Comparison of Numerical Methods of the SEIR Epidemic Model of Fractional Order

Anwar Zeb[a], Madad Khan[a], Gul Zaman[b], Shaher Momani[c], and Vedat Suat Ertürk[d]

[a] Department of Mathematics, COMSATS Institute of Information Technology Abbottabad Campus, Pakistan

[b] Department of Mathematics, University of Malakand, Chakdara Dir (Lower) Khyber Pakhtunkhawa, Pakistan

[c] The University of Jordan, Faculty of Science, Department of Mathermatics, Amman 1194, Jordan

[d] Department of Mathematics, Faculty of Arts and Sciences, Ondokuz May"s University, 55139, Samsun, Turkey

Reprint requests to M. K.; E-mail: madadmath@yahoo.com

Z. Naturforsch. **69a,** 81 – 89 (2014) / DOI: 10.5560/ZNA.2013-0073
Received January 4, 2013 / revised September 26, 2013 / published online December 18, 2013

In this paper, we consider the SEIR (Susceptible-Exposed-Infected-Recovered) epidemic model by taking into account both standard and bilinear incidence rates of fractional order. First, the non-negative solution of the SEIR model of fractional order is presented. Then, the multi-step generalized differential transform method (MSGDTM) is employed to compute an approximation to the solution of the model of fractional order. Finally, the obtained results are compared with those obtained by the fourth-order Runge-Kutta method and non-standard finite difference (NSFD) method in the integer case.

Key words: Fractional Differential Equations; Epidemic Model; Iterative Method; Non-Standard Scheme; Differential Transform Method.

1. Introduction

Mathematical modelling has proved its importance in understanding the dynamics of many infectious diseases. In compartmental models, the time t is an independent variable, and derivatives with respect to time of the sizes of the compartments are the rates of transfer between compartments. One of the early models in epidemiology was introduced in 1927 by Kermack and McKendrick [1] which is the starting point of epidemic models. In their proposed model, they divided the total population into three classes: the susceptible denoted by S, the infectious denoted by I, and the recovered denoted by R. This model, known as the susceptible-infectious-recovered (SIR) model, could be used to describe an influenza epidemic and was developed early in the 20th century. In many infectious diseases there is an exposed period after the transmission of infection from susceptible to potentially infective members but before these potential invectives develop symptoms and can transmit infection. That means a disease may have a latent or incubation time, when the susceptible has become infected but it is not yet infectious. The incubation period for measles, for example, is 8 – 13 days. For AIDS, on the other hand, it can be anything from a few months to many years. This can be included in the model as a delay, or by introducing a new class, say E, in which the susceptible remains for a given time before moving to I class. Many infectious diseases in nature have both horizontal and vertical transmission routes. These include such human diseases as Rubella, Herpes Simplex, Hepatitis B, and the HIV/AIDS. Horizontal transmission of diseases among humans and animals occurs through physical contact with hosts or through disease vectors like mosquitos, flies, etc. A vertical transmission is the transmission of an infection from parents to child during the perinatal period. In [2], a detailed analysis for integer order SEIR models with vertical transmission within a constant population can be found. Also other researchers did a lot of work on epidemic models of integer order (see for example [3 – 5]).

Nowadays, researchers are working on fractional-order differential equations because these give a better presentation of many phenomena. The fractional calculus represents a generalization of the ordinary differentiation and integration to non-integer and complex order and is used to establish new models in many fields not only in Mathematics. Mathematical models, using ordinary differential equations with integer order, have been proven valuable in understanding the dynamics of biological systems. However, the behaviour of most biological systems has memory or after-effects. The modelling of these systems by fractional-order differential equations has more advantages than classical integer-order mathematical modelling, in which such effects are neglected. Accordingly, the subject of fractional calculus (that is, calculus of integral and derivatives of arbitrary order) has gained popularity and importance, mainly due to its demonstrated applications in numerous diverse and widespread fields of science and engineering. In some situations, the fractional-order differential equations (FODEs) models look more suitable with the real phenomena than the integer-order models. This is due to the fact that fractional derivatives and integrals enable the description of memory and hereditary properties inherent in various materials and processes. Hence there is a growing need to study and use the fractional-order differential and integral equations (see for example [6 – 10]). In this paper, we consider a SEIR model taking into account both bilinear and standard incidence rates. First, we show that the SEIR model of fractional order has a positive solution. Then, we use the multi-step generalized differential transform method (MSGDTM) to approximate the numerical solution. Finally, we compare our numerical results with the results obtained by the non-standard finite difference (NSFD) method and fourth-order Runge–Kutta method. This paper is organized as follows.

In Section 2, we present the formulation of the model with some basic definitions and notations related to this work, and in Section 3 we show the non-negative solution and uniqueness of the model. In Section 4, the MSGDTM is applied to the model. Numerical simulations are presented graphically in Section 5. A conclusion is given in the last section.

