Exact Travelling Wave Solutions of two Important Nonlinear Partial Differential Equations

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Coupled nonlinear partial differential equations describing the spatio-temporal dynamics of predator–prey systems and nonlinear telegraph equations have been widely applied in many real world problems. So, finding exact solutions of such equations is very helpful in the theories and numerical studies. In this paper, the Kudryashov method is implemented to obtain exact travelling wave solutions of such physical models. Further, graphic illustrations in two and three dimensional plots of some of the obtained solutions are also given to predict their behaviour. The results reveal that the Kudryashov method is very simple, reliable, and effective, and can be used for finding exact solution of many other nonlinear evolution equations.

Key words: Exact Travelling Wave Solutions; Nonlinear Physical Models; Homogeneous Balance; Kudryashov Method.
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1. Introduction

Nonlinear partial differential equations (PDEs) are widely used as models to describe complex physical phenomena in various fields of science and engineering such as solid state physics, plasma wave, thermodynamics, soil mechanics, civil engineering, population ecology, infectious disease epidemiology, neural networks, and so on. Therefore, for the past few decades, much attention has been paid to the problem of finding exact solutions of nonlinear PDEs. With the use of these solutions, one may give better insight into the physical aspects of the particular nonlinear models considered. Recently, a considerable number of analytic methods have been successfully developed and applied for constructing exact travelling wave solutions to nonlinear evolution equations such as extended Jacobi elliptic function expansion method [1], simplified Hirota's method [2], (G'/G) expansion method [3, 4], modified F-expansion method [5], trial equation method [6], ansatz method [7], modified tanh–coth function method [8], Legendre spectral element method [9], homotopy perturbation method [10], and so on. However, there is no unified method that can be used to find solutions of all types of nonlinear evolution equations.

The Kudryashov method is a very powerful technique for finding exact solutions of nonlinear PDEs which was first developed by Kudryashov [11] and used successfully for finding exact solutions of nonlinear evolution equations arising in mathematical physics [12 – 17]. Using the Kudryashov method, exact solutions of the Benjamin–Bona–Mahony–Peregrine equation are obtained in [18] with power-law and dual power-law nonlinearities. Ryabov [19, 20] obtained exact solutions of the Kudryashov–Sinelshchikov equation and higher-order nonlinear evolution equations by using the Kudryashov method. Subsequently, Vitanov [21, 22] proposed a modified Kudryashov method to obtain exact solutions of some nonlinear PDEs. Kabir et al. [23] used the modified Kudryashov method to construct the solitary travelling wave solutions of the Kuramoto–Sivashinsky and seventh-order Sawada–Kotera equations. Kudryashov and Kochanov [24] obtained quasi-exact solutions of the Kuramoto–Sivashinsky, the Korteweg–de Vries–

Burgers, and the Kawahara equations. Moreover, processes of self-organization described by a nonlinear evolution equation of sixth order are considered on substrate surfaces after ion beam bombardment in [25] and subsequently exact solutions are found with use of the Kudryashov method. One of the main advantages of this technique is that it is possible to construct more effectively exact solutions of high-order nonlinear evolution equations in comparison with other methods [26].

Finding explicit solutions to nonlinear PDEs is of fundamental importance. In this paper, we consider a system of two coupled nonlinear PDEs describing the spatio-temporal dynamics of a predator–prey system where the prey per capita growth rate is subject to the Allee effect [27]. Further, we consider the nonlinear telegraph equations which appear in the propagation of electrical signals along a telegraph line, digital image processing, telecommunication, signals, and systems (see [28, and references therein]). The solutions of predator–prey systems have been studied in various aspects [27, 29, 30]. Dehghan and Sabouri [30] developed a Legendre spectral element method for solving a one-dimensional predator–prey system on a large spatial domain. Kraenkel et al. [29] used the (G'/G) expansion method to obtain exact solutions for a diffusive predator–prey system and also for each wave velocity three different forms of solutions are reported. Recently, Mirzazadeh and Eslami [31] obtained some exact travelling wave solutions of the nonlinear telegraph equation by using the first integral method. In this paper, the Kudryashov method is employed to obtain some exact solutions of a system of predator–prey equations and the nonlinear telegraph equation.

