrm{C}}D_t^{\alpha} X(t) = f\big(X(t),u(t),t\big)\,,\\ &X(0) = X_0\,,\end{aligned} \tag{8}$$

where $0 < \alpha < 1$, $X(t)$ is a n-dimensional state vector, $u(t)$ is a m-dimensional control vector, and f is a n-dimensional vector-valued function with $t \in [0,t_\mathrm{f}]$. Here $t_\mathrm{f} > 0$ is the terminal time of the control process. Suppose that the performance index is given by

$$J[u(t)] = \theta\left[X(t_\mathrm{f}),t_\mathrm{f}\right] + \int_0^{t_\mathrm{f}} \phi\left[X(t),u(t),t\right]\mathrm{d}t\,, \tag{9}$$

where t_f is fixed and $X(t_\mathrm{f})$ is free. The fractional optimal control process is to find the optimal control law $u(t)$ for the FDS (8) that minimizes the performance index (9). For $\alpha = 1$, the mentioned problem reduces to the standard optimal control problem.

Proposition 1. The necessary condition for the optimality of FDS (8) is given by

$$\begin{aligned}&{}_tD_{t_\mathrm{f}}^{\alpha}\lambda(t) = \frac{\partial\phi}{\partial X} + \lambda^{\mathrm{T}}\frac{\partial f}{\partial X}\,,\\ &{}_0^{\mathrm{C}}D_t^{\alpha} X(t) = f\big(X(t),u(t),(t)\big)\,,\\ &\frac{\partial H}{\partial u} = \frac{\partial\phi}{\partial u} + \lambda^{\mathrm{T}}\frac{\partial f}{\partial u}\,,\\ &{}_tD_{t_\mathrm{f}}^{\alpha-1}\lambda(t_\mathrm{f}) = \left.\frac{\partial\theta}{\partial X}\right|_{t_\mathrm{f}}\,,\\ &X(0) = X_0\,.\end{aligned} \tag{10}$$

In the next section, we will use this theory to develop an optimal control strategy in the fractional-order giving up smoking model.

3. Control Strategy in Fractional-Order Model

In this section, we apply the optimal control strategy to system (1) in fractional order. To control the spread of smoking in the community, we use three control variables. These three control variables represent education campaign $u_1(t)$, anti-smoking gum $u_2(t)$, and anti-nicotive drug/medicine $u_3(t)$. The control variables satisfy the conditions u_1, $u_3 \in [0,1]$ and $u_2 \in [0,0.9]$ with $u_1(t) \le u_2(t) \le u_3(t)$. The new system with optimal control is

$$\begin{aligned}
{}_0^{\mathrm{C}}D_t^{\alpha}P(t) &= \alpha - \beta_1 P(t)L(t) \\
&\quad - \big(d_1+\mu+u_1(t)\big)P(t) + \tau Q(t)\,, \\
{}_0^{\mathrm{C}}D_t^{\alpha}L(t) &= \beta_1 P(t)L(t) - \beta_2\big(1-u_2(t)\big)L(t)S(t) \\
&\quad - (d_2+\mu)L(t)\,, \\
{}_0^{\mathrm{C}}D_t^{\alpha}S(t) &= \beta_2\big(1-u_2(t)\big)L(t)S(t) \\
&\quad - \big(\gamma+d_3+\mu+u_3(t)\big)S(t)\,, \\
{}_0^{\mathrm{C}}D_t^{\alpha}Q(t) &= \big(\gamma+u_3(t)\big)S(t) \\
&\quad - (\mu+d_4+\tau)Q(t) + u_1(t)P(t)\,,
\end{aligned} \tag{11}$$

where ${}_0^{\mathrm{C}}D_t^{\alpha}$ is the Caputo fractional derivative of α-order $\in (0,1]$.

We can write system (11) in vector form by

$${}_0^{\mathrm{C}}D_t^{\alpha}X(t) = f\big[X(t),u(t)\big]\,, \tag{12}$$

where

$$X^{\mathrm{T}}(t) = \big[P(t),L(t),S(t),Q(t)\big]$$

is the state vector and $u^{\mathrm{T}}(t) = \big[u_1(t),u_2(t),u_3(t)\big]$ is the control vector.

