Stochastic Soliton Solutions of the High-Order Nonlinear Schrödinger Equation in the Optical Fiber with Stochastic Dispersion and Nonlinearity

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In this paper, the high-order nonlinear Schrödinger (HNLS) equation driven by the Gaussian white noise, which describes the wave propagation in the optical fiber with stochastic dispersion and nonlinearity, is studied. With the white noise functional approach and symbolic computation, stochastic one- and two-soliton solutions for the stochastic HNLS equation are obtained. For the stochastic one soliton, the energy and shape keep unchanged along the soliton propagation, but the velocity and phase shift change randomly because of the effects of Gaussian white noise. Ranges of the changes increase with the increase in the intensity of Gaussian white noise, and the direction of velocity is inverted along the soliton propagation. For the stochastic two solitons, the effects of Gaussian white noise on the interactions in the bound and unbound states are discussed: In the bound state, periodic oscillation of the two solitons is broken because of the existence of the Gaussian white noise, and the oscillation of stochastic two solitons forms randomly. In the unbound state, interaction of the stochastic two solitons happens twice because of the Gaussian white noise. With the increase in the intensity of Gaussian white noise, the region of the interaction enlarges.

Key words: Stochastic Solitons; Gaussion White Noise; High-Order Nonlinear Schrödinger Equation; Symbolic Computation; White Noise Functional Approach.

1. Introduction

Nonlinear Schrödinger (NLS) equations describe the wave propagation in different nonlinear media, such as the nonlinear fibers [1], photonic crystals [2], Bose-Einstein condensates [3], and ion plasmas [4]. Some NLS solitons have been studied to analyse the interaction between the nonlinearity and dispersion [5, 6]. In reality, such models in the nonlinear media involve certain uncertainties [7]. Having considered the effects of stochastic coefficients or initial conditions on the soliton solutions, people are also able to work on the NLS equations driven by the noises [8 – 14]. Evolution of the NLS solitons with stochastic cubic nonlinearity has been investigated via the numerical simulations [9]. NLS solitons with white noise dispersion which describe the propagation of a signal in an optical fiber with dispersion management have been studied [10], and dispersion-managed vector solitons in the birefringent optical fibers with stochastic birefringence have been discussed numerically [11].

Propagation of the soliton pulse in a single-mode fiber with delayed Raman response and random parameters is described by the stochastic NLS equation with high-order nonlinearity and dispersion [1, 15 – 17],

$$\frac{\partial U}{\partial z}+\mathrm{i}\sum_{k\geq 2}^{3}\frac{\mathrm{i}^k\beta_k(z)}{k!}\frac{\partial^k U}{\partial t^k}=\mathrm{i}\gamma(z)\left(1+\frac{\mathrm{i}}{\omega_0}\frac{\partial}{\partial t}\right)\cdot\left[U\left(z,t\right)\int_{-\infty}^{t}R\left(t'\right)\left|U\left(z,t-t'\right)\right|^2\mathrm{d}t'\right], \tag{1}$$

where $U(z,t)$ is the slowly varying normalized envelope of the ultrashort pulse, z is the normalized distance along the direction of the propagation, t is the retarded time, t' is the formal variable, ω_0 is the center frequency of the ultrashort pulse spectrum, $R(t)$ is the Raman response function, while the kth-order group-velocity-dispersion (GVD) coefficient $\beta_k(z)$ and nonlinearity coefficient $\gamma(z)$ are considered as stochastic functions:

$$\beta_k(z)=\beta_k'\left[1+m_{\beta_k}(z)\right],\ \ \gamma(z)=\gamma_0\left[1+m_\gamma(z)\right],$$

where β_k' and γ_0 are respectively the mean values of $\beta_k(z)$ and $\gamma(z)$, $m_{\beta_k}(z)$ and $m_\gamma(z)$ are the zero-mean stochastic processes of the Gaussian white noise,

$$\langle m_{\beta_k}\rangle=\langle m_\gamma\rangle=0,\ \ \langle m_{\beta_k}(z)m_{\beta_k}(z')\rangle=2\sigma_{\beta_k}^2\delta(z-z'),$$
$$\langle m_\gamma(z)m_\gamma(z')\rangle=2\sigma_\gamma^2\delta(z-z'),$$

where $\langle\cdots\rangle$ denotes the statistical average, z' is the formal variable, and σ_{β_k} and σ_γ are respectively the variances of stochastic processes m_{β_k} and m_γ. Modulational instability of the periodic pluse arrays described by (1) in the optical fiber with stochastic parameters of high-order nonlinearity and dispersion has been studied [15]. In the nonlocal focusing and defocusing Kerr media with stochastic dispersion and nonlinearity, effects of the noises on the modulational instability for (1) have been discussed [16, 17]. In the case when the pulse envelope evolves slowly along the fiber, (1) can be approximately written as [1]

$$\frac{\partial U}{\partial z}+\frac{\mathrm{i}\beta_2(z)}{2}\frac{\partial^2 U}{\partial t^2}-\frac{\beta_3(z)}{6}\frac{\partial^3 U}{\partial t^3}=\mathrm{i}\gamma(z)\left[|U|^2U+\frac{\mathrm{i}}{\omega_0}\frac{\partial}{\partial t}\left(|U|^2U\right)-\mathrm{i}\tau_R U\frac{\partial|U|^2}{\partial t}\right],\tag{2}$$

where τ_R is the Raman resonant time constant. Although the periodic-like solutions [18] and numerical simulation [19] of (2) without high-order nonlinearity and dispersion have been obtained, to our knowledge, stochastic soliton solutions for (2), in the optical fiber with stochastic high-order nonlinearity and dispersion, have not been obtained as yet. Effects of the noises on the optical solitons have been discussed in the optical fiber communication systems [20], fiber lasers [21, 22], and fiber optical parametric amplifiers [23], so that the stochastic optical solitons, which we will obtain based on (2), might mirror the effects of the Gaussian white noise on those studies with the high-order dispersion and nonlinearity.

In this paper, we will work on (2) via the white noise functional approach[1] and symbolic computation [5, 31, 32]. In Section 2, the Wick product, Hida test function space, and Hida distribution space will be constructed, then (2) will be transformed into the Wick-type stochastic NLS equation. Via the Hermite transformation, wick-type stochastic NLS equation will be transformed into an equation under certain conditions to obtain the solutions of (2). In Section 3, stochastic one- and two-soliton solutions will be obtained via the symbolic computation and inverse Hermite transform, and the effects of Gaussian white noise on the dynamic properties of the solitons will be discussed. In Section 4, the stabilities of stochastic solitons will be studied through the numerical simulation. Finally, our conclusions will be given in Section 5.

2. White Noise Analysis of Equation (2)

In (2), the Gaussian-white-noise $m_{\beta_2}(z)$, $m_{\beta_3}(z)$, and $m_\gamma(z)$ have the following properties:

$$\langle m_{\beta_2}(z)\rangle=\langle m_{\beta_3}(z)\rangle=\langle m_\gamma(z)\rangle=0,$$
$$m_{\beta_2}(z)=h_1W(z),\ \ m_{\beta_3}(z)=h_2W(z),$$
$$m_\gamma(z)=h_3W(z),$$

where h_1, h_2, and h_3 are all non-zero constants, which represent the intensity coefficients of the Gaussian white noises, respectively, $W(z)$ is the standard Gaussian white noise, $W(z)=\frac{\mathrm{d}B(z)}{\mathrm{d}z}$, and $B(z)$ is the standard Brownian motion [24].

The white noise functional approach to study the stochastic partial differential equations in the Wick versions has been given in [24]. For (2), $U=U(z,t,W)$ is the generalized stochastic process and $t\in\mathbb{R}^d$. Let $(S(\mathbb{R}^d))$ and $(S(\mathbb{R}^d))^*$ be the Hide test function space and Hide function space on $\mathbb{R}^d$, respectively [33]. Let $h_n(x)$ be the dth-order Hermite polynomials. Setting $\xi_n(x)=\mathrm{e}^{-\frac{1}{2}x^2}h_n(\sqrt{2}x)/(\pi(n-1)!)^{\frac{1}{2}}$, we denote $\alpha=(\alpha_1,\cdots,\alpha_d)$ being the d-dimensional multi-indices with $\alpha_j's$ $(j=1,\cdots,d)\in\mathbb{N}$ ($\mathbb{N}$ denotes the set of the natural numbers), and let $\alpha^{(i)}=(\alpha_1^{(i)},\cdots,\alpha_d^{(i)})$ be the ith multi-index number in some fixed ordering of all the d-dimensional multi-indices $\alpha=(\alpha_1,\cdots,\alpha_d)\in\mathbb{N}^d$ [34–36]. For α, we define

$$H_\alpha(\omega)=\prod_{i=1}^{\infty}h_{\alpha_i}(\langle\omega,\eta_i\rangle),\ \ \omega\in\left(S\left(\mathbb{R}^d\right)\right)^*$$
$$\eta_i=\xi_{\alpha^{(i)}}=\xi_{\alpha_1^{(i)}}\otimes\cdots\otimes\xi_{\alpha_d^{(i)}},$$

where $\otimes$ is the tensor product.