2. Kudryashov Method and its Applications to Nonlinear Physical Models

Let us present the main steps of the Kudryashov method which is described in [11, 12] for finding exact solutions of nonlinear partial differential equations.

Step 1: Consider a general form of nonlinear PDE in the form

$$P(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0 \,. \tag{1}$$

Step 2: To find the exact solutions of (1), we consider the variable transformation

$$u(x,t) = y(\eta) \,, \quad \eta = lx - wt \,, \tag{2}$$

where l and w are constants to be determined later. By using the transformation in (2), (1) converted to an ordinary differential equation (ODE)

$$E\big(y, -wy_\eta, ly_\eta, w^2 y_{\eta\eta}, l^2 y_{\eta\eta\eta}, \ldots\big) = 0 \,, \tag{3}$$

where $y = y(\eta)$ is an unknown function and E is a polynomial in variable y and its derivatives. To find the dominant terms, we substitute $y(\eta) = \eta^{-p}$, $p > 0$, into all terms of (3). Then we compare degrees of all terms in (3) and choose two or more terms with smallest degree. The maximum value of p is the pole of (3), and we denote it as N. It should be mentioned that the method can be applied when N is integer. If the value N is noninteger, we have to use the transformation of solution $y(\eta)$.

Step 3: The Kudryashov method consists in expanding the solutions $y(\eta)$ of (3) in a finite series

$$\begin{aligned} y(\eta) = a_0 + a_1 Q(\eta) + a_2 Q(\eta)^2 \\ + \cdots + a_N Q(\eta)^N \,, \end{aligned} \tag{4}$$

where a_i, $i = 1, 2, \ldots, N$, are unknown constants, $a_N \neq 0$, and $Q(\eta)$ has the form

$$Q(\eta) = \frac{1}{1 + \mathrm{e}^\eta} \tag{5}$$

which is a solution to the Riccati equation

$$Q_\eta = Q^2 - Q \,. \tag{6}$$

Step 4: We should calculate the necessary number of derivatives of function y. For instance, we consider the general case when N is arbitrary. Differentiating (4) with respect to η and taking into account (6), we have

$$\begin{aligned} y_\eta &= \sum_{i=1}^{N} a_i i (Q-1) Q^i \,, \\ y_{\eta\eta} &= \sum_{i=1}^{N} a_i i \Big((i+1) Q^2 \\ &\quad - (2i+1) Q + i \Big) Q^i \,. \end{aligned} \tag{7}$$

The high-order derivative of $y(\eta)$ can be found as in [11, 12]. Next, substitute expressions (7) in (1). Then we collect all terms with the same powers of function $Q(z)$ and equate the resulting expression equal to zero. Finally, we obtain a system of algebraic equations, and solving the resulting system, we can obtain exact solutions of (1).

Example 1. Diffusive predator–prey system

Let us demonstrate the application of the Kudryashov method for finding the exact travelling wave solutions of the diffusive predator–prey system. Consider the diffusive predator–prey system [27] in the form

$$\begin{aligned} u_t &= u_{xx} - \beta u + (1+\beta)u^2 - u^3 - uv\,, \\ v_t &= v_{xx} + \kappa uv - mv - \delta v^3\,, \end{aligned} \tag{8}$$

where κ, δ, m, and β represent positive parameters, subscripts x and t denote partial derivatives. To have the equation in a simple form, the equations are expressed in dimensionless variables and also the biological meaning of the each term has been discussed in [27, 29]. In order to investigate the dynamics of the diffusive predator–prey system, the relations between the parameters, namely $m = \beta$ and $\kappa + \frac{1}{\sqrt{\delta}} = \beta + 1$, have been defined in [27]. Under this relation, (8) can be written as

$$\begin{aligned} u_t &= u_{xx} - \beta u + \left(\kappa + \frac{1}{\sqrt{\delta}}\right)u^2 - u^3 - uv\,, \\ v_t &= v_{xx} + \kappa uv - \beta v - \delta v^3\,. \end{aligned} \tag{9}$$