2. Formulation of the Model with Preliminaries

For the formulation of the model, we consider the SEIR model by taking into account both bilinear and standard incidence rates. The model is as follows:

$$\begin{aligned}
\frac{\mathrm{d}S(t)}{\mathrm{d}t} &= bN(t) - \beta_1 E(t)S(t) - \beta_2 \frac{I(t)S(t)}{N(t)} - \mu S(t)\,,\\
\frac{\mathrm{d}E(t)}{\mathrm{d}t} &= \beta_1 E(t)S(t) - (\gamma_1 + \mu)E(t)\,,\\
\frac{\mathrm{d}I(t)}{\mathrm{d}t} &= \beta_2 \frac{I(t)S(t)}{N(t)} + \gamma_1 E(t) - (\gamma_2 + \mu)I(t)\,,\\
\frac{\mathrm{d}R(t)}{\mathrm{d}t} &= \gamma_2 I(t) - \mu R(t)
\end{aligned} \tag{1}$$

with

$$S(0) = S_0\,,\ E(0) = E_0\,,\ I(0) = I_0\,,\ R(0) = R_0\,.$$

The total population size is $N(t) = S(t) + E(t) + I(t) + R(t)$.

By adding all equations of system (1), we obtain

$$\frac{\mathrm{d}N}{\mathrm{d}t} = (b - \mu)N. \tag{2}$$

Here b is the birth rate, β_1 is the transmission rate from susceptible to exposed class due to infected class, β_2 is the transmission rate from susceptible to infected class.

Now we introduce fractional order into the system (1) which consists of ordinary differential equations. The new system is described by the following set of fractional-order differential equations:

$$\begin{aligned}
D_t^\alpha S(t) &= bN(t) - \beta_1 E(t)S(t) - \beta_2 \frac{I(t)S(t)}{N(t)} - \mu S(t)\,,\\
D_t^\alpha E(t) &= \beta_1 E(t)S(t) - (\gamma_1 + \mu)E(t)\,,\\
D_t^\alpha I(t) &= \beta_2 \frac{I(t)S(t)}{N(t)} + \gamma_1 E(t) - (\gamma_2 + \mu)I(t)\,,\\
D_t^\alpha R(t) &= \gamma_2 I(t) - \mu R(t)\,,\\
D_t^\alpha N(t) &= (b - \mu)N(t)\,.
\end{aligned} \tag{3}$$

Here D_t^α is the fractional derivative in the Caputo sense, and α is a parameter describing the order of the fractional time derivative with $0 < \alpha \le 1$. For $\alpha = 1$, the system reduces to ordinary differential equations. The system (3) is the generalization of system (1). Now we give some basic definitions and properties of the fractional calculus theory which are used further in this paper [5 – 9].

Definition 1. A function $f(x)\,(x > 0)$ is said to be in the space $C_\alpha\,(\alpha \in \mathbb{R})$ if it can be written as $f(x) = x^p f_1(x)$ for some $p > \alpha$ where $f_1(x)$ is continuous in $[0,\infty)$, and it is said to be in the space C_α^m if $f^{(m)} \in C_\alpha$ and $m \in \mathbb{N}$.

Definition 2. The Riemann–Liouville integral operator of order $\alpha > 0$ with $a \geq 0$ is defined as

$$(J_a^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,\mathrm{d}t\,, \quad x > a\,, \tag{4}$$

$$(J_a^0 f)(x) = f(x)\,. \tag{5}$$

Properties of the operator can be found in [11]. We only need here the following:

For $f \in C_\alpha$, $\alpha, \beta > 0$, $a \geq 0$, $c \in \mathbb{R}$ and $\gamma > -1$, we have

$$(J_a^\alpha J_a^\beta f)(x) = (J_a^\beta J_a^\alpha f)(x) = (J_a^{\alpha+\beta} f)(x)\,, \tag{6}$$

$$J_a^\alpha x^\gamma = \frac{x^{\gamma+\alpha}}{\Gamma(\alpha)} B_{\frac{x-a}{x}}(\alpha, \gamma+1)\,, \tag{7}$$

where $B_\tau(\alpha, \gamma+1)$ is the incomplete beta function which is defined as

$$B_\tau(\alpha, \gamma+1) = \int_0^\tau t^{\alpha-1}(1-t)^\gamma \,\mathrm{d}t\,, \tag{8}$$

$$J_a^\alpha \mathrm{e}^{cx} = \mathrm{e}^{ac}(x-a)^\alpha \sum_{k=0}^\infty \frac{[c(x-a)]^k}{\Gamma(\alpha+k+1)}\,. \tag{9}$$

The Riemann–Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. The Riemann–Liouville derivative of a constant is not zero. In addition, if an arbitrary function is a constant at the origin, its fractional derivation has a singularity at the origin for instant exponential and Mittag–Leffler functions. Theses disadvantages reduce the field of application of the Riemann–Liouville fractional derivative. One of the great advantages of the Caputo fractional derivative is that it allows traditional initial and boundary conditions to be included in the formulation of the problem [12, 13]. Therefore, we shall introduce a modified fractional differential operator ${}_0^cD_a^\alpha$ proposed by Caputo in his work on the theory of viscoelasticity.