Our goal in this optimal control problem is to minimize the number of potential and smoking individuals and maximize the number of ex-smoking individuals by applying the above three control variables. To do this, we construct the following objective functional:

$$\begin{aligned}
J[u(t)] = \min \Bigg(&\int_0^{t_{\mathrm{f}}} \bigg[A_1 S(t) - A_2 Q(t) + \frac{1}{2}\Big(r_1 u_1^2(t) \\
&+ r_2 u_2^2(t) + r_3 u_3^2(t) \Big) \bigg] \mathrm{d}t + A_3 P(t_{\mathrm{f}}) + A_4 L(t_{\mathrm{f}}) \Bigg)\,.
\end{aligned} \tag{13}$$

Here t_{f} is the final time, A_i for $i = 1,2,3,4$ and r_i for $i = 1,2,3$ are positive weights parameters to balance the control factors.

We can write the objective functional in the following form:

$$J\big[u(t)\big] = \theta\big[X(t_{\mathrm{f}})\big] + \int_0^{t_{\mathrm{f}}} \phi\big[X(t)+u(t)\big]\,\mathrm{d}t\,. \tag{14}$$

Applying the result of (7), we can obtain the necessary conditions for the optimality system (8):

$$\begin{aligned}
&{}_0^{\mathrm{C}}D_t^{\alpha}X(t) = f\big(X(t),u(t)\big)\,, \\
&X(0) = X_0\,,
\end{aligned} \tag{15}$$

$$\begin{aligned}
&{}_tD_{t_{\mathrm{f}}}^{\alpha}\lambda(t) = \frac{\partial \phi}{\partial X} + \lambda^{\mathrm{T}}\frac{\partial f}{\partial X}\,, \\
&{}_tD_{t_{\mathrm{f}}}^{\alpha-1}\lambda(t_{\mathrm{f}}) = \frac{\partial \theta}{\partial X}\Big|_{t_{\mathrm{f}}}\,,
\end{aligned} \tag{16}$$

$$\frac{\partial H}{\partial u} = \frac{\partial \phi}{\partial u} + \lambda^{\mathrm{T}}\frac{\partial f}{\partial u}\,, \tag{17}$$

where

$$\lambda^{\mathrm{T}}(t) = \big[\lambda_1(t),\lambda_2(t),\lambda_3(t),\lambda_4(t)\big]$$

$$f(t) = \begin{cases}
\alpha - \beta_1 P(t)L(t) - (d_1+\mu)P(t) \\
\quad +\tau Q(t) - u_1(t)P(t)\,, \\
\beta_1 P(t)L(t) - \beta_2(1-u_2(t))L(t)S(t) \\
\quad -(d_2+\mu)L(t)\,, \\
\beta_2(1-u_2(t))L(t)S(t) \\
\quad -\big(\gamma+d_3+\mu+u_3(t)\big)S(t)\,, \\
\big(\gamma+u_3(t)\big)S(t) - (\mu+d_4+\tau)Q(t) \\
\quad +u_1(t)P(t)\,.
\end{cases}$$

By considering the terminal time, we can write

$$\theta\big[X(t_{\mathrm{f}})\big] = A_3 P(t_{\mathrm{f}}) + A_4 L(t_{\mathrm{f}})\,.$$

The objective functional can be written as

$$\begin{aligned}
\phi\big[X(t),u(t)\big] &= A_1 S(t) - A_2 Q(t) \\
&\quad + \frac{1}{2}\Big[r_1 u_1^2(t) + r_2 u_2^2(t) + r_3 u_3^2(t) \Big]\,.
\end{aligned}$$

We use Pontryagin's maximum principle to find the adjoint system

$$\begin{aligned}
{}_tD_{t_{\mathrm{f}}}^{\alpha}\lambda_1(t) &= \lambda_1\big(-\beta_1 L(t) - (d_1+\mu)\big) \\
&\quad + \beta_1\lambda_2 L(t)\,, \\
{}_tD_{t_{\mathrm{f}}}^{\alpha}\lambda_2(t) &= -\beta_1\lambda_1 P(t) + \lambda_2\big(\beta_1 P(t) - \beta_2 S(t) \\
&\quad - d_2 - \mu\big) + \beta_2\lambda_3 S(t)\,, \\
{}_tD_{t_{\mathrm{f}}}^{\alpha}\lambda_3(t) &= A_1 - \beta_2\lambda_2 L(t) + \lambda_3(\beta_2 L(t) - \gamma \\
&\quad - d_3 - \mu) + \gamma\lambda_4\,, \\
{}_tD_{t_{\mathrm{f}}}^{\alpha}\lambda_4(t) &= \tau\lambda_1 - \lambda_4(\tau + d_4 + \mu) - A_2\,,
\end{aligned} \tag{18}$$