By applying the wave transformation defined as in (2), (9) becomes a system of ODEs which can be written as

$$\begin{aligned} &l^2u'' + \omega u' - \beta u + \left(\kappa + \frac{1}{\sqrt{\delta}}\right)u^2 - u^3 - uv = 0\,, \\ &l^2v'' + \omega v' + \kappa uv - \beta v - \delta v^3 = 0\,. \end{aligned} \tag{10}$$

In order to solve (10), let us consider the transformation

$$v = \frac{1}{\sqrt{\delta}}u\,. \tag{11}$$

Substituting transformation (11) in (10), we can obtain

$$l^2u'' + \omega u' - \beta u + \kappa u^2 - u^3 = 0\,. \tag{12}$$

Now, we employ the Kudryashov technique to solve (12) and as a result, we obtain the exact solutions of coupled PDE (8). To determine the parameter N, we balance the linear terms of highest order in (12) with the highest-order nonlinear terms. The balancing procedure yields $N = 1$, so the solution of the ordinary differential equation (12) is of the form

$$u(\eta) = a_0 + a_1 Q(\eta)\,. \tag{13}$$

By substituting (13) into (12) and making use of (6), we obtain the following system of algebraic equations for a_0, a_1, ω, and l by equating all coefficients of the functions $Q(\eta)$ to zero:

$$\begin{aligned} &2l^2a_1 - a_1^3 = 0\,,\ \ -\beta a_0 + \kappa a_0^2 - a_0^3 = 0\,, \\ &-3l^2a_1 + \omega a_1 + \kappa a_1^2 - 3a_0a_1^2 = 0\,, \\ &l^2a_1 - \omega a_1 - \beta a_1 + 2\kappa a_0 a_1 - 3a_0^2 a_1 = 0\,. \end{aligned} \tag{14}$$

Solving the above system of algebraic equations with the aid of MAPLE, four possible sets of solutions were obtained:

Case 1.

$$\begin{aligned} a_1 &= \frac{\kappa^2 + \sqrt{\kappa^4 - 4\beta\kappa^2}}{2\kappa}\,,\ \ a_0 = 0\,, \\ l &= \pm\frac{\kappa^2 - 2\beta + \sqrt{\kappa^4 - 4\beta\kappa^2}}{2}\,, \\ \omega &= \frac{\kappa^2}{4} - \frac{3\beta}{2} + \frac{\sqrt{\kappa^4 - 4\beta\kappa^2}}{4}\,. \end{aligned} \tag{15}$$

Substituting (15) into (13), the solitary wave solution of (9) can be obtained as

$$\begin{aligned} u_1(x,t) &= \frac{\kappa^2 + \sqrt{\kappa^4 - 4\beta\kappa^2}}{2\kappa}\left(\frac{1}{1+\mathrm{e}^{\eta}}\right), \\ v_1(x,t) &= \frac{1}{\sqrt{\delta}}u_1(x,t)\,, \end{aligned} \tag{16}$$

where $\eta = \pm\frac{\kappa^2 - 2\beta + \sqrt{\kappa^4 - 4\beta\kappa^2}}{2}x - \left(\frac{\kappa^2}{4} - \frac{3\beta}{2} + \frac{\sqrt{\kappa^4 - 4\beta\kappa^2}}{4}\right)t$ and in which β, κ, and δ are arbitrary constants.

Case 2.

$$
\begin{aligned}
a_1 &= \frac{\kappa^2-\sqrt{\kappa^4-4\beta\kappa^2}}{2\kappa}, \quad a_0 = 0, \\
l &= \pm\frac{\kappa^2-2\beta-\sqrt{\kappa^4-4\beta\kappa^2}}{2}, \\
\omega &= \frac{\kappa^2}{4}-\frac{3\beta}{2}-\frac{\sqrt{\kappa^4-4\beta\kappa^2}}{4}.
\end{aligned} \tag{17}
$$

According to (17) and (13), we obtain the exact travelling wave solution in the form

$$
\begin{aligned}
u_2(x,t) &= \frac{\kappa^2-\sqrt{\kappa^4-4\beta\kappa^2}}{2\kappa}\left(\frac{1}{1+\mathrm{e}^{\eta}}\right), \\
v_2(x,t) &= \frac{1}{\sqrt{\delta}}u_2(x,t),
\end{aligned} \tag{18}
$$

where $\eta = \pm\frac{\kappa^2-2\beta-\sqrt{\kappa^4-4\beta\kappa^2}}{2}x - \left(\frac{\kappa^2}{4}-\frac{3\beta}{2}-\frac{\sqrt{\kappa^4-4\beta\kappa^2}}{4}\right)t$ and in which β, κ, and δ are arbitrary constants.