Definition 3. The Caputo fractional derivative of $f(x)$ of order $\alpha > 0$ with $a \geq 0$ is defined as

$${}_0^cD_a^\alpha(f(x)) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} \frac{\mathrm{d}^m f(t)}{\mathrm{d}t^m}\,\mathrm{d}t\,, \tag{10}$$

$m-1 < \alpha \leq m\,.$

For the case of Riemann–Liouville, we have the following definition:

$$(D_a^\alpha f)(x) = (J_a^{m-\alpha} f^{(m)})(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{f^{(m)}(t)}{(x-t)^{\alpha+1-m}}\,\mathrm{d}t$$

for $m-1 < \alpha \leq m$, $m \in \mathbb{N}, x \geq a$, $f(x) \in C_{-1}^m$.

The Caputo fractional derivative was investigated by many authors. For $m-1 < \alpha \leq m$, $f(x) \in C_\alpha^m$, and $\alpha \geq -1$, we have

$$\begin{aligned}(J_a^\alpha D_a^\alpha f)(x) &= J^m D^m f(x) \\ &= f(x) - \sum_{k=0}^{m-1} f^{(k)}(a) \frac{(x-a)^k}{k!}\,.\end{aligned} \tag{11}$$

For mathematical properties of fractional derivatives and integrals one can consult the mentioned references [11–15].

3. Non-Negative solutions

Definition 4. For $T > 0$ a real function $u : [0,T] \to \mathbb{R}_n$ is a non-negative function if $u(t) \geq 0$ on the interval $[0,T]$.

Let $\mathbb{R}_+^5 = \{X \in \mathbb{R}^5 : X \geq 0\}$ and $X(t) = (P(t), L(t), S(t), Q(t), N(t))^\mathrm{T}$. For the proof of the theorem about non-negative solutions, we shall need the following Lemma [15].

Lemma 1 (Generalized Mean Value Theorem). *Let $f(x) \in C[a,b]$ and $D^\alpha f(x) \in C[a,b]$ for $0 < \alpha \leq 1$. Then we have*

$$f(x) = f(a) + \frac{1}{\Gamma(\alpha)} D^\alpha f(\xi)(x-a)^\xi$$

with $0 \leq \xi \leq x$, for all $x \in (a,b]$.

Remark 1. Suppose $f(x) \in C[0,b]$ and $D^\alpha f(x) \in C[0,b]$ for $0 < \alpha \leq 1$. It is clear from Lemma 1 that if $D^\alpha f(x) \geq 0$ for all $x \in (0,b)$, then the function f is non-decreasing, and if $D^\alpha f(x) \leq 0$ for all $x \in (0,b)$, then the function f is non-increasing.

Theorem 1. *There is a unique solution for the initial value problem given by (3), and the solution remains in $\mathbb{R}_+^5$.*

Proof. The existence and uniqueness of the solution of system (3) in $(0,\infty)$ can be obtained from [7, Theorem 3.1 and Remark 3.2]. We need to show that the domain $\mathbb{R}_+^5$ is positively invariant. Since

$$D_t^\alpha S(t)|_{S=0} = bN(t) \geq 0\,, \quad D_t^\alpha E(t)|_{E=0} = 0\,,$$

$D_t^\alpha I(t)|_{I=0} = \gamma_1 E(t) \geq 0$, $D_t^\alpha R(t)|_{R=0} = \gamma_2 I(t) \geq 0$, $D_t^\alpha N(t)|_{N=0} = 0$.

On each hyperplane bounding the non-negative orthant, the vector field points into $\mathbb{R}_+^5$. □

4. Multi-Step Generalized Differential Transform Method (MSGDTM)

Now, we apply the MSGDTM to find the approximate solution of system (3), which gives an accurate solution over a longer time frame as compared to the standard generalized differential transform method (GDTM) [16–18]. Taking the differential transform of system (3) with respect to time, we obtain

$$\begin{aligned}
S(k+1) &= \frac{\Gamma(\alpha k+1)}{\Gamma((\alpha k+1)+1)} \cdot \left(bN(k) - \beta_1 \sum_{s=0}^{k} E(k-s)S(s) - \beta_2 \mathrm{ISN}(k) - \mu S(k)\right), \\
E(k+1) &= \frac{\Gamma(\alpha k+1)}{\Gamma((\alpha k+1)+1)} \cdot \left(\beta_1 \sum_{s=0}^{k} E(k-s)S(s) - (\gamma_1+\mu)E(k)\right), \\
I(k+1) &= \frac{\Gamma(\alpha k+1)}{\Gamma((\alpha k+1)+1)} \cdot (\beta_2 \mathrm{ISN}(k) + \gamma_1 E(k) - (\gamma_2+\mu)I(k)), \\
R(k+1) &= \frac{\Gamma(\alpha k+1)}{\Gamma((\alpha k+1)+1)}(\gamma_2 I(k) - \mu R(k)), \\
N(k+1) &= \frac{\Gamma(\alpha k+1)}{\Gamma((\alpha k+1)+1)}(b-\mu)N(k).
\end{aligned} \tag{12}$$

Here $S(k), E(k), I(k), R(k)$, and $N(k)$ are the differential transformations of $S(t), E(t), I(t), R(t)$, and $N(t)$, respectively. Also, $\mathrm{ISN}(k)$ is the differential transformation of the function $\mathrm{ISN}(t) = \frac{I(t)S(t)}{N(t)}$ and is defined as

$$\mathrm{ISN}(k) = \frac{1}{N(0)} \cdot \left[\sum_{s=0}^{k} I(s)S(k-s) - \sum_{s=0}^{k-1} \mathrm{ISN}(s)N(k-s)\right].$$

The differential transforms of the initial conditions are

$S(0) = S_0$, $E(0) = E_0$, $I(0) = I_0$, $R(0) = R_0$ and $N(0) = N_0$.