$$\begin{aligned}
{}_tD_{t_{\mathrm{f}}}^{\alpha-1}\lambda_1(t) &= A_3\,, \\
{}_tD_{t_{\mathrm{f}}}^{\alpha-1}\lambda_2(t) &= A_4\,, \\
{}_tD_{t_{\mathrm{f}}}^{\alpha-1}\lambda_3(t) &= 0\,, \\
{}_tD_{t_{\mathrm{f}}}^{\alpha-1}\lambda_4(t) &= 0\,.
\end{aligned} \tag{19}$$

From the optimality conditions, we can find the optimal control variables

$$u_1(t) = \frac{\big(\lambda_1(t) - \lambda_4(t)\big)P(t)}{r_1},$$
$$u_2(t) = \frac{(\lambda_3 - \lambda_2)\beta_2 L(t)S(t)}{r_2}, \tag{20}$$
$$u_3(t) = \frac{(\lambda_3 - \lambda_4)S(t)}{r_3}.$$

4. Numerical Simulations

In this section, we present some numerical simulations that illustrate the theoretical results derived in the previous section. To achieve this, a program is developed in MATLAB to integrate the optimality system and the output was comprehensively verified using a detailed output from a number of runs. In our control problem, we obtain the optimality system from the state and adjoint equations. The optimal control problem strategy is obtained by solving the optimal system, which consists of eight ordinary differential equations and boundary conditions. Our choice of numerical method is a forward time/backward space finite difference method. Starting with an initial guess for the adjoint variables, the state equations are solved by a forward time and backward space finite difference method. The key is to replace the system (12) by the following equivalent fractional integral equation:

$$X(t) = X(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f\big(X(\tau), u(\tau)\big)\, \mathrm{d}\tau,$$

and then apply the generalized Adams-type predictor-corrector method [5, 6]. Then those state values are used to solve the adjoint equations by a backward time and forward space finite difference method, because of the transversality conditions. The system (13) becomes equivalent to the following fractional integral equation:

$$\lambda(t) = \frac{{}_tD_{t_\mathrm{f}}^{\alpha-1}\lambda(t_\mathrm{f})}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \cdot \int_t^{t_\mathrm{f}} (\tau - t)^{\alpha-1} \left[\frac{\partial \phi}{\partial X} + \lambda^\mathrm{T} \frac{\partial f}{\partial X}\right] \mathrm{d}\tau.$$

We use a steepest-method to generate a successive approximation of the optimal control form, then continue the iterations until the convergence is achieved. For numerical simulation, we use the parameter values

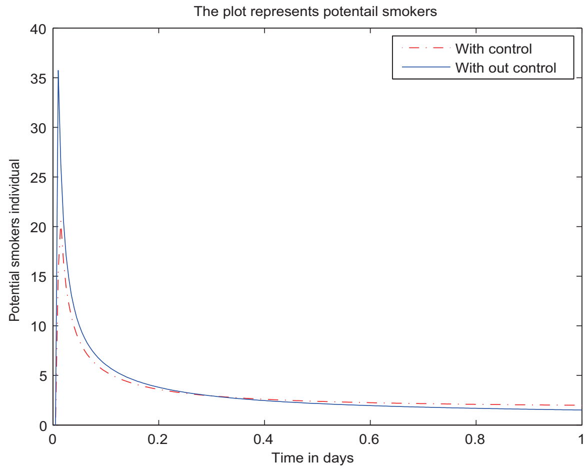

Fig. 1 (colour online). Potential smokers both with control and without control for time $\alpha = 0.3, 0.5, 0.7, 1$.

Table 1. Parameter values used in the numerical simulations.