[1]With the white noise functional approach [24], certain analytic solutions for the stochastic Korteweg–de Vries (KdV) [25–27], KdV–Burgers [28, 29], and (2 + 1)-dimensional Broer–Kaup equations [30] have been obtained.

Fix $n \in \mathbb{N}$. Let $(S)_1^n$ consist of those $x = \sum_\alpha c_\alpha H_\alpha$ with $c_\alpha \in \mathbb{R}^n$ such that $\|x\| = \sum_\alpha c_\alpha^2 (\alpha!)^2 (2\mathbb{N})^{k\alpha}$, $\forall k \in \mathbb{N}$ with $c_\alpha^2 = |c_\alpha|^2 = \sum_{k=1}^n (c_\alpha^{(k)})^2$ if $c_\alpha = (c_\alpha^{(1)}, \cdots, c_\alpha^{(n)}) \in \mathbb{R}^n$.

The Wick product can be defined as [24, 37, 38]

$$F \diamond G = \sum_{\alpha,\beta} (a_\alpha, b_\beta) H_{\alpha+\beta}\,, \tag{3}$$

where β is the d-dimensional multi-indices, $F = \sum_\alpha a_\alpha H_\alpha$, $G = \sum_\beta b_\beta H_\beta \in (S)_{-1}^n$ with $a_\alpha, b_\beta \in \mathbb{R}^n$. By interpreting Wick versions, we can write (2) as

$$\begin{aligned}&\frac{\partial U}{\partial z} + \frac{\mathrm{i}}{2} \diamond H_1(z) \diamond \frac{\partial^2 U}{\partial t^2} - \frac{1}{6} \diamond H_2(z) \diamond \frac{\partial^3 U}{\partial t^3} = \\ &i \diamond H_3(z) \diamond \Bigg[|U|^{\diamond 2} \diamond U + \frac{\mathrm{i}}{\omega_0} \diamond \frac{\partial}{\partial t} \left(|U|^{\diamond 2} \diamond U \right) \\ &- \mathrm{i}\, \tau_R \diamond U \diamond \frac{\partial |U|^{\diamond 2}}{\partial t} \Bigg],\end{aligned} \tag{4}$$

where $H_1(z) = \beta_2' [1 + h_1 W(z)]$, $H_2(z) = \beta_3' [1 + h_2 W(z)]$, and $H_3(z) = \gamma_0 [1 + h_3 W(z)]$.

For $F = \sum_\alpha a_\alpha H_\alpha \in (S)_{-1}^n$ with $a_\alpha \in \mathbb{R}^n$, the Hermite transform of F is defined as [24, 37, 38]

$$\widetilde{F}(z) = \mathcal{H}(F) = \sum_\alpha a_\alpha \omega^\alpha \in \mathbb{C}^n, \tag{5}$$

where $\omega \in \mathbb{C}^n$ is a complex variable. For $F, G \in (S)_{-1}^n$, by the Wick product definition, we have

$$\begin{aligned}\mathcal{H}(F(W) \diamond G(W)) &= \widetilde{F(W) \diamond G(W)} \\ &= \widetilde{F(\omega)} \cdot \widetilde{G(\omega)}\,.\end{aligned} \tag{6}$$

With the Hermite transformation of (4), the Wick products are turned into the ordinary products and (4) is written as

$$\begin{aligned}&\frac{\partial \widetilde{u}(z,t,\omega)}{\partial z} + \frac{\mathrm{i}}{2} \widetilde{H_1}(z,\omega) \frac{\partial^2 \widetilde{u}(z,t,\omega)}{\partial t^2} \\ &- \frac{1}{6} \widetilde{H_2}(z,\omega) \frac{\partial^3 \widetilde{u}(z,t,\omega)}{\partial t^3} = \\ &\mathrm{i} \widetilde{H_3}(z,\omega) \Bigg[|\widetilde{u}(z,t,\omega)|^2 \widetilde{u}(z,t,\omega) \\ &+ \frac{\mathrm{i}}{\omega_0} \frac{\partial}{\partial t} \left(|\widetilde{u}(z,t,\omega)|^2 \widetilde{u}(z,t,\omega) \right) \\ &- \mathrm{i}\, \tau_R \widetilde{u}(z,t,\omega) \frac{\partial |\widetilde{u}(z,t,\omega)|^2}{\partial t} \Bigg],\end{aligned} \tag{7}$$

where $\widetilde{u}(z,t,\omega) = \mathcal{H}[U(z,t,W)]$, $\widetilde{H_1}(z,\omega) = \beta_2'[1 + h_1 \widetilde{W}(z,\omega)]$, $\widetilde{H_2}(z,\omega) = \beta_3'[1 + h_2 \widetilde{W}(z,\omega)]$, and $\widetilde{H_3}(z,\omega) = \gamma_0[1 + h_3 \widetilde{W}(z,\omega)]$ with $\widetilde{W}(z,\omega) = \mathcal{H}[W(z)] = \sum_{k=1}^\infty \eta_k(z)\omega$. Once the solutions $\widetilde{u}(z,t,\omega)$ for (7) is obtained, the solutions $U(z,t,W)$ for (2) can be obtained by the inverse Hermite transform of $\widetilde{u}(z,t,\omega)$.

3. Stochastic Soliton Solutions and Discussions

3.1. Bilinear Forms and Stochastic Soliton Solutions

To obtain the bilinear forms for (7), we introduce the dependent variable transformation,

$$\widetilde{u}(z,t,\omega) = \frac{g(z,t,\omega)}{f(z,t,\omega)}\,, \tag{8}$$

where $g(z,t,\omega)$ is a complex differentiable function and $f(z,t,\omega)$ is a real one. With $h_2 = h_3$ and $\beta_3' = \tau_R = \frac{1}{\omega_0}$, the bilinear forms for (7) can be obtained as

$$\left[D_z + \frac{\mathrm{i}}{2} \widetilde{H_1}(z,\omega) D_t^2 - \frac{1}{6} \widetilde{H_2}(z,\omega) D_t^3 \right] g \cdot f = 0\,, \tag{9}$$

$$\widetilde{H_1}(z,\omega) D_t^2 f \cdot f = \widetilde{H_3}(z,\omega)\, g \cdot g^*\,, \tag{10}$$

with "*" representing the complex conjugate and the Hirota operators D_z and D_t defined by [31, 32]

$$\begin{aligned}D_z^m D_t^n (a \cdot b) = &\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z'} \right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n \\ &\cdot a(z,t)\, b(z',t') \Big|_{z'=z, t'=t}\,,\end{aligned} \tag{11}$$

z' and t' being the formal variables, $a(z,t)$ as the function of z and t, $b(z',t')$ as the function of z' and t', $m = 0, 1, 2, \cdots$ and $n = 0, 1, 2, \cdots$.

Via the Hirota method [31, 32], $g(z,t,\omega)$ and $f(z,t,\omega)$ can be expanded as

$$\begin{aligned}g(z,t,\omega) = &\,\varepsilon\, g_1(z,t,\omega) + \varepsilon^3 g_3(z,t,\omega) \\ &+ \varepsilon^5 g_5(z,t,\omega) + \cdots\,,\end{aligned} \tag{12}$$

$$\begin{aligned}f(z,t,\omega) = &\,1 + \varepsilon^2 f_2(z,t,\omega) + \varepsilon^4 f_4(z,t,\omega) \\ &+ \varepsilon^6 f_6(z,t,\omega) + \cdots\,,\end{aligned} \tag{13}$$

with ε is a small parameter. $g_n(z,t,\omega)$ and $f_n(z,t,\omega)$ $(n = 1, 2, \cdots)$ are the functions to be determined.

Substituting Expressions (12) and (13) into Bilinear Forms (9) and (10) and equating the coefficients of the same powers of ε to zero yield the recursion relations for $g_n(z,t,\omega)$ and $f_n(z,t,\omega)$ $(n=1,2,\cdots)$.

