Group-Theoretic Approach to Boundary Layer Equations of an Oldroy-B Fluid

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Boundary layer equations are derived for the first time for an Oldroy-B fluid. The symmetry analysis of the equations is performed using Lie Group theory and the partial differential system is transferred to an ordinary differential system via symmetries. Resulting equations are numerically solved for the case of the stretching sheet problem. Effects of non-Newtonian parameters on the solutions are discussed.

Key words: Non-Newtonian Fluid; Oldroy-B Fluid; Boundary Layer Theory; Lie Group Analysis; Stretching Sheet Problem.

1. Introduction

The Newtonian fluid model predicts a linear relationship between the stress tensor and the velocity gradient. Stress constitutive relations of some real fluids however do not obey this linear relationship and many different models which are classified as non-Newtonian fluids were proposed to explain the complex behaviour between stress and velocity gradient. Usually, the stress constitutive relations of such models inherit complexities which lead to highly nonlinear equations of motion with many terms. To simplify the extremely complex equations, one alternative is to use the boundary layer theory which is known to effectively reduce the complexity of Navier–Stokes equations and reduce drastically the computational time. Since there are many non-Newtonian models and new models are being proposed continuously, the boundary layer theory for each proposed model also appear in the literature. Some of the example boundary layer models corresponding to different non-Newtonian fluids were proposed and solved already [1 – 23].

In this work, boundary layer equations are systematically developed for the Oldroy-B fluid, a well known non-Newtonian fluid model which combines the effects of relaxation and retardation observed in many real fluids. A complete symmetry analysis of the boundary layer equations is presented. Using one of the symmetries, the partial differential system is transformed into an ordinary differential system. Stretching sheet boundary conditions are taken in the analysis. The resulting ordinary differential system is numerically solved by a finite difference algorithm. The effect of non-Newtonian parameters on the velocity profiles are shown in the graphs.

Some of the recent works on Oldroy-B fluids are as follows: Bhatnagar et al. [16] investigated the flow of an Oldroy-B fluid due to a stretching sheet in the presence of a free stream velocity and found that their numerical solutions agree well with their perturbation solutions. Hayat et al. [24] obtained an exact solution for magnetohydrodynamic flow over an infinite oscillatory plate with the entire system rotating about an axis normal to the plate. Fetecau and Kannan [25] studied the one-dimensional unsteady flow of an Oldroy-B fluid induced by the motion of a flat plate. Haroun [26] considered the peristaltic flow of an Oldroy-B fluid and presented an approximate solution using perturbation theory. Hayat et al. [27] examined the effect of Hall current on the rotating flow of an Oldroy-B fluid in a porous medium taking into consideration the modified Darcy law. To the best of authors' knowledge, boundary layer equations in terms of velocity components and symmetry reductions of Oldroy-B fluids do not exist in the literature.

2. Equations of Motion and Boundary Layer Equations

The Cauchy stress tensor for an Oldroy-B fluid is

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \tag{1}$$

where the constitutive relation for extra stress tensor is

$$\mathbf{S} + \lambda_1 \frac{D\mathbf{S}}{Dt} = \mu\left[\mathbf{A}_1 + \lambda_2 \frac{D\mathbf{A}_1}{Dt}\right], \tag{2}$$

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^{\mathrm{T}},\ \ \mathbf{L} = \operatorname{grad}\mathbf{V}, \quad \frac{D\mathbf{S}}{Dt} = \frac{\mathrm{d}\mathbf{S}}{\mathrm{d}t} - \mathbf{LS} - \mathbf{SL}^{\mathrm{T}}. \tag{3}$$

$\mathbf{V}$ is the velocity vector, $\mathbf{A}_1$ is the first Rivlin-Ericksen tensor, and μ is the viscosity, λ_1 and λ_2 are relaxation and retardation time, respectively. Steady-state equations of motion in two dimensions in the absence of body force including mass conservation can be written as

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0, \tag{4}$$

$$\rho\left(u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right) = -\frac{\partial p^*}{\partial x^*} + \frac{\partial S_{xx}}{\partial x^*} + \frac{\partial S_{xy}}{\partial y^*}, \tag{5}$$

$$\rho\left(u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*}\right) = -\frac{\partial p^*}{\partial y^*} + \frac{\partial S_{xy}}{\partial x^*} + \frac{\partial S_{yy}}{\partial y^*}, \tag{6}$$

where x^* is the spatial coordinate along the surface, y^* is vertical to it, u^* and v^* are the velocity components in the x^* and y^* coordinates. The fluid is assumed to be incompressible. Using (1)–(3) in the equations of motion, (4)–(6) one finally has