Notation	Value	Notation	Value
λ	1	d_2	0.045
β_1	0.014	d_3	0.054
β_2	0.014	d_4	0.061
μ	0.0165	A_1	0.03
γ	0.0021	A_2	0.01
τ	0	A_3	0.01
d_1	0.034	A_4	0.01

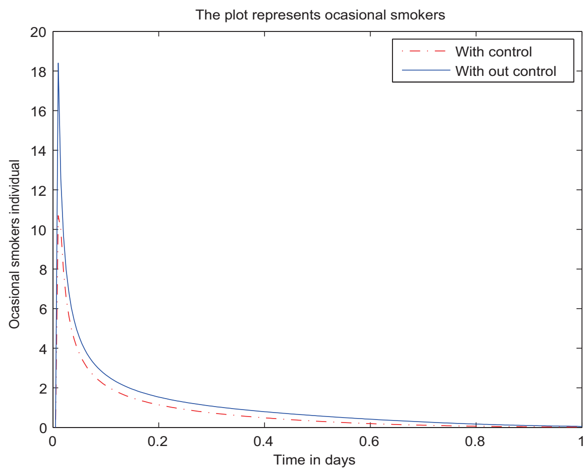

Fig. 2 (colour online). Occasional smokers both with control and without control for $\alpha = 0.3, 0.5, 0.7, 1$.

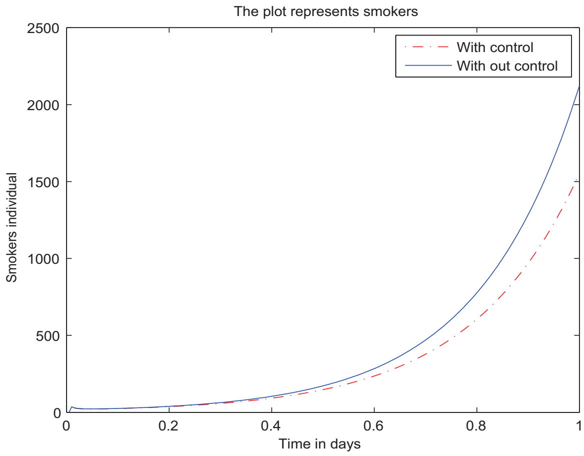

Fig. 3 (colour online). Smoking population both with control and without control for $\alpha = 0.3, 0.5, 0.7, 1$.

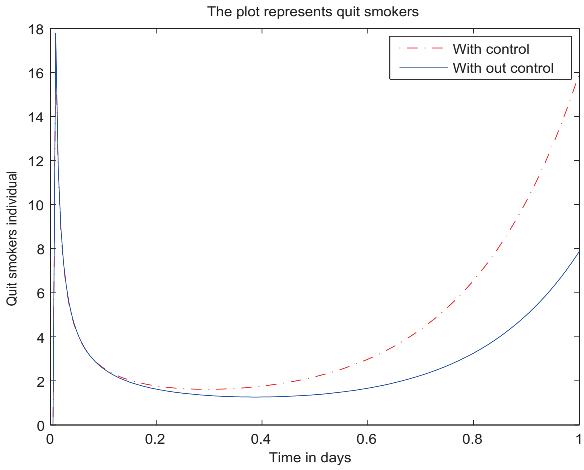

Fig. 4 (colour online). Ex-smokers both with control and without control for $\alpha = 0.3, 0.5, 0.7, 1$.

in Table 1 with initial values $P(0) = 153$, $L(0) = 68$, $S(0) = 79$, and $Q(0) = 55$.

In Figure 1, we plotted the potential smokers in the two systems (1) and (11). The solid line denotes the population of potential smokers in system (1) without control while the dash-dotted line denotes the population of potential smokers in system (11) with control. The population of potential smokers is 153. Figure 2 represents the occasional smokers in the two systems (1) and (11). The number of occasional smokers is 68. The solid line denotes that there are more occasional smokers when the control (treatment) is not implemented to the system. Figure 3 represents the smoking population in the two systems (1) without control and (11) with control. The solid line denotes that there are more smokers when the control (treatment) is not implemented to the smoking population. The number of ex-smoker is 79. Figure 4 represents the number of ex-smokers in both systems. Their number is 55. The dash-dotted line denotes that there are more non-smokers when the control (treatment) is implemented to the smoking population. Figure 5 represents the control functions u_1, u_2, u_3.