To obtain the one-soliton solutions for (7), we truncate Expressions (12) and (13) for $g_1(z,t,\omega)$ and $f_2(z,t,\omega)$, respectively. Setting

$$g_1(z,t,\omega)=n_1\,\mathrm{e}^{\theta},\ \ f_2(z,t,\omega)=m_1\,\mathrm{e}^{\theta+\theta^*}, \tag{14}$$

where $\theta=k(z,\omega)+b_1t=k(z,\omega)+(b_{11}+\mathrm{i}b_{12})t$ and $n_1=n_{11}+\mathrm{i}n_{12}$ with n_{11}, n_{12}, b_{11}, b_{12}, and m_1 as the real constants, $k(z,\omega)$ as the complex function. Substituting Expressions (14) into Bilinear Forms (9) and (10), we can get the constraints on the parameters,

$$k(z,\omega)=\int_{-\infty}^{z}\left(b_1^3-\mathrm{i}b_1^2\right)\widetilde{H_2}(\xi,\omega)\,\mathrm{d}\xi\,,$$

$$m_1=-\frac{\left(n_{11}^2+n_{12}^2\right)}{8b_{11}^2},\ \ \alpha=\frac{2\gamma_0}{\beta_2'}\ \ h_3=h_1\,.$$

With $\varepsilon=1$, the one-soliton solutions for (7) can be expressed as

$$\begin{aligned}\widetilde{u}(z,t,\omega)&=\frac{n_1\,\mathrm{e}^{b_1t+k(z,\omega)}}{1-\frac{(n_{11}^2+n_{12}^2)\gamma_0}{4\beta_2'b_{11}^2}\,\mathrm{e}^{\left(b_1+b_1^*\right)t+k(z,\omega)+k^*(z,\omega)}}\\&=\frac{n_1\,\mathrm{e}^{b_1t+\int_0^z\left(b_1^3-\mathrm{i}b_1^2\right)\widetilde{H_2}(\xi,\omega)\,\mathrm{d}\xi}}{1-\frac{(n_{11}^2+n_{12}^2)\gamma_0}{4\beta_2'b_{11}^2}\,\mathrm{e}^{\left(b_1+b_1^*\right)t+\int_0^z\left(b_1^3-\mathrm{i}b_1^2\right)\widetilde{H_2}(\xi,\omega)\,\mathrm{d}\xi+\int_0^z\left(b_1^{*3}+\mathrm{i}b_1^{*2}\right)\widetilde{H_2}(\xi,\omega)\,\mathrm{d}\xi}}\,.\end{aligned} \tag{15}$$

To obtain the two-soliton solutions for (7), we assume that

$$\begin{aligned}g_1(z,t)&=m_1\,\mathrm{e}^{\theta_1}+m_2\,\mathrm{e}^{\theta_2},\\ f_2(z,t)&=n_1\,\mathrm{e}^{\theta_1+\theta_1^*}+n_2\,\mathrm{e}^{\theta_2+\theta_2^*}+n_3\,\mathrm{e}^{\theta_1+\theta_2^*}\\&\quad+n_4\,\mathrm{e}^{\theta_2+\theta_1^*},\\ g_3(z,t)&=m_3\,\mathrm{e}^{\theta_1+\theta_2+\theta_1^*}+m_4\,\mathrm{e}^{\theta_1+\theta_2+\theta_2^*},\\ f_4(z,t)&=O_1\,\mathrm{e}^{\theta_1+\theta_2+\theta_1^*+\theta_2^*},\end{aligned}$$

where $O_1=O_{11}+\mathrm{i}O_{12}$, $m_j=m_{j1}+\mathrm{i}m_{j2}$, $n_j=n_{j1}+\mathrm{i}n_{j2}$, and $\theta_l=k_l(z,\omega)+b_lt=k_l(z,\omega)+(b_{l1}+b_{l2})t$ with O_{11}, O_{12}, m_{j1}, m_{j2}, n_{j1}, n_{j2}, b_{l1}, and b_{l2} as the real constants, and $k_l(z,\omega)$ as the complex functions $(j=1,2\cdots4, l=1,2)$. Substituting them into Bilinear Forms (9) and (10), we can get the constraints on the parameters,

$$\begin{aligned}k_1(z,\omega)&=\int_{-\infty}^{z}\left(b_{11}+\mathrm{i}b_{12}\right)^2\left(b_{11}+\mathrm{i}b_{12}-\mathrm{i}\alpha_1\right)\\&\quad\cdot\widetilde{H_2}(\xi,\omega)\,\mathrm{d}\xi\,,\end{aligned}$$

$$\begin{aligned}k_2(z,\omega)&=\int_{-\infty}^{z}\left(b_{21}+\mathrm{i}b_{22}\right)^2\left(b_{21}+\mathrm{i}b_{22}-\mathrm{i}\alpha_1\right)\\&\quad\cdot\widetilde{H_2}(\xi,\omega)\,\mathrm{d}\xi,\\ n_1&=\frac{-\left(m_{11}^2+m_{12}^2\right)\alpha_2}{8b_{11}^2},\\ n_2&=\frac{-\left(m_{21}^2+m_{22}^2\right)\alpha_2}{8b_{21}^2},\\ n_3&=\frac{-\left(m_{11}+\mathrm{i}m_{12}\right)\left(m_{21}-\mathrm{i}m_{22}\right)\alpha_2}{2\left(b_{11}+\mathrm{i}b_{12}+b_{21}-\mathrm{i}b_{22}\right)^2},\\ n_4&=\frac{-\left(m_{11}-\mathrm{i}m_{12}\right)\left(m_{21}+\mathrm{i}m_{22}\right)\alpha_2}{2\left(b_{11}-\mathrm{i}b_{12}+b_{21}+\mathrm{i}b_{22}\right)^2},\\ m_3&=-\frac{m_2\left(m_{11}^2+m_{12}^2\right)\alpha_2\left(b_1-b_2\right)^2}{8b_{11}^2\left(b_1+b_2^*\right)^2},\\ m_4&=-\frac{m_1\left(m_{21}^2+m_{22}^2\right)\alpha_2\left(b_1-b_2\right)^2}{8b_{21}^2\left(b_1+b_2^*\right)^2},\\ \alpha_1&=\frac{3\beta_2'}{\beta_3'}\ \ \alpha_2=\frac{6\gamma_0}{\beta_3'}\ \ h_1=h_2,\end{aligned}$$

$$O_1=\frac{\left(m_{11}^2+m_{12}^2\right)\left(m_{21}^2+m_{22}^2\right)\alpha_2^2\left[\left(b_{11}-b_{21}\right)^2+\left(b_{12}-b_{22}\right)^2\right]^2}{64b_{11}^2b_{21}^2\left[\left(b_{11}+b_{21}\right)^2+\left(b_{12}-b_{22}\right)^2\right]^2}\,.$$

With $\varepsilon = 1$, the two-soliton solutions for (7) can be expressed as

$$\begin{aligned}
\widetilde{u}(z,t,\omega) &= \frac{m_1\,\mathrm{e}^{b_1 t+} + m_2\,\mathrm{e}^{\theta_2} + m_3\,\mathrm{e}^{\theta_1+\theta_2+\theta_1^*} + m_4\,\mathrm{e}^{\theta_1+\theta_2+\theta_2^*}}{n_1\,\mathrm{e}^{\theta_1+\theta_1^*} + n_2\,\mathrm{e}^{\theta_2+\theta_2^*} + n_3\,\mathrm{e}^{\theta_1+\theta_2^*} + n_4\,\mathrm{e}^{\theta_2+\theta_1^*} + O_1\,\mathrm{e}^{\theta_1+\theta_2+\theta_1^*+\theta_2^*}} \\
&= \Big\{ m_1\,\mathrm{e}^{b_1 t + b_1^2(b_1-\mathrm{i}\alpha_1)\int_{-\infty}^{z}\widetilde{H_2}(\xi,\omega)\,\mathrm{d}\xi} + m_2\,\mathrm{e}^{b_2 t + b_2^2(b_2-\mathrm{i}\alpha_1)\int_{-\infty}^{z}\widetilde{H_2}(\xi,\omega)\,\mathrm{d}\xi} \\
&\quad + m_3\,\mathrm{e}^{(b_2+2b_{11})t+\left[2b_{11}(b_{11}^2+2\alpha_1 b_{12}-3b_{12}^2)+b_2^2(b_2-\mathrm{i}\alpha_1)\right]\int_{-\infty}^{z}\widetilde{H_2}(\xi,\omega)\,\mathrm{d}\xi} \\
&\quad + m_4\,\mathrm{e}^{(b_1+2b_{21})t+\left[2b_{21}(b_{21}^2+2\alpha_1 b_{22}-3b_{22}^2)+b_1^2(b_1-\mathrm{i}\alpha_1)\right]\int_{-\infty}^{z}\widetilde{H_2}(\xi,\omega)\,\mathrm{d}\xi} \Big\} \Big/ \\
&\quad \Big\{ n_1\,\mathrm{e}^{2b_{11}t+2b_{11}(b_{11}^2+2\alpha_1 b_{12}-3b_{12}^2)\int_{-\infty}^{z}\widetilde{H_2}(\xi,\omega)\,\mathrm{d}\xi} \\
&\quad + n_2\,\mathrm{e}^{2b_{21}t+2b_{21}(b_{21}^2+2\alpha_1 b_{22}-3b_{22}^2)\int_{-\infty}^{z}\widetilde{H_2}(\xi,\omega)\,\mathrm{d}\xi} \\
&\quad + n_3\,\mathrm{e}^{(b_1+b_2^*)t+\left[b_1^2(b_1-\mathrm{i}\alpha_1)+b_2^{*2}(b_2^*+\mathrm{i}\alpha_1)\right]\int_{-\infty}^{z}\widetilde{H_2}(\xi,\omega)\,\mathrm{d}\xi} \\
&\quad + n_4\,\mathrm{e}^{(b_2+b_1^*)t+\left[b_2^2(b_2-\mathrm{i}\alpha_1)+b_1^{*2}(b_1^*-\mathrm{i}\alpha_1)\right]\int_{-\infty}^{z}\widetilde{H_2}(\xi,\omega)\,\mathrm{d}\xi} \\
&\quad + O_1\,\mathrm{e}^{2(b_{11}+b_{21})t+2\left[b_{11}^3+b_{21}^3+b_{11}(2\alpha_1-3b_{12})b_{12}+b_{21}(2\alpha_1-3b_{22})b_{22}\right]\int_{-\infty}^{z}\widetilde{H_2}(\xi,\omega)\,\mathrm{d}\xi} \Big\}.
\end{aligned} \tag{16}$$