Case 3.

$$
\begin{aligned}
a_1 &= -\frac{-12\beta+3\kappa^2\pm\kappa\sqrt{\kappa^2-4\beta}}{\kappa\pm3\sqrt{\kappa^2-4\beta}}, \\
a_0 &= \frac{\kappa\pm\sqrt{\kappa^2-4\beta}}{2}, \\
l &= \pm\sqrt{\frac{\kappa^2-4\beta}{2}}, \quad \omega = \mp\frac{\kappa\sqrt{\kappa^2-4\beta}}{2}.
\end{aligned} \tag{19}
$$

Further, using (19) into (13), we obtain the exact wave solution of (9) in the form

$$
\begin{aligned}
u_3(x,t) &= -\frac{-12\beta+3\kappa^2\pm\kappa\sqrt{\kappa^2-4\beta}}{\kappa\pm3\sqrt{\kappa^2-4\beta}} \\
&\quad\cdot\left(\frac{1}{1+\mathrm{e}^{\eta}}\right)+\frac{\kappa\pm\sqrt{\kappa^2-4\beta}}{2}, \\
v_3(x,t) &= \frac{1}{\sqrt{\delta}}u_3(x,t),
\end{aligned} \tag{20}
$$

where $\eta = \pm\sqrt{\frac{\kappa^2-4\beta}{2}}x \pm \frac{\kappa\sqrt{\kappa^2-4\beta}}{2}t$ and in which β, κ, and δ are arbitrary constants.

Case 4.

$$
\begin{aligned}
a_1 &= -\frac{-6\beta+2\kappa\left(\kappa\pm\sqrt{\kappa^2-4\beta}\right)}{\kappa\pm3\sqrt{\kappa^2-4\beta}}, \\
a_0 &= \frac{\kappa\pm\sqrt{\kappa^2-4\beta}}{2}, \\
l &= \pm\sqrt{\frac{\kappa\left(\kappa\pm\sqrt{\kappa^2-4\beta}\right)-2\beta}{4}}, \\
\omega &= \frac{3\beta}{2}-\frac{\kappa\left(\kappa\pm\sqrt{\kappa^2-4\beta}\right)}{4}.
\end{aligned} \tag{21}
$$

Finally, (21) leads to the exact travelling wave solution in the form

$$
\begin{aligned}
u_4(x,t) &= -\frac{-6\beta+2\kappa\left(\kappa\pm\sqrt{\kappa^2-4\beta}\right)}{\kappa\pm3\sqrt{\kappa^2-4\beta}} \\
&\quad\cdot\left(\frac{1}{1+\mathrm{e}^{\eta}}\right)+\frac{\kappa\pm\sqrt{\kappa^2-4\beta}}{2}, \\
v_4(x,t) &= \frac{1}{\sqrt{\delta}}u_4(x,t),
\end{aligned} \tag{22}
$$

where $\eta = \pm\sqrt{\frac{\kappa\left(\kappa\pm\sqrt{\kappa^2-4\beta}\right)-2\beta}{4}}x - \left(\frac{3\beta}{2}-\frac{\kappa\left(\kappa\pm\sqrt{\kappa^2-4\beta}\right)}{4}\right)t$ and in which β, κ, and δ are arbitrary constants.

Example 2. Nonlinear telegraph equation

In this example, we are concerned about exact solutions of nonlinear telegraph equations. The simplest and well-known nonlinear telegraph model is given by the partial differential equation [28]

$$
u_{tt}-u_{xx}+u_t+\alpha u+\beta u^3=0. \tag{23}
$$

Equation (23) is referred to as second-order hyperbolic telegraph equation with constant coefficients which models a mixture between diffusion and wave propagation by introducing a term that accounts for effects of finite velocity to standard heat or mass transport equation [32]. But (23) is commonly used in signal analysis for transmission and propagation of electrical signals [32].