In view of the differential inverse transform, the differential transform series solution for system (3) can be obtained as

$$\begin{aligned}
&S(t) = \sum_{k=0}^{K} S(k)t^{\alpha k}, \quad E(t) = \sum_{k=0}^{K} E(k)t^{\alpha k}, \\
&I(t) = \sum_{k=0}^{K} I(k)t^{\alpha k}, \quad R(t) = \sum_{k=0}^{K} R(k)t^{\alpha k}, \\
&N(t) = \sum_{k=0}^{K} N(k)t^{\alpha k}.
\end{aligned} \tag{13}$$

Now according to the MSGDTM, the series solutions for system (3) is suggested by

$$S(t) = \begin{cases} \sum_{k=0}^{K} S_1(k)t^{\alpha k}, & t \in [0, t_1] \\ \sum_{k=0}^{K} S_2(k)(t-t_1)^{\alpha k}, & t \in [t_1, t_2] \\ \vdots \\ \sum_{k=0}^{K} S_M(k)(t-t_{M-1})^{\alpha k}, & t \in [t_{M-1}, t_M] \end{cases} \tag{14}$$

$$E(t) = \begin{cases} \sum_{k=0}^{K} E_1(k)t^{\alpha k}, & t \in [0, t_1] \\ \sum_{k=0}^{K} E_2(k)(t-t_1)^{\alpha k}, & t \in [t_1, t_2] \\ \vdots \\ \sum_{k=0}^{K} E_M(k)(t-t_{M-1})^{\alpha k}, & t \in [t_{M-1}, t_M] \end{cases} \tag{15}$$

$$I(t) = \begin{cases} \sum_{k=0}^{K} I_1(k)t^{\alpha k}, & t \in [0, t_1] \\ \sum_{k=0}^{K} I_2(k)(t-t_1)^{\alpha k}, & t \in [t_1, t_2] \\ \vdots \\ \sum_{k=0}^{K} I_M(k)(t-t_{M-1})^{\alpha k}, & t \in [t_{M-1}, t_M] \end{cases} \tag{16}$$

$$R(t) = \begin{cases} \sum_{k=0}^{K} R_1(k)t^{\alpha k}, & t \in [0, t_1] \\ \sum_{k=0}^{K} R_2(k)(t-t_1)^{\alpha k}, & t \in [t_1, t_2] \\ \vdots \\ \sum_{k=0}^{K} R_M(k)(t-t_{M-1})^{\alpha k}, & t \in [t_{M-1}, t_M] \end{cases} \tag{17}$$

$$N(t)=\begin{cases}\sum\limits_{k=0}^{K} N_1(k)t^{\alpha k},\ t\in[0,t_1]\\ \sum\limits_{k=0}^{K} N_2(k)(t-t_1)^{\alpha k},\ t\in[t_1,t_2]\\ \vdots\\ \sum\limits_{k=0}^{K} N_M(k)(t-t_{M-1})^{\alpha k},\ t\in[t_{M-1},t_M]\end{cases} \tag{18}$$

Here $S_i(k), E_i(k), I_i(k), R_i(k)$, and $N_i(k)$ for $i = 1,2,\ldots,M$ satisfy the following recurrence relations:

$$S_i(k+1)=\frac{\Gamma(\alpha k+1)}{\Gamma((\alpha k+1)+1)}\Bigg(bN_i(k) - \beta_1\sum_{s=0}^{k}E_i(k-s)S_i(s)-\beta_2\,\mathrm{ISN}_i(k)-\mu S_i(k)\Bigg),$$

$$E_i(k+1)=\frac{\Gamma(\alpha k+1)}{\Gamma((\alpha k+1)+1)} \cdot\Bigg(\beta_1\sum_{s=0}^{k}E_i(k-s)S_i(s)-(\gamma_1+\mu)E_i(k)\Bigg),$$

$$I_i(k+1)=\frac{\Gamma(\alpha k+1)}{\Gamma((\alpha k+1)+1)} \cdot\left(\beta_2\mathrm{ISN}_i(k)+\gamma_1E_i(k)-(\gamma_2+\mu)I_i(k)\right),$$

$$R_i(k+1)=\frac{\Gamma(\alpha k+1)}{\Gamma((\alpha k+1)+1)}\left(\gamma_2I_i(k)-\mu R_i(k)\right),$$

$$N_i(k+1)=\frac{\Gamma(\alpha k+1)}{\Gamma((\alpha k+1)+1)}(b-\mu)N_i(k)$$

such that

$$S_i(t_{i-1})=S_{i-1}(t_{i-1}),\ E_i(t_{i-1})=E_{i-1}(t_{i-1}),$$
$$I_i(t_{i-1})=I_{i-1}(t_{i-1}),\ R_i(t_{i-1})=R_{i-1}(t_{i-1})$$