Taking the inverse Hermite transformation of solutions (15), the one-soliton solutions for (4) are obtained as

$$\begin{aligned}
U(z,t) &= \frac{n_1\,\mathrm{e}^{\diamond\int_0^z\left(b_1^3-\mathrm{i}b_1^2\right)H_2(\xi)\,\mathrm{d}\xi+b_1 t}}{1-\frac{\left(n_{11}^2+n_{12}^2\right)\gamma_0}{4\beta_2' b_{11}^2}\,\mathrm{e}^{\diamond\left[\int_0^z\left(b_1^3-\mathrm{i}b_1^2\right)H_2(\xi)\,\mathrm{d}\xi+\int_0^z\left(b_1^{*3}+\mathrm{i}b_1^{*2}\right)H_2(\xi)\,\mathrm{d}\xi\right]+\left(b_1+b_1^*\right)t}} \\
&= \frac{n_1\,\mathrm{e}^{\left(b_1^3-\mathrm{i}b_1^2\right)\beta_3' z+b_1 t}\,\mathrm{e}^{\diamond\beta_3' h_2\left(b_1^3-\mathrm{i}b_1^2\right)\int_{-\infty}^{z}W(\xi)\,\mathrm{d}\xi+b_1 t}}{1-\frac{\left(n_{11}^2+n_{12}^2\right)\gamma_0}{4\beta_2' b_{11}^2}\,\mathrm{e}^{\left(b_1^3-\mathrm{i}b_1^2+b_1^{*3}+\mathrm{i}b_1^{*2}\right)\beta_3' z+\left(b_1+b_1^*\right)t}\,\mathrm{e}^{\diamond\beta_3' h_2\left(b_1^3-\mathrm{i}b_1^2+b_1^{*3}+\mathrm{i}b_1^{*2}\right)\int_{-\infty}^{z}W(\xi)\,\mathrm{d}\xi}} \\
&= \frac{n_1\,\mathrm{e}^{\left(b_1^3-\mathrm{i}b_1^2\right)\beta_3' z+b_1 t}\,\mathrm{e}^{\diamond\beta_3' h_2\left(b_1^3-\mathrm{i}b_1^2\right)B(z)}}{1-\frac{\left(n_{11}^2+n_{12}^2\right)\gamma_0}{4\beta_2' b_{11}^2}\,\mathrm{e}^{\left(b_1^3-\mathrm{i}b_1^2+b_1^{*3}+\mathrm{i}b_1^{*2}\right)\beta_3' z+\left(b_1+b_1^*\right)t}\,\mathrm{e}^{\diamond\beta_3' h_2\left(b_1^3-\mathrm{i}b_1^2+b_1^{*3}+\mathrm{i}b_1^{*2}\right)B(z)}}.
\end{aligned} \tag{17}$$

Since $\mathrm{e}^{\diamond B(z)} = \mathrm{e}^{B(z)-\frac{1}{2}z^2}$ with $\diamond$ representing the Wick products and $B(z)$ representing the standard Brownian motion [24], the stochastic one-soliton solutions for (2) can be expressed as

$$U(z,t) = \frac{n_1\,\mathrm{e}^{\left(b_1^3-\mathrm{i}b_1^2\right)\beta_3' z+\left(b_1^3-\mathrm{i}b_1^2\right)\beta_3' h_2 B(z)-\frac{1}{2}\left(b_1^3-\mathrm{i}b_1^2\right)\beta_3' h_2 z^2+b_1 t}}{1-\frac{\left(n_{11}^2+n_{12}^2\right)\gamma_0}{4\beta_2' b_{11}^2}\,\mathrm{e}^{\left(b_1^3-\mathrm{i}b_1^2+b_1^{*3}+\mathrm{i}b_1^{*2}\right)\beta_3' z+\beta_3' h_2\left(b_1^3-\mathrm{i}b_1^2+b_1^{*3}+\mathrm{i}b_1^{*2}\right)\left[B(z)-\frac{1}{2}z^2\right]+\left(b_1+b_1^*\right)t}}. \tag{18}$$

Similarly, through the inverse Hermite transformation, Solutions (16) for (4) is transformed into

$$\begin{aligned}
U(z,t) &= \Big\{ m_1\,\mathrm{e}^{b_1 t+b_1^2(b_1-\mathrm{i}\alpha_1)\beta_3' z}\,\mathrm{e}^{\diamond b_1^2(b_1-\mathrm{i}\alpha_1)\beta_3' h_2 B(z)} \\
&\quad + m_2\,\mathrm{e}^{b_2 t+b_2^2(b_2-\mathrm{i}\alpha_1)\beta_3' z}\,\mathrm{e}^{\diamond b_2^2(b_2-\mathrm{i}\alpha_1)\beta_3' h_2 B(z)} \\
&\quad + m_3\,\mathrm{e}^{(b_2+2b_{11})t+\left[2b_{11}(b_{11}^2+2\alpha_1 b_{12}-3b_{12}^2)+b_2^2(b_2-\mathrm{i}\alpha_1)\right]\beta_3' z} \\
&\quad \cdot \mathrm{e}^{\diamond\left[2b_{11}(b_{11}^2+2\alpha_1 b_{12}-3b_{12}^2)+b_2^2(b_2-\mathrm{i}\alpha_1)\right]\beta_3' h_2 B(z)} \\
&\quad + m_4\,\mathrm{e}^{(b_1+2b_{21})t+\left[2b_{21}(b_{21}^2+2\alpha_1 b_{22}-3b_{22}^2)+b_1^2(b_1-\mathrm{i}\alpha_1)\right]\beta_3' z} \\
&\quad \cdot \mathrm{e}^{\diamond\left[2b_{21}(b_{21}^2+2\alpha_1 b_{22}-3b_{22}^2)+b_1^2(b_1-\mathrm{i}\alpha_1)\right]\beta_3' h_2 B(z)} \Big\} \Big/ \\
&\quad \Big\{ n_1\,\mathrm{e}^{2b_{11}t+2b_{11}(b_{11}^2+2\alpha_1 b_{12}-3b_{12}^2)\beta_3' z}\,\mathrm{e}^{\diamond 2b_{11}(b_{11}^2+2\alpha_1 b_{12}-3b_{12}^2)\beta_3' h_2 B(z)} \\
&\quad + n_2\,\mathrm{e}^{2b_{21}t+2b_{21}(b_{21}^2+2\alpha_1 b_{22}-3b_{22}^2)\beta_3' z}\,\mathrm{e}^{\diamond 2b_{21}(b_{21}^2+2\alpha_1 b_{22}-3b_{22}^2)\beta_3' h_2 B(z)}
\end{aligned}$$