Equations of this kind arise in the study of heat transfer, transmission lines, chemical kinetics, biological population dispersal, random walks (see [28, and references therein]). Using the wave transformation $u(x,t) = u(\eta)$, $\eta = lx - \omega t$, (23) turns into the nonlinear ordinary differential equation

$$\left(\omega^2 - l^2\right)u'' - \omega u' + \alpha u + \beta u^3 = 0\,, \tag{24}$$

where prime denotes the derivative with respect to the variable η.

To determine the parameter M, we balance the linear terms of highest order in (24) with the highest-order nonlinear terms. Considering the homogeneous balance between u'' and u^3 in (24), we obtain that $M + 2 = 3M \Rightarrow M = 1$. As a result, the Kudryashov method admits the solution of (24) in the form

$$u(\eta) = a_0 + a_1 Q(\eta)\,, \tag{25}$$

where a_0 and a_1 are constants to be determined later, $Q(\xi)$ is the solution in (6).

By substituting $u(\eta)$ of (25) along with (6) in (24) and then setting the coefficients of powers of Q to be zero in the resulting expression, we obtain a set of algebraic equations involving a_0, a_1, ω, and l as

$$\begin{aligned}
&\alpha a_0 + \beta a_0^3 = 0\,,\\
&2a_1\omega^2 - 2a_1\kappa^2 + \beta a_1^3 = 0\,,\\
&-3a_1\omega^2 + 3a_1\kappa^2 - \omega a_1 + 3\beta a_0 a_1^2 = 0\,,\\
&a_1\omega^2 - a_1\kappa^2 + \omega a_1 + \alpha a_1 + 3\beta a_0^2 a_1 = 0\,.
\end{aligned} \tag{26}$$

Solving the above system of algebraic equations, we can obtain three sets of solutions:

$$\begin{aligned}
&a_1 = \pm 2\sqrt{-\frac{\alpha}{\beta}}\,,\ a_0 = \pm\frac{\alpha}{\beta\sqrt{-\frac{\alpha}{\beta}}}\,,\\
&l = \pm\sqrt{-2\alpha}\,,\ \omega = 0\,,
\end{aligned} \tag{27}$$

$$\begin{aligned}
&a_1 = \pm\sqrt{-\frac{\alpha}{\beta}}\,,\ a_0 = \pm\frac{\alpha}{\beta\sqrt{-\frac{\alpha}{\beta}}}\,,\\
&l = \pm\frac{\sqrt{9\alpha^2 - 2\alpha}}{2}\,,\ \omega = \frac{3\alpha}{2}\,,
\end{aligned} \tag{28}$$

$$\begin{aligned}
&a_1 = \pm\sqrt{-\frac{\alpha}{\beta}}\,,\ a_0 = 0\,,\\
&l = \pm\frac{\sqrt{9\alpha^2 - 2\alpha}}{2}\,,\ \omega = -\frac{3\alpha}{2}\,,
\end{aligned} \tag{29}$$

where α and β are arbitrary constants.

The first two sets (27) and (28) give the exact travelling wave solutions of (23) in the following form:

$$u_1(x,t) = \pm 2\sqrt{-\frac{\alpha}{\beta}}\left(\frac{1}{1+\mathrm{e}^{\eta}}\right) \pm \frac{\alpha}{\beta\sqrt{-\frac{\alpha}{\beta}}}\,, \tag{30}$$

where $\eta = \pm\sqrt{-2\alpha}x$, α, and β are arbitrary constants, and

$$u_2(x,t) = \pm\sqrt{-\frac{\alpha}{\beta}}\left(\frac{1}{1+\mathrm{e}^{\eta}}\right) \pm \frac{\alpha}{\beta\sqrt{-\frac{\alpha}{\beta}}}\,, \tag{31}$$

where $\eta = \pm\frac{\sqrt{9\alpha^2-2\alpha}}{2}x - \frac{3\alpha}{2}t$, α, and β are arbitrary constants.