and

$$N_i(t_{i-1})=N_{i-1}(t_{i-1}),\ i=2,3,\ldots,M.$$

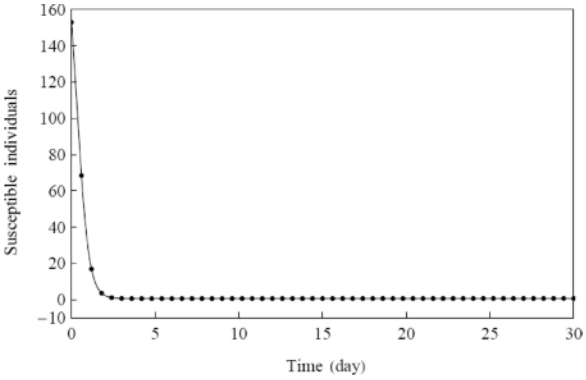

Fig. 1. Solid line: MSGDTM, dotted line: Runge–Kutta method.

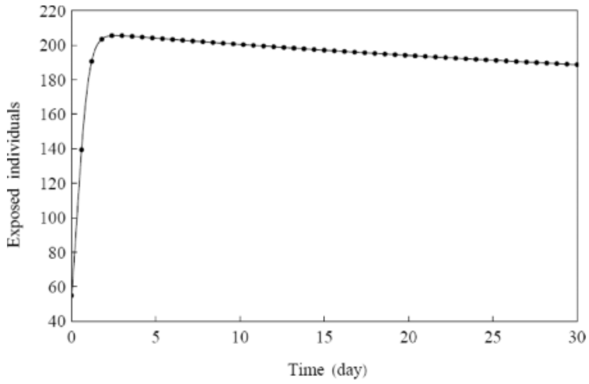

Fig. 2. Solid line: MSGDTM, dotted line: Runge–Kutta method.

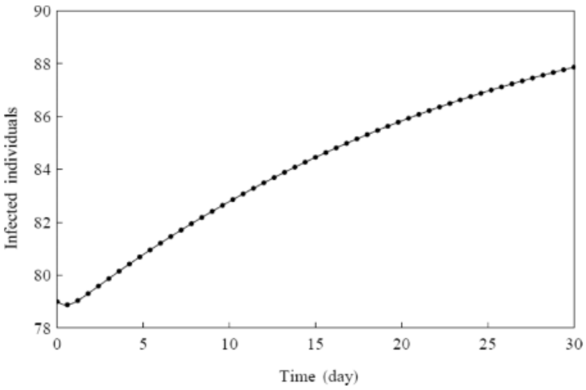

Fig. 3. Solid line: MSGDTM, dotted line: Runge–Kutta method.

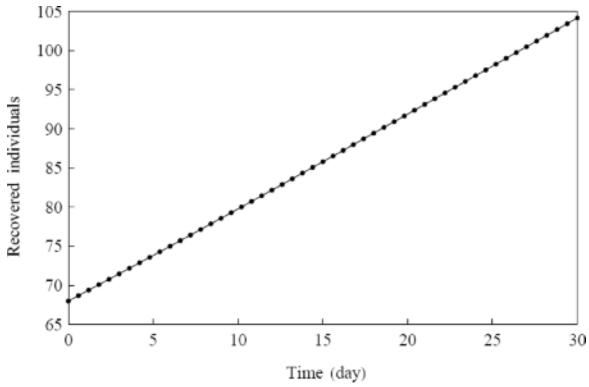

Fig. 4. Solid line: MSGDTM, dotted line: Runge–Kutta method.

Finally, we start with the initial conditions

$$S_0(0) = S_0\,,\ \ E_0(0) = E_0\,,\ \ I_0(0) = I_0\,,\ \ R_0(0) = R_0\,,$$

and

$$N_0(0) = N_0\,.$$

With the use of the recurrence relation given in the above system, we can obtained the MSGDTM solution given in (14)–(18).

5. Numerical Methods and Simulations

We solved analytically the system (3) with transform initial conditions by using the MSGDTM. We also used NSFD method and the fourth-order Runge–Kutta method for numerical results. For numerical simulation, we used a set of parameters given in Table 1. To demonstrate the effectiveness of the proposed algorithm as an approximate tool for solving the nonlinear system of fractional differential equations descrided in system (3) for large time t, we applied this algorithm

Notation	Parameter description	Value/day
μ	Natural death rate	0.0021
β_1	Transmission rate from susceptible to exposed class due to infected class	0.014
β_2	Transmission rate from susceptible to infected class	0.014
γ_1	Recovery rate of exposed class	0.0095
γ_2	Recovery rate of infected class	0.0165
b	Natural birth rate	0.0045

Table 1. Parameter values for the numerical simulation.