$$
\begin{aligned}
&+ n_3\, \mathrm{e}^{(b_1+b_2^*)t+\left[b_1^2(b_1-\mathrm{i}\alpha_1)+b_2^{*2}(b_2^*+\mathrm{i}\alpha_1)\right]\beta_3' z}\, \mathrm{e}^{\diamond\left[b_1^2(b_1-\mathrm{i}\alpha_1)+b_2^{*2}(b_2^*+\mathrm{i}\alpha_1)\right]\beta_3' h_2 B(z)} \\
&+ n_4\, \mathrm{e}^{(b_2+b_1^*)t+\left[b_2^2(b_2-\mathrm{i}\alpha_1)+b_1^{*2}(b_1^*-\mathrm{i}\alpha_1)\right]\beta_3' z}\, \mathrm{e}^{\diamond\left[b_2^2(b_2-\mathrm{i}\alpha_1)+b_1^{*2}(b_1^*-\mathrm{i}\alpha_1)\right]\beta_3' h_2 B(z)} \\
&+ O_1\, \mathrm{e}^{2(b_{11}+b_{21})t+2\left[b_{11}^3+b_{21}^3+b_{11}(2\alpha_1-3b_{12})b_{12}+b_{21}(2\alpha_1-3b_{22})b_{22}\right]\beta_3' z} \\
&\cdot \mathrm{e}^{\diamond 2\left[b_{11}^3+b_{21}^3+b_{11}(2\alpha_1-3b_{12})b_{12}+b_{21}(2\alpha_1-3b_{22})b_{22}\right]\beta_3' h_2 B(z)}\Big\}\,. \qquad (19)
\end{aligned}
$$

The stochastic two-soliton solutions for (2) can be expressed as

$$
\begin{aligned}
U(z,t) = \Big\{ & m_1\, \mathrm{e}^{b_1 t+b_1^2(b_1-\mathrm{i}\alpha_1)\beta_3' z}\, \mathrm{e}^{b_1^2(b_1-\mathrm{i}\alpha_1)\beta_3' h_2\left[B(z)-\frac{1}{2}z^2\right]} \\
&+ m_2\, \mathrm{e}^{b_2 t+b_2^2(b_2-\mathrm{i}\alpha_1)\beta_3' z}\, \mathrm{e}^{b_2^2(b_2-\mathrm{i}\alpha_1)\beta_3' h_2\left[B(z)-\frac{1}{2}z^2\right]} \\
&+ m_3\, \mathrm{e}^{(b_2+2b_{11})t+\left[2b_{11}(b_{11}^2+2\alpha_1 b_{12}-3b_{12}^2)+b_2^2(b_2-\mathrm{i}\alpha_1)\right]\beta_3' z} \\
&\cdot \mathrm{e}^{\left[2b_{11}(b_{11}^2+2\alpha_1 b_{12}-3b_{12}^2)+b_2^2(b_2-\mathrm{i}\alpha_1)\right]\beta_1 h_2\left[B(z)-\frac{1}{2}z^2\right]} \\
&+ m_4\, \mathrm{e}^{(b_1+2b_{21})t+\left[2b_{21}(b_{21}^2+2\alpha_1 b_{22}-3b_{22}^2)+b_1^2(b_1-\mathrm{i}\alpha_1)\right]\beta_3' z} \\
&\cdot \mathrm{e}^{\left[2b_{21}(b_{21}^2+2\alpha_1 b_{22}-3b_{22}^2)+b_1^2(b_1-\mathrm{i}\alpha_1)\right]\beta_3' h_2\left[B(z)-\frac{1}{2}z^2\right]}\Big\}\Big/ \\
\Big\{ & n_1\, \mathrm{e}^{2b_{11}t+2b_{11}(b_{11}^2+2\alpha_1 b_{12}-3b_{12}^2)\beta_3' z}\, \mathrm{e}^{2b_{11}(b_{11}^2+2\alpha_1 b_{12}-3b_{12}^2)\beta_3' h_2\left[B(z)-\frac{1}{2}z^2\right]} \\
&+ n_2\, \mathrm{e}^{2b_{21}t+2b_{21}(b_{21}^2+2\alpha_1 b_{22}-3b_{22}^2)\beta_3' z}\, \mathrm{e}^{2b_{21}(b_{21}^2+2\alpha_1 b_{22}-3b_{22}^2)\beta_3' h_2\left[B(z)-\frac{1}{2}z^2\right]} \\
&+ n_3\, \mathrm{e}^{(b_1+b_2^*)t+\left[b_1^2(b_1-\mathrm{i}\alpha_1)+b_2^{*2}(b_2^*+\mathrm{i}\alpha_1)\right]\beta_3' z}\, \mathrm{e}^{\left[b_1^2(b_1-\mathrm{i}\alpha_1)+b_2^{*2}(b_2^*+\mathrm{i}\alpha_1)\right]\beta_3' h_2\left[B(z)-\frac{1}{2}z^2\right]} \\
&+ n_4\, \mathrm{e}^{(b_2+b_1^*)t+\left[b_2^2(b_2-\mathrm{i}\alpha_1)+b_1^{*2}(b_1^*-\mathrm{i}\alpha_1)\right]\beta_3' z}\, \mathrm{e}^{\left[b_2^2(b_2-\mathrm{i}\alpha_1)+b_1^{*2}(b_1^*-\mathrm{i}\alpha_1)\right]\beta_3' h_2\left[B(z)-\frac{1}{2}z^2\right]} \\
&+ O_1\, \mathrm{e}^{2(b_{11}+b_{21})t+2\left[b_{11}^3+b_{21}^3+b_{11}(2\alpha_1-3b_{12})b_{12}+b_{21}(2\alpha_1-3b_{22})b_{22}\right]\beta_3' z} \\
&\cdot \mathrm{e}^{2\left[b_{11}^3+b_{21}^3+b_{11}(2\alpha_1-3b_{12})b_{12}+b_{21}(2\alpha_1-3b_{22})b_{22}\right]\beta_3' h_2\left[B(z)-\frac{1}{2}z^2\right]}\Big\}\,. \qquad (20)
\end{aligned}
$$

3.2. Discussions for Solutions (18) and (20)

For the effects of the Gaussian white noise on the stochastic one soliton, solution (18) is rewritten as

$$
U(z,t) = A\,\mathrm{sech}\left[\eta_1 z+\eta_2 t+\eta_3 B(z)-\frac{1}{2}\eta_3 z^2+C\right] \cdot \mathrm{e}^{\mathrm{i}\eta_4 z+\mathrm{i}\eta_5 t+\mathrm{i}\eta_6 B(z)-\frac{\mathrm{i}}{2}\eta_6 z^2},
$$

with

$$
\begin{aligned}
&A = 2n_1 C\,,\quad \eta_1 = \left(b_{11}^3+2b_{11}b_{12}-3b_{11}b_{12}^2\right)\beta_3'\,, \\
&\eta_2 = b_{11}\,, \\
&\eta_3 = \left(b_{11}^3+2b_{11}b_{12}-3b_{11}b_{12}^2\right)\beta_3' h_2\,, \\
&\eta_4 = \left(-b_{11}^2+3b_{11}^2 b_{12}+b_{12}^2-b_{12}^3\right)\beta_3'\,, \\
&\eta_5 = b_{12}\,, \\
&\eta_6 = \left(-b_{11}^2+3b_{11}^2 b_{12}+b_{12}^2-b_{12}^3\right)\beta_3' h_2\,, \\
&C = \sqrt{\frac{-\beta_2' b_{11}^2}{4\left(n_{11}^2+n_{12}^2\right)\gamma_0}}\,,
\end{aligned}
$$

where, of the stochastic one soliton, A and C are the amplitude and initial position, η_2 and η_5 are respectively related to the pulsewidth and frequency, η_1 and

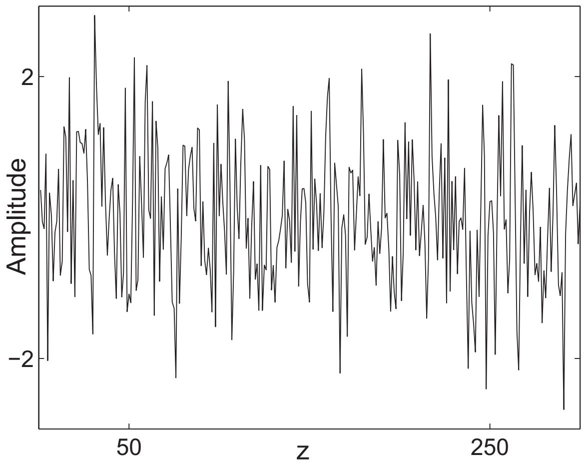

Fig. 1. Plot of the stochastic function $B(z)$ via stochastic number sequence.