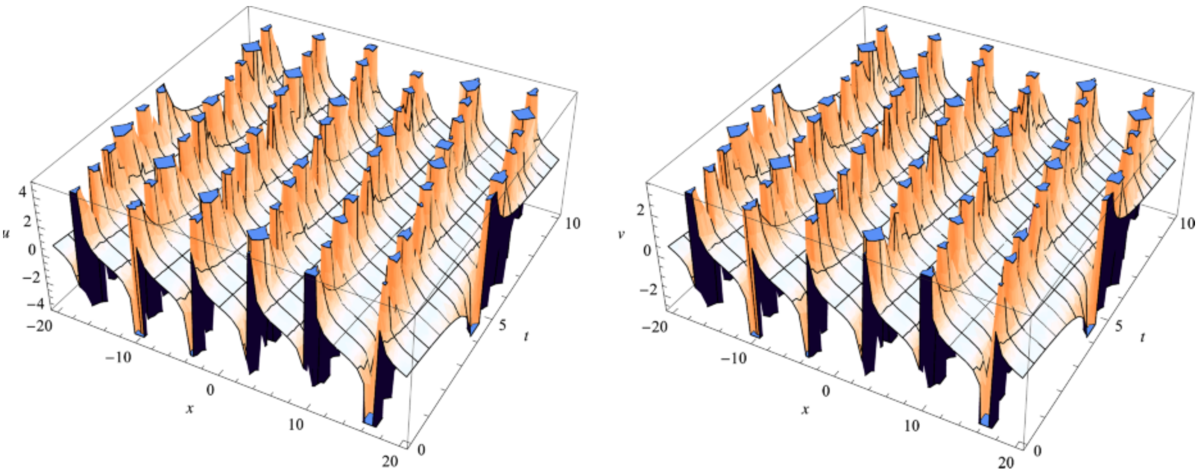

Fig. 1 (colour online). Profiles of soliton solutions u and v of (20) under the given parameters $\kappa = 1$, $\beta = 0.7$, $\delta = 2.04082$, $t \in [0,10]$, and $x \in [-20,20]$.