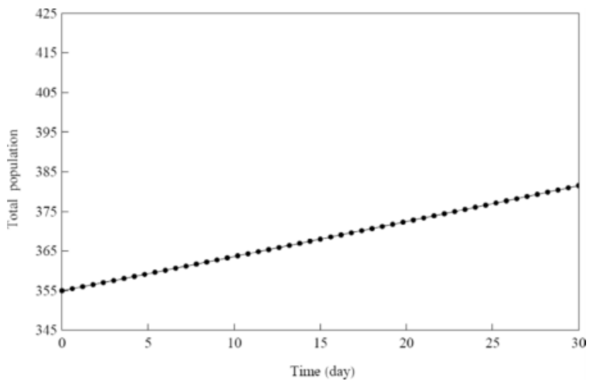

Fig. 5. Solid line: MSGDTM, dotted line: Runge–Kutta method.

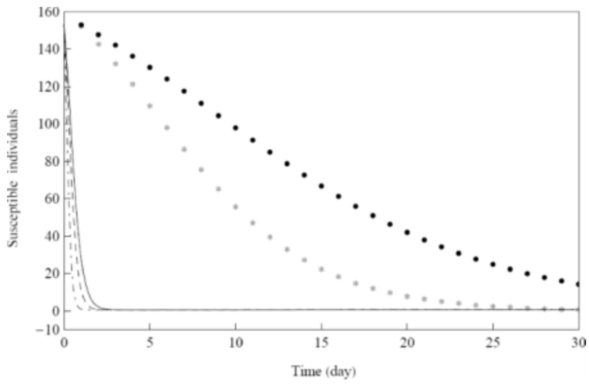

Fig. 6. Solid line: $\alpha = 1.0$, dashed line: $\alpha = 0.95$, dot-dashed line: $\alpha = 0.85$, dotted line: $\alpha = 0.75$, and star line: $\alpha = 0.65$.

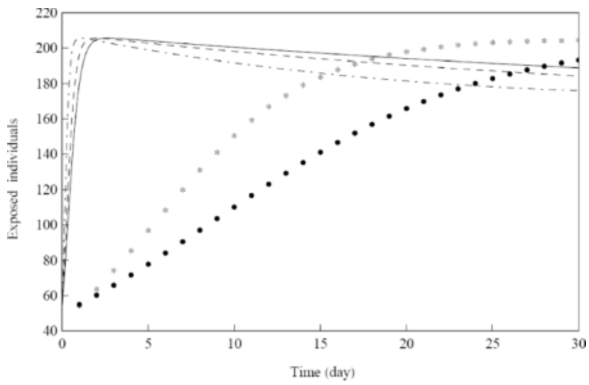

Fig. 7. Solid line: $\alpha = 1.0$, dashed line: $\alpha = 0.95$, dot-dashed line: $\alpha = 0.85$, dotted line: $\alpha = 0.75$, and star line: $\alpha = 0.65$.

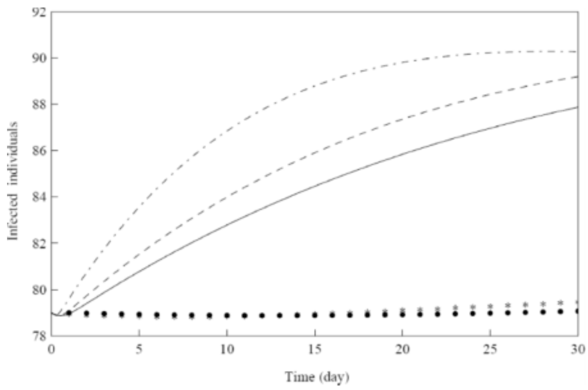

Fig. 8. Solid line: $\alpha = 1.0$, dashed line: $\alpha = 0.95$, dot-dashed line: $\alpha = 0.85$, dotted line: $\alpha = 0.75$, and star line: $\alpha = 0.65$.

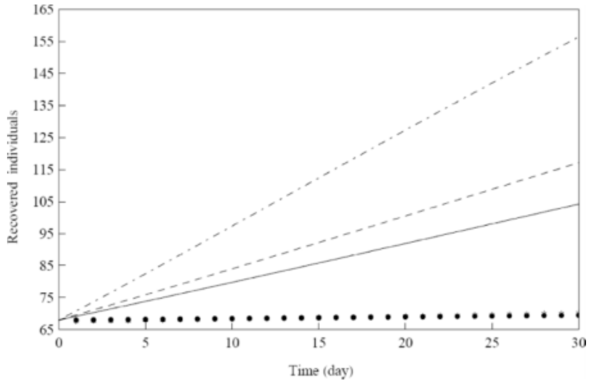

Fig. 9. Solid line: $\alpha = 1.0$, dashed line: $\alpha = 0.95$, dot-dashed line: $\alpha = 0.85$, dotted line: $\alpha = 0.75$, and star line: $\alpha = 0.65$.