η_3 are both related to the velocity, and η_4 and η_6 are related to the phase shifting. Therefore, the velocity and phase shifting of the stochastic one soliton are obtained as $\eta_1 + \eta_3 \dot{B}(z) - \eta_3 z$ and $\eta_4 z + \eta_6 B(z) - \frac{1}{2}\eta_6 z^2$. It shows that the Gaussian white noise affects the velocity and phase shifting of the stochastic one soliton, while the amplitude, energy, and shape of the stochastic one soliton are unaffected by the Gaussian white noise. The average value of the velocity can be obtained as $\eta_1 - \eta_3 z$. It means that the acceleration is ex-

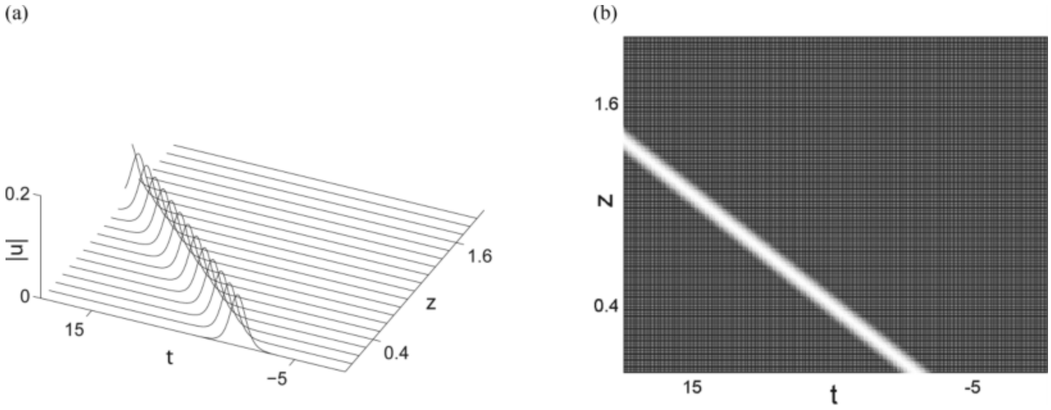

Fig. 2. (a) Stochastic one soliton via Solutions (18). (b) Energy of the stochastic one soliton via Solutions (18). Parameters are: $n_1 = 1 - \mathrm{i}$, $b_1 = 1 - 2\mathrm{i}$, $\alpha = -20$, $\beta_3' = 1$, and $h_2 = 0$.

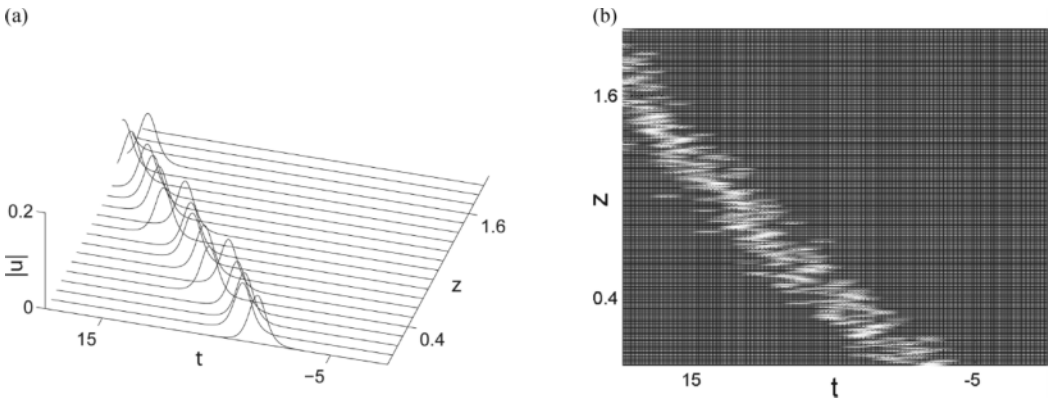

Fig. 3. (a) Stochastic one soliton via Solutions (18). (b) Energy of the stochastic one soliton via Solutions (18). Parameters are: $n_1 = 1 - \mathrm{i}$, $b_1 = 1 - 2\mathrm{i}$, $\alpha = -20$, $\beta_3' = 1$, and $h_2 = 0.2$.

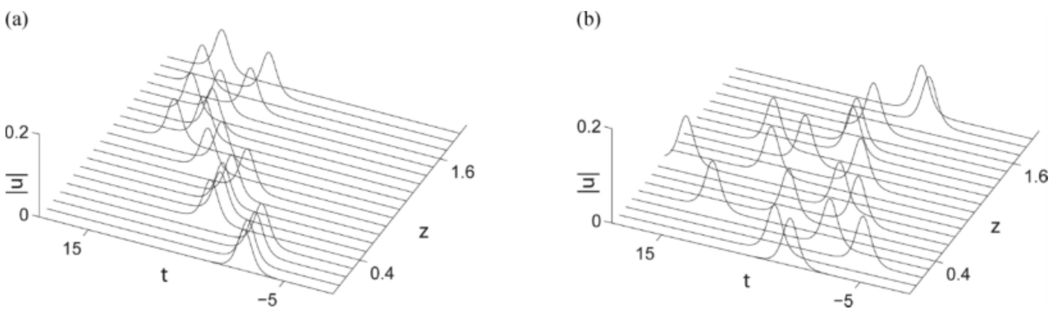

Fig. 4. (a) Stochastic one soliton via Solutions (18) with $h_2 = 0.5$. (b) Stochastic one soliton via Solutions (18) with $h_2 = 0.8$. Parameters are: $n_1 = 1 - \mathrm{i}$, $b_1 = 1 - 2\mathrm{i}$, $\alpha = -20$, and $\beta_3' = 1$.

istent in the stochastic one soliton. When $\eta_3 > 0$, the value of acceleration is negative. The velocity direction of the stochastic one soliton is inverted.

The stochastic function $B(z)$ via stochastic number sequence is generated by the stochastic algorithm simulator in MATLAB, as shown in Figure 1. With such stochastic function, compared with the soliton without stochastic perturbation, effects of the Gaussian white noise on the solitons are discussed. In Figures 2 and 3, the energy and shape of the stochastic one soliton are not affected by the Gaussian white noise and are the same as the ones of one soliton without the stochastic perturbation, but the velocity of the stochastic one soliton is affected by the Gaussian white noise, and the effect is related to the intensity of the Gaussian white noise. When the intensity h_2 of the Gaussian white noise is zero, the velocity and phase shift of the one soliton are only related to β_1, and the energy and velocity of the one soliton keep unchanged, as seen in Figure 2. When the intensity h_2 of the Gaussian white noise is not zero, velocity change of the stochastic one soliton is positively correlated with the intensity h_2, as seen in Figures 3 and 4.

Similarly, effects of the Gaussian white noise on the dynamics of the stochastic two solitons are discussed. Solutions (20) are rewritten as

$$U(z,t) = \frac{1}{Q}\Big[\chi_1 \operatorname{sech}(a_2 + \ln\sqrt{n_2} + \mathrm{i}\psi_1)\,\mathrm{e}^{\mathrm{i}\phi_1} + \chi_2 \operatorname{sech}(a_1 + \ln\sqrt{n_1} + \mathrm{i}\psi_2)\,\mathrm{e}^{\mathrm{i}\phi_2}\Big] \tag{21}$$

with

$$\begin{aligned}
Q &= \kappa_1\cosh(a_1+a_2+\kappa_1) + \kappa_2\cosh\left(a_1-a_2+\ln\sqrt{\frac{n_1}{n_2}}\right) + \kappa_3\cos(a_3-a_4) + \kappa_4\cos\left(a_3-a_4+\frac{\pi}{2}\right),\\
a_1 &= b_{11}t + b_{11}\left(b_{11}^2 - 3b_{12}^2 + 2b_{12}\alpha_1\right)\cdot\left[z + h_2B(z) - h_2\frac{z^2}{2}\right],\\
a_2 &= b_{21}t + b_{21}\left(b_{21}^2 - 3b_{22}^2 + 2b_{22}\alpha_1\right)\cdot\left[z + h_2B(z) - h_2\frac{z^2}{2}\right],\\
a_3 &= b_{12}t + \left[3b_{11}^2b_{12} - b_{12}^3 + \left(b_{12}^2 - b_{11}^2\right)\alpha_1\right]\cdot\left[z + h_2B(z) - h_2\frac{z^2}{2}\right],\\
a_4 &= b_{22}t + \left[3b_{21}^2b_{22} - b_{22}^3 + \left(b_{22}^2 - b_{21}^2\right)\alpha_1\right]\cdot\left[z + h_2B(z) - h_2\frac{z^2}{2}\right],\\
\chi_1 &= \sqrt{\frac{n_2}{\left(m_{11}^2+m_{12}^2\right)\left(w_1^2+w_2^2\right)}},\\
\chi_2 &= \sqrt{\frac{n_1}{\left(m_{21}^2+m_{22}^2\right)\left(w_1^2+w_2^2\right)}},\\
\psi_1 &= \psi_2 = \frac{1}{2}\arcsin\frac{w_2}{\sqrt{w_1^2+w_2^2}} + \frac{\pi}{2},\\
\kappa_1 &= \frac{(b_{11}-b_{21})^2 + (b_{12}-b_{22})^2}{(b_{11}+b_{21})^2 + (b_{12}-b_{22})^2}\sqrt{n_1n_2},\\
\kappa_2 &= \sqrt{n_1n_2},
\end{aligned}$$