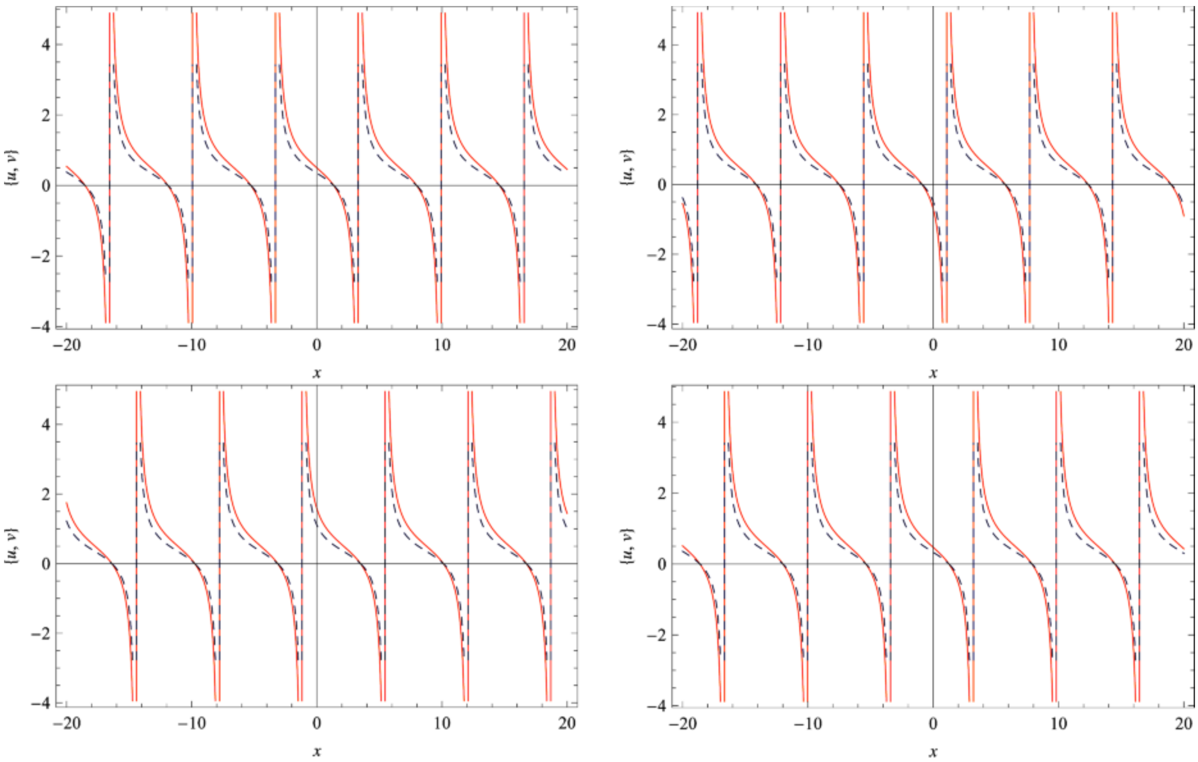

Fig. 2 (colour online). Densities of prey (solid line) and predator (dashed line) as given by (20) shown for different time parameters $t = 0, 50, 100, 150$.

Finally, substituting (29) into (25), we can get the following travelling wave solution:

$$u_3(x,t) = \pm\sqrt{-\frac{\alpha}{\beta}}\left(\frac{1}{1+\mathrm{e}^{\eta}}\right), \tag{32}$$

where $\eta = \pm\frac{\sqrt{9\alpha^2-2\alpha}}{2}x + \frac{3\alpha}{2}t$, α, and β are arbitrary constants.

When we take $\nu_1 = \omega^2 - l^2$, $\nu_2 = -\omega$, $\mu_1 = \alpha$, $\mu_2 = 0$, $\mu_3 = \beta$, and $a = -1$, $b = 1$ in [33, solution (3.14)], we can get the solution (32) in our paper, so our solution (32) is a particular case of the solution obtained in [33]. But the obtained other two solution forms (30) and (31) are different from the solutions of [33].

Figure 1 shows the profile of soliton solutions u and v of (20) with $\kappa = 1$, $\beta = 0.7$, and $\delta = 2.04082$. Figure 2 represents the densities of prey (solid line) and predator (dashed line) as given by the exact solution (20) with $\kappa = 1$, $\beta = 0.7$, $\delta = 2.04082$, and for different time parameters $t = 0, 50, 100, 150$. Besides the solitary wave solutions, (22) admits kink travelling waves. The kink waves are travelling waves which rise or descend from one asymptotic state to another. In particular, the kink solution approaches a constant at infinity. Figure 3 shows the space-time plot of the numerical kink wave solution (22) with $\kappa = 4.16667$, $\beta = 4$, $\delta = 1.44$, and x in the interval $[-20, 20]$. The initial conditions corresponding to the exact solution can be obtained from (22) by letting $t \to 0$. In general, the rate of convergence to the solution (22) and the form of the transients can depend on the initial conditions. It can be seen that for some values of parameters the kink wave solution (22) generates hills in terms of the surface. It is noted that the plots of (22) also represent antikink solitons when the negative sign is taken. Equation (32) provides kink waves of the nonlinear telegraph equation. The space-time graph of the solution (32) up to $t = 1$ is presented in Figure 4. The corresponding values of parameters are given as $\alpha = -1$, $\beta = 0.1$, and x in the interval $[-10, 10]$. Figure 5 represents the travelling wave solutions (31) of the nonlinear telegraph equation for some different times $t = 0, 0.5, 1$ with $\alpha = -1$, $\beta = 0.1$, and $x \in [-10, 10]$.