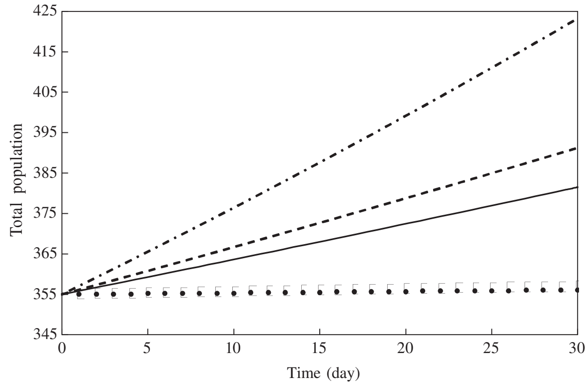

Fig. 10. Solid line: $\alpha = 1.0$, dashed line: $\alpha = 0.95$, dot-dashed line: $\alpha = 0.85$, dotted line: $\alpha = 0.75$, and star line: $\alpha = 0.65$.

on the interval [0-30]. It is to be noted that the MSGDTM results are obtained for $K = 10$ and $M = 3000$.

We also assumed the initial conditions to be $S_0 = 153$, $E_0 = 55$, $I_0 = 79$, $R_0 = 68$, and $N_0 = 355$.

Figures 1 – 5 show the approximate solutions for $S(t), E(t), I(t), R(t)$, and $N(t)$ obtained by using the MSGDTM and the classical Runge–Kutta method when α is one.

From the graphical results in Figures 1 – 5, it can be seen that the results obtained by using the MSGDTM match the results of the classical Runge–Kutta method very well, which implies that the presented method can predict the behaviour of these variables accurately for the region under consideration.

Figures 6 – 10 show the approximate solutions for $S(t), E(t), I(t), R(t)$, and $N(t)$ obtained for different values of α using the MSGDTM.

From the numerical results in Figures 6 – 10, it is clear that the approximate solutions depend continuously on the time-fractional derivative α. It is evident that the efficiency of this approach can be dramatically enhanced by decreasing the step size and computing further terms or further components of $S(t), E(t), I(t), R(t)$, and $N(t)$.

Figures 11 – 15 show the approximate solutions for $S(t), E(t), I(t), R(t)$, and $N(t)$ via NSFD [12], the classical Runge–Kutta method, and the algorithm ode45 when α is one.

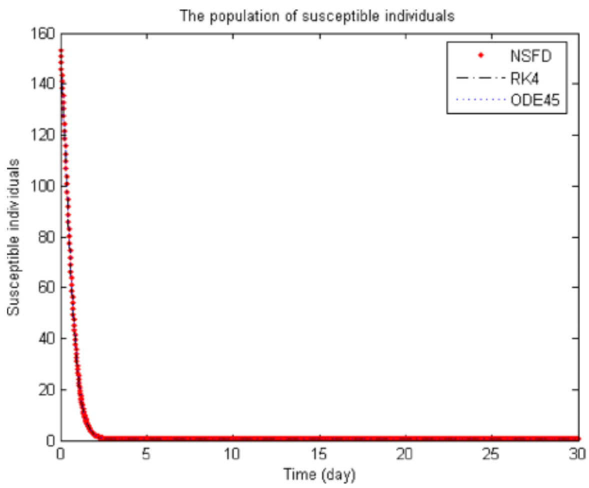

Fig. 11 (colour online). Plot of the approximation for $S(t)$ with respect to time t.

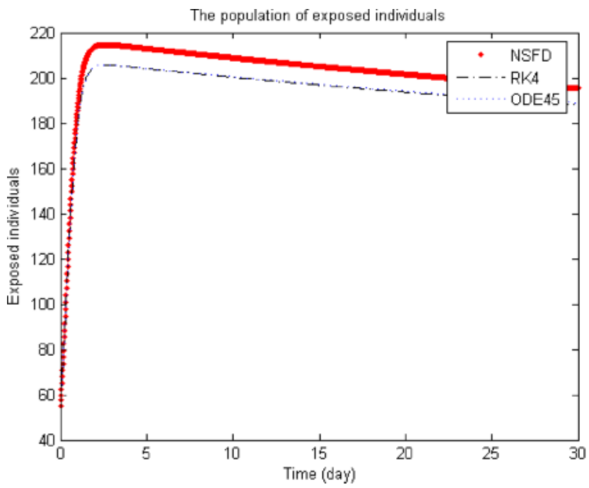

Fig. 12 (colour online). Plot of the approximation for $E(t)$ with respect to time t.

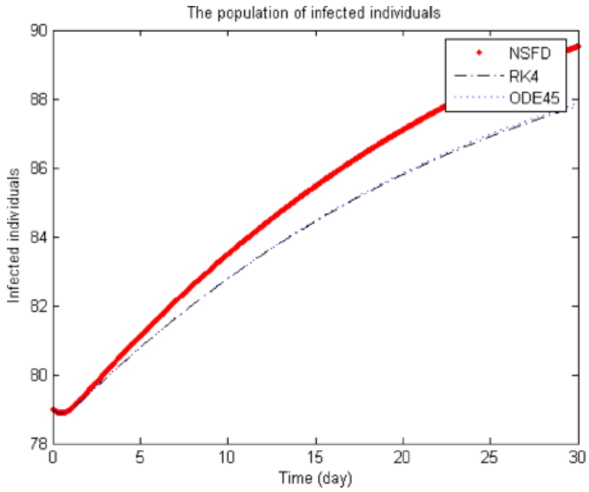

Fig. 13 (colour online). Plot of the approximation for $I(t)$ with respect to time t.