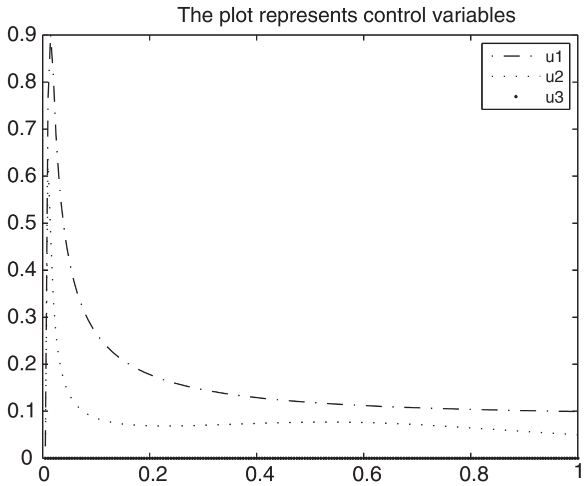

Fig. 5 (colour online). Control variables.

5. Conclusion

In this paper, we applied the optimal control strategies in the giving up smoking model. In order to control the smoking in a community, we introduced three control variables: education campaign, anti-smoking gum, and anti-nicotive drugs/medicine. The optimality condition from the optimal control problem are derived. We minimized the number of potential and occasional smokers, and we maximized the number of ex-smokers.

[1] C. Castillo, Elec. J. Diff. Eqs. **125**, 1 (2006).
[2] M. L. Brandeau, G. S. Zeric, and A. Richter, J. H. E. **22**, 575 (2003).
[3] N. G. Becker and D. N. Starczak, Math. Biosci. **139**, 1 (2006).
[4] A. H. Mokdad, J. S. Marks, D. F. Stroup, and J. L. Gerberding, J. Am. Med. Assoc. **291**, 1238 (2004).

[5] O. K. Ham, West J. Nurs. Res. **29**, 301 (2007).
[6] B. Adhikari, J. Kahende, A. Malarcher, T. Pechacek, and V. Tong, Morb. Mortal. Wkly. Rep. **57**, 1226 (2008).
[7] C. Castillo-Garsow, G. Jordan-Salivia, and A. R. Herrera, Mathematical Models for the Dynamics of Tobacco Use, Recovery, and Relapse, Technical Report Series BU 1505-M, Cornell University, Ithaca, NY, USA 2000.
[8] G. Zaman, Bull. Malays. Sci. Soc. **34**, 403 (2011).
[9] T. J. Anastasio, Biolog. Cybern. **72**, 69 (1994).
[10] O. P. Agrawal, Nonlin. Dyn. **38**, 323 (2004).
[11] K. Diethelm, N. J. Ford, and A. D. Freed, Nonlin. Dyn. **29**, 3 (2002).
[12] K. Diethelm, N. J. Ford, and A. D. Freed, Numer. Algo. **36**, 31 (2004).
[13] O. P. Agrawal, J. Dyn. Syst. Meas. Con. **130**, 1 (2008).
[14] C. Tricaud and Y. Chen, Comput. Math. Appl. **59**, 1644 (2010).
[15] R. K. Biswas and S. Sen, J. Comput. Non. Dyn. **6**, 1 (2011).
[16] V. S. Erturk, G. Zaman, and S. Momani, Comput. Math. Appl. **64**, 3065 (2012).
[17] D. Kirschner, S. Lenhart, and S. Serbin, J. Math. Biol. **35**, 775 (1997).
[18] O. Sharomi and A. B. Gumel, Appl. Math. Comput. **195**, 475 (2008).
[19] G. Zaman, Y. H. Kang, and I. H. Jung, Biosyst. **98**, 43 (2009).
[20] W. Lin, J. Am. Med. Assoc. **332**, 709 (2007).
[21] L. Debnath, Int. J. Math. Math. Sci. **54**, 3413 (2003).
[22] M. I. Kamien and N. L. Schwartz, Dynamics Optimization, The Calculus of Variations and Optimal Control in Economics and Management, Vol. 31 of Advanced Textbooks in Economics, 2nd edn., Elsevier B.V., Amsterdam 1991.
[23] I. Podlubny, Fractional Differential Equations, Academic Presss, London 1999.
[24] G. Zaman and H. Jung, Proc. Korean Soc. Indust. Appl. Math. **3**, 31 (2007).
[25] S. Miller and B. Ross, An Introduction to the Fractional calculas and Fractional Differential Equations, Willey, New York 1993.
[26] F. Riewe, Phys. Rev. E. **53**, 1890 (1996).
[27] N. Ozalp and E. Demirci, Math. Comput. Model. **54**, 1 (2011).