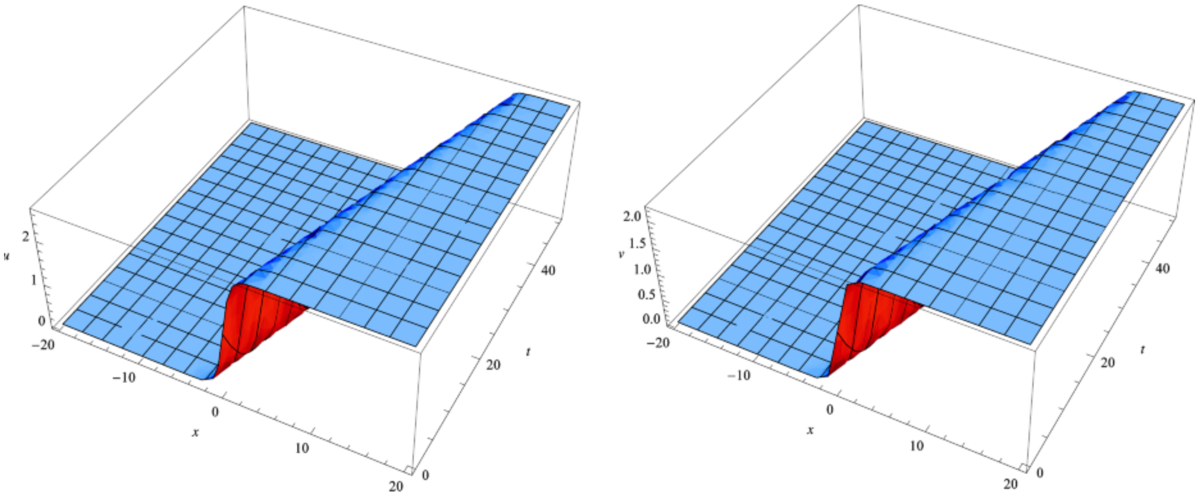

Fig. 3 (colour online). Above figures represent the solutions u and v of (22) for the predator–prey system with the values $\kappa = 4.16667$, $\beta = 4$, and $\delta = 1.44$.

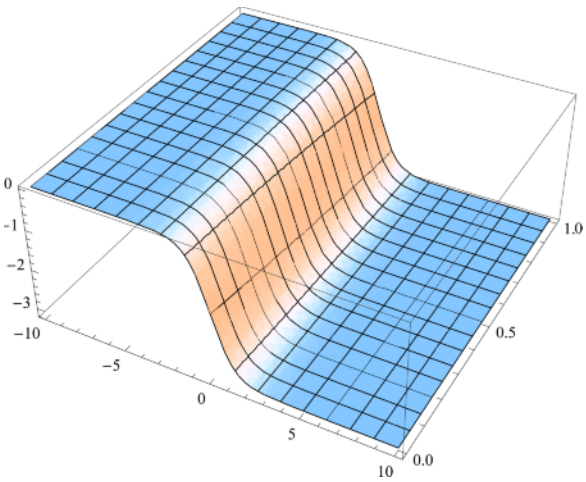

Fig. 4 (colour online). Three dimensionalt plot of solution (32) with $\alpha = -1$ and $\beta = 0.1$.

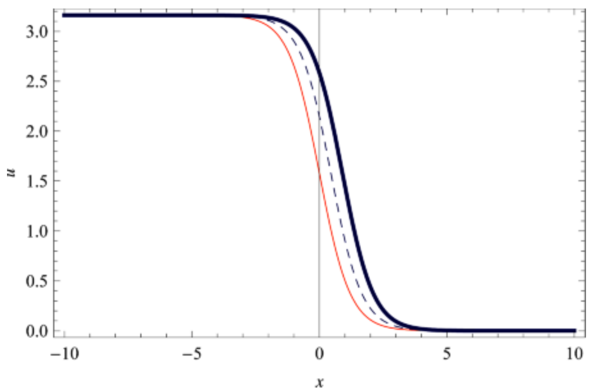

Fig. 5 (colour online). Solution of (31) for different values of t. The thin, dashed and thick curves represent the solutions for $t = 0, 0.5, 1$ respectively, when $\alpha = -1$ and $\beta = 0.1$.

3. Conclusion

In this paper, the Kudryashov method has been implemented to obtain exact travelling wave solutions of two important nonlinear partial differential equations. In particular, with the aid of symbolic computation system MAPLE, we obtain a wider class of exact travelling wave solutions of the considered equations. The obtained solutions are potentially significant and important for the explanation of better insight of physical aspects of the considered nonlinear models. It should be mentioned that the Kudryashov method can be more suitable to the nonlinear PDEs with higher-order nonlinearity. Also, all the obtained solutions are verified by putting them back into the original equations. The Kudryashov method can be extended to solve nonlinear coupled systems which arise in the theory of solitons and other areas.

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