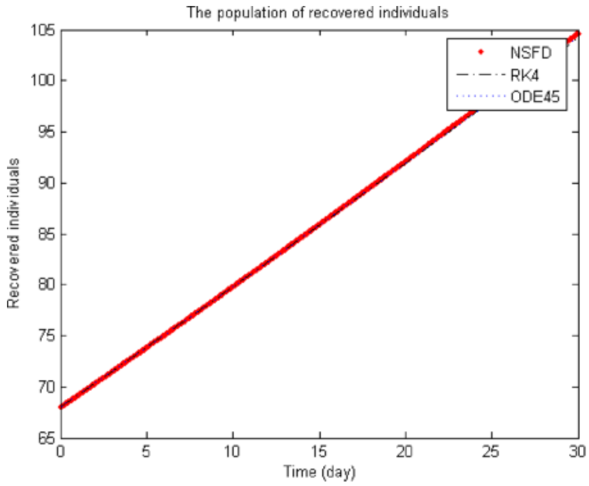

Fig. 14 (colour online). Plot of the approximation for $R(t)$ with respect to time t.

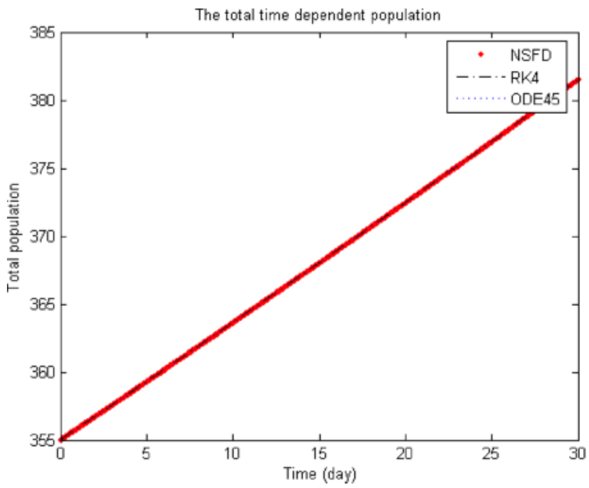

Fig. 15 (colour online). Plot of the approximation for $N(t)$ with respect to time t.

6. Conclusion

In this paper, a fractional-order system for SEIR (susceptible-exposed-infected-recovered) epidemic model is studied and its approximate solutions are presented by using the MSGDTM. The approximate solutions obtained by this method are highly accurate and valid for a long time in the integer case. The MSGDTM introduces a new idea for constructing the approximate solution. In this approximation, the interval is divided into subintervals of equal length, and initial conditions are chosen as mentioned in Section 4. That is, in every subinterval a new initial condition is determined by functions obtained in these subintervals. This gives a way of finding accurate approximate solutions at points far from the first initial point. This is the main advantage of the present method. But there is no such a situation in built function precedures provided by Mathematica or Maple softwares. That is, the first initial condition is never changed. For this reason, generally accurate approximate solutions can not be obtained at points far from the first initial point. This method is very applicable and is also a good approach for obtaining the solutions of differential equations of such order. This tool is the best one for modelling in science and engineering.

[1] W. O. Kermack and A. G. McKendrick, Bull. Math. Bio. **53**, 33 (1991).
[2] M. M. A. El-Sheikh and S. A. A. El-Marouf, Int. J. Math. Math. Sci. **56**, 2971 (2004).
[3] H. W. Hethcote, Math. Biosci. **7**, 335 (1976).
[4] B. Shulgin, L. Stone, and Z. Agur, Bull. Math. Biol. **60**, 1123 (1998).
[5] C. M. Kribs-Zaleta and J. X. Velasco-Hernandez, Math. Biosci. **164**, 183 (2000).

[6] N. Özalp and E. Demirci, Math. Comput. Mod. **54**, 1 (2011).
[7] W. Lin, J. Math. Anal. Appl. **332**, 709 (2007).
[8] V. S. Erturk, Z. M. Odibait, and S. Momani, Comput. Math. Appl. **62**, 996 (2011).
[9] S. Pooseh, H. S. Rodrigues, and D. F. M. Torres, American Isnt. Phys. Conf. Proc. **1389**, 739 (2012).
[10] K. Diethelm, Nonlin. Dyn. **71**, 613 (2013).
[11] I. Podlubny, Fractional Differential Equations, Academic Presss, London 1999.
[12] M. Caputo, Geophys. J. Int. **13**, 529 (1967).
[13] A. Atangana and A. Kilicman, Math. Prob. Eng. **2013**, (2013), Article ID 853127.
[14] S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Willey, New York 1993.
[15] Z. M. Odibat and N. T. Shawafeh, Comput. Math. Appl. **186**, 286 (2007).
[16] V. Erturk, S. Momani, and Z. Odibat, Commun. Nonlin. Sci. Numer. Simul. **13**, 1642 (2008).
[17] Z. Odibat and S. Momani, Appl. Math. Lett. **21**, 194 (2008).
[18] S. Momani and Z. Odibat, J. Comput. Appl. Math. **220**, 85 (2008).