$$\kappa_3 = -\alpha_2\frac{\left(b_{21}^2-b_{22}^2\right)(m_{11}m_{21}+m_{12}m_{22}) + 2(m_{11}m_{22}-m_{12}m_{21})b_{21}b_{22} + 2b_{12}(b_{22}m_{11}m_{21} + b_{21}m_{12}m_{21} - b_{21}m_{11}m_{22} + b_{22}m_{12}m_{22}) + \left(b_{11}^2-b_{12}^2\right)(m_{11}m_{21}+m_{12}m_{22})}{2\left[(b_{11}+b_{21})^2 + (b_{12}-b_{22})^2\right]^2},$$

$$\kappa_4 = \alpha_2\frac{-2b_{21}b_{22}m_{11}m_{21} - b_{21}^2m_{12}m_{21} + b_{22}^2m_{12}m_{21} + b_{21}^2m_{11}m_{22} - b_{22}^2m_{11}m_{22} + b_{12}^2(m_{12}m_{21}-m_{11}m_{22}) + b_{11}^2(-m_{12}m_{21}+m_{11}m_{22}) - 2b_{21}b_{22}m_{12}m_{22} + 2b_{12}(b_{21}m_{11}m_{21} - b_{22}m_{12}m_{21} + b_{22}m_{11}m_{22} + b_{21}m_{12}m_{22})}{2\left[(b_{11}+b_{21})^2 + (b_{12}-b_{22})^2\right]^2}$$

$$\frac{+2b_{11}\left[(b_{12}-b_{22})(m_{11}m_{21}+m_{12}m_{22})-b_{21}m_{12}m_{21}+b_{21}m_{11}m_{22}\right]}{},$$

$$\phi_1 = a_1 + \arcsin\frac{m_{12}}{\sqrt{(m_{11}^2+m_{12}^2)}} - \frac{1}{2}\arcsin\frac{w_2}{\sqrt{w_1^2+w_2^2}},$$

$$\phi_2 = a_2 + \arcsin\frac{m_{12}}{\sqrt{(m_{11}^2+m_{12}^2)}} - \frac{1}{2}\arcsin\frac{w_2}{\sqrt{w_1^2+w_2^2}},$$

$$w_1 = \frac{b_{11}^4+b_{12}^4+b_{21}^4-4b_{12}^3b_{22}-6b_{21}^2b_{22}^2+b_{22}^4-6b_{12}^2\left(b_{21}^2-b_{22}^2\right)+4b_{12}\left(3b_{21}^2b_{22}-b_{22}^3\right)}{\left[(b_{11}+b_{21})^2+(b_{12}-b_{22})^2\right]^2}$$

$$\frac{+2b_{11}^2\left(b_{12}^2-b_{21}^2-2b_{12}b_{22}+b_{22}^2\right)}{},$$

$$w_2 = \frac{4b_{21}(b_{12}-b_{22})\left(b_{11}^2+b_{12}^2-b_{21}^2-2b_{12}b_{22}+b_{22}^2\right)}{\left[(b_{11}+b_{21})^2+(b_{12}-b_{22})^2\right]^2},$$

where the two hyperbolic secant functions in solutions (21) respectively represent the stochastic two solitons, and Q reflects the interaction of the stochastic two solitons. Because of the nonlinear form of Q, the interaction of the two solitons is nonlinear. The interaction of two optical solitons can be classified into the different statuses, according to the different initial optical pulses [39]. When the two initial optical pulses have the equal initial phases, interaction of the stochastic two solitons in a bound state has the property of the period oscillation. Period of the oscillation is related to the cosine functions of Q. For the effects of Gaussian white noise, the period changes randomly with the propagation distance z. The function form of the period can be obtained as

$$P(z) = \left[3b_{11}^2b_{12}-b_{12}^3+3b_{21}^2b_{22}-b_{22}^3 + \left(b_{12}^2-b_{11}^2+b_{22}^2-b_{21}^2\right)\alpha_1\right]\left[1+h_2\dot{B}(z)-h_2\frac{z}{2}\right].$$

When the intensity of the Gaussian white noise is zero, the periodic oscillation of the optical solitons in the bound state is stable, as shown in Figure 5a. When the intensity of the Gaussian white noise is not zero, the oscillation of the stochastic two solitons is also stochastic, as shown in Figure 5b.

When the two initial optical pulses have different initial phases, the two optical solitons form the unbound state. When the intensity of the Gaussian white noise is zero, the velocities of the two solitons have

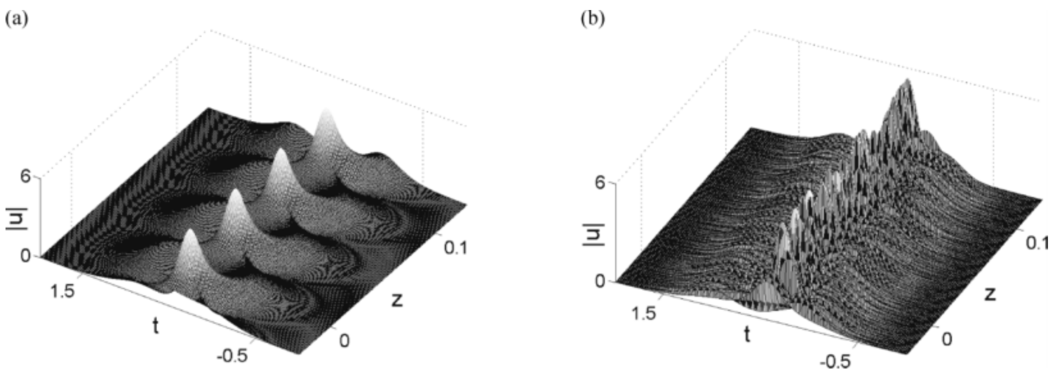

Fig. 5. (a) Two solitons via Solutions (19) with $h_2 = 0$. (b) Two stochastic solitons via Solutions (19) with $h_2 = 0.2$. Parameters are: $n_1 = 1+\mathrm{i}$, $n_2 = 1+\mathrm{i}$, $b_1 = 1+\mathrm{i}$, $b_2 = 2\sqrt{6}+4\mathrm{i}$, $\alpha_1 = -20$, $\alpha_2 = 3$, and $\beta_3' = 1$.

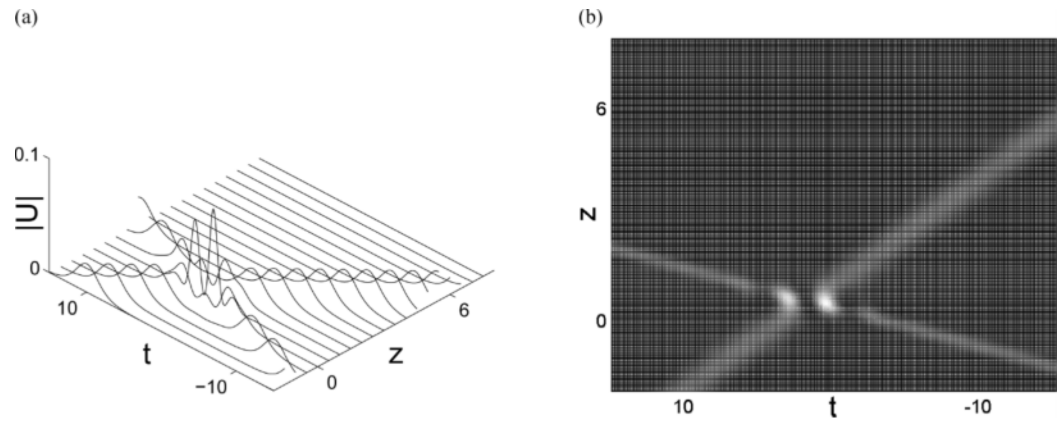

Fig. 6. (a) Stochastic two solitons via Solutions (19). (b) Energy of the stochastic two solitons via Solutions (19). Parameters are: $n_1 = -1 + \mathrm{i}$, $n_2 = -1 + \mathrm{i}$, $b_1 = 0.5 - \mathrm{i}$, $b_2 = -0.5 + \mathrm{i}$, $\alpha_1 = -20$, $\alpha_2 = 3$, $\beta_3' = 1$, and $h_2 = 0$.

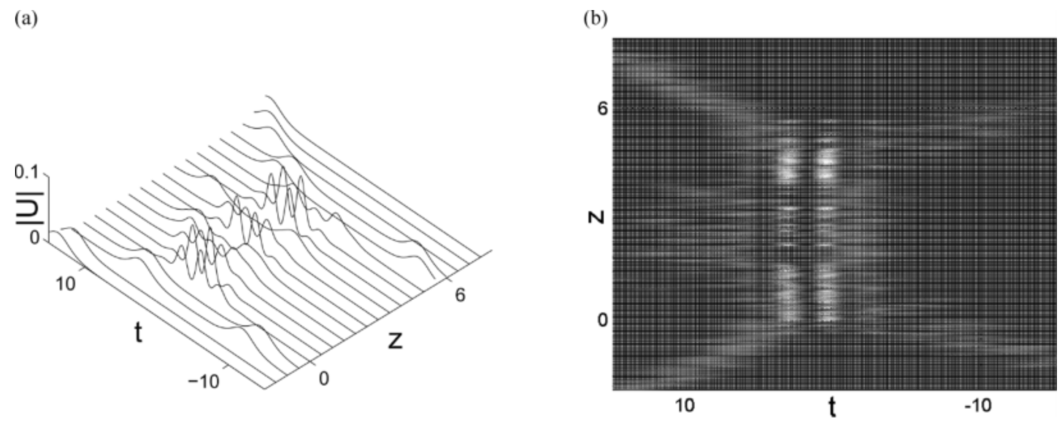

Fig. 7. (a) Stochastic two solitons via Solutions (19). (b) Energy of the stochastic two solitons via Solutions (19). Parameters are: $n_1 = -1 + \mathrm{i}$, $n_2 = -1 + \mathrm{i}$, $b_1 = 0.5 - \mathrm{i}$, $b_2 = -0.5 + \mathrm{i}$, $\alpha_1 = -20$, $\alpha_2 = 3$, $\beta_3' = 1$, and $h_2 = 0.4$.

opposite directions, as shown in Figure 6. When the intensity of the Gaussian white noise is not zero, the velocity functions of two stochastic solitons can be obtained as

$$\begin{aligned} V_1(z) &= b_{11}\left(b_{11}^2 - 3b_{12}^2 + 2b_{12}\alpha_1\right) \\ &\quad \cdot \left[1 + h_2\dot{B}(z) - h_2\frac{z}{2}\right], \\ V_2(z) &= b_{21}\left(b_{21}^2 - 3b_{22}^2 + 2b_{22}\alpha_1\right) \\ &\quad \cdot \left[1 + h_2\dot{B}(z) - h_2\frac{z}{2}\right]. \end{aligned}$$

The velocities $V_1(z)$ and $V_2(z)$ change randomly because of the effects of Gaussian white noise, and the velocity directions of stochastic two solitons are inverted along the soliton propagation. After the interaction of stochastic two solitons, the interaction happens again because of the inverted velocity directions, as shown in Figure 7, while the region of the interaction changes with the increase of h_2, and the distance between the two solitons changes randomly. When the intensity h_2 of the Gaussian noise increases, the region of interaction enlarges, as shown in Figures 6 and 7.

4. Stability Analysis of the Stochastic Soliton Solutions

For NLS equations in nonlinear optical fibers, there are two numerical-simulation methods for the pulse-propagation problems, which are the finite difference method [9] and the split-step Fourier method [40–42].

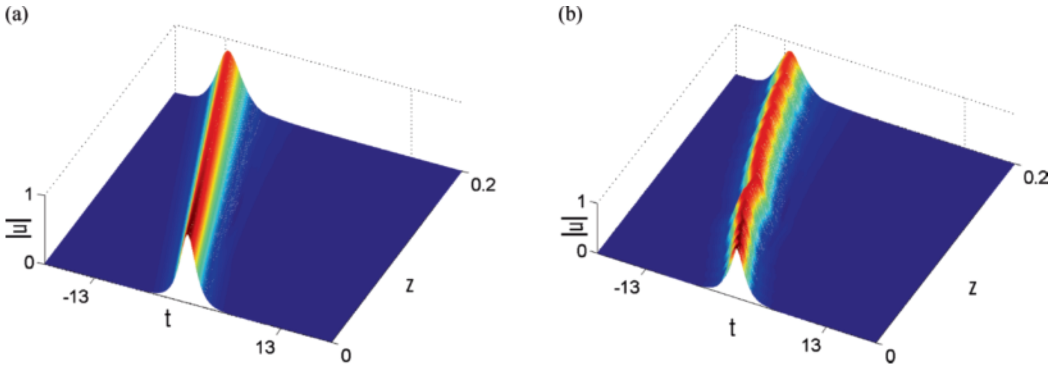

Fig. 8 (colour online). (a) Numerical simulation on the one soliton of (2) without Gaussian white noise. (b) The case of (a) with Gaussian white noise.

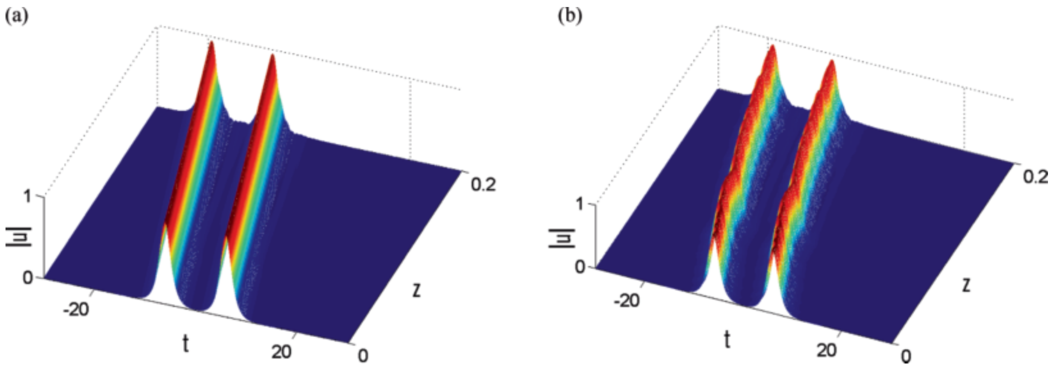

Fig. 9 (colour online). (a) Numerical simulation on the two solitons of (2) without the Gaussian white noise. (b) The case of (a) with the Gaussian white noise.

In this paper, we will make use of the split-step Fourier method in the MATLAB software environment to get the stability of stochastic soliton solutions for (2). For the stability analysis, dispersion and nonlinearity of (2) are added to the Gaussian white noise which is generated by the stochastic algorithm simulator. In the region of t, 1024 points are simulated. The 2000 iterative times along the z-axis are taken with the step size of 0.0001. The one- and two-soliton solutions are tested. Figure 8a shows the one soliton for (2) without the Gaussian white noise, and the one for (2) with the Gaussian white noise is shown in Figure 8b. Comparing Figure 8a and Figure 8b, we find that the solitons with the Gaussian white noise show the smaller changes compared with the solitons without the Gaussian white noise along the propagation. Similarly, the stability of two solitons is discussed, as seen in Figure 9. With the numerical simulation, one- and two-soliton solutions for (2) with the Gaussian white noise can propagate stably.

5. Conclusions

Equation (2), the HNLS equation driven by the Gaussian white noise, describes the wave propagation in the optical fiber with stochastic dispersion and nonlinearity. In this paper, the stochastic solitons for (2) have been investigated analytically. With symbolic computation and white noise functional approach, the stochastic one- and two-soliton solutions (18) and (20) have been obtained. For the stochastic one soliton, the velocity and phase shift change randomly because of the effects of Gaussian white noise, and the ranges of the changes increase with the increase in the intensity h_2 of Gaussian white noise. The energy and shape keep unchanged along the soliton propagation, as shown in Figures 2, 3, and 4, but the direction of velocity for the stochastic one soliton is inverted. For the stochastic two solitons, the interactions in the bound and unbound states have been discussed. The periodic oscillation of the two solitons in the bound state is broken because of the existence of Gaussian white noise, and the oscillation of the stochastic two solitons forms randomly, as shown in Figure 5. In the unbound state, interaction of the stochastic two solitons happens twice because of the Gaussian white noise, as shown in Figure 7. With the increase in the intensity of Gaussian white noise, the region of the interaction enlarges, as shown in Figures 6 and 7. Through the numerical simulation performed via the split-step Fourier method, the stochastic one- and two-soliton solutions for (2) have been shown to be stable in Figures 8 and 9.

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