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0, \tag{7}$$

$$\begin{aligned}
&\rho\left(u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right) + \rho\lambda_1\left[u^*\frac{\partial}{\partial x^*}\left(u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right) + v^*\frac{\partial}{\partial y^*}\left(u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right)\right.\\
&\left.-\frac{\partial u^*}{\partial x^*}\left(u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right) - \frac{\partial u^*}{\partial y^*}\left(u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*}\right)\right]\\
&+\frac{\partial p^*}{\partial x^*} + \lambda_1\left[u^*\frac{\partial^2 p^*}{\partial x^{*2}} + v^*\frac{\partial^2 p^*}{\partial x^*\partial y^*} - \frac{\partial u^*}{\partial x^*}\frac{\partial p^*}{\partial x^*} - \frac{\partial u^*}{\partial y^*}\frac{\partial p^*}{\partial y^*}\right] = \mu\left[2\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}} + \frac{\partial^2 v^*}{\partial x^*\partial y^*}\right.\\
&+\lambda_2\frac{\partial}{\partial x^*}\left(2u^*\frac{\partial^2 u^*}{\partial x^{*2}} + 2v^*\frac{\partial^2 u^*}{\partial x^*\partial y^*} - 4\left(\frac{\partial u^*}{\partial x^*}\right)^2 - 2\frac{\partial u^*}{\partial y^*}\left(\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*}\right)\right) + \lambda_2\frac{\partial}{\partial y^*}\left(u^*\frac{\partial^2 u^*}{\partial x^*\partial y^*}\right.\\
&\left.\left.+v^*\frac{\partial^2 u^*}{\partial y^{*2}} + u^*\frac{\partial^2 v^*}{\partial x^{*2}} + v^*\frac{\partial^2 v^*}{\partial x^*\partial y^*} - 2\left(\frac{\partial u^*}{\partial y^*}\frac{\partial v^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*}\frac{\partial u^*}{\partial x^*}\right)\right)\right],
\end{aligned} \tag{8}$$

$$\begin{aligned}
&\rho\left(u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*}\right) + \rho\lambda_1\left[u^*\frac{\partial}{\partial x^*}\left(u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*}\right) + v^*\frac{\partial}{\partial y^*}\left(u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*}\right)\right.\\
&\left.-\frac{\partial v^*}{\partial x^*}\left(u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right) - \frac{\partial v^*}{\partial y^*}\left(u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*}\right)\right]\\
&+\frac{\partial p^*}{\partial y^*} + \lambda_1\left[u^*\frac{\partial^2 p^*}{\partial x^*\partial y^*} + v^*\frac{\partial^2 p^*}{\partial y^{*2}} - \frac{\partial v^*}{\partial x^*}\frac{\partial p^*}{\partial x^*} - \frac{\partial v^*}{\partial y^*}\frac{\partial p^*}{\partial y^*}\right] = \mu\left[\frac{\partial^2 u^*}{\partial x^*\partial y^*} + \frac{\partial^2 v^*}{\partial x^{*2}} + 2\frac{\partial^2 v^*}{\partial y^{*2}}\right.\\
&+\lambda_2\frac{\partial}{\partial x^*}\left(u^*\frac{\partial^2 u^*}{\partial x^*\partial y^*} + v^*\frac{\partial^2 u^*}{\partial y^{*2}} + u^*\frac{\partial^2 v^*}{\partial x^{*2}} + v^*\frac{\partial^2 v^*}{\partial x^*\partial y^*} - 2\left(\frac{\partial u^*}{\partial y^*}\frac{\partial v^*}{\partial y^*} + \frac{\partial u^*}{\partial x^*}\frac{\partial v^*}{\partial x^*}\right)\right)\\
&\left.+\lambda_2\frac{\partial}{\partial y^*}\left(2u^*\frac{\partial^2 v^*}{\partial x^*\partial y^*} + 2v^*\frac{\partial^2 v^*}{\partial y^{*2}} - 4\left(\frac{\partial v^*}{\partial y^*}\right)^2 - 2\frac{\partial v^*}{\partial x^*}\left(\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*}\right)\right)\right].
\end{aligned} \tag{9}$$

The usual boundary layer assumptions are made, i.e. $x^* \sim \mathrm{O}(1)$, $y^* \sim \mathrm{O}(\delta)$, $u^* \sim \mathrm{O}(1)$, $v^* \sim \mathrm{O}(\delta)$, and $p^* \sim \mathrm{O}(1)$. The highest-order terms are retained, and the momentum equations become

$$\begin{aligned}
&\rho\left(u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right) + \rho\lambda_1\left[u^{*2}\frac{\partial^2 u^*}{\partial x^{*2}} + 2u^*v^*\frac{\partial^2 u^*}{\partial x^*\partial y^*} + v^{*2}\frac{\partial^2 u^*}{\partial y^{*2}}\right] = -\frac{\partial p^*}{\partial x^*} + \mu\left[\frac{\partial^2 u^*}{\partial y^{*2}} + \lambda_2\left(u^*\frac{\partial^3 u^*}{\partial x^*\partial y^{*2}}\right.\right.\\
&\left.\left.+v^*\frac{\partial^3 u^*}{\partial y^{*3}} - \frac{\partial u^*}{\partial y^*}\frac{\partial^2 u^*}{\partial x^*\partial y^*} - \frac{\partial v^*}{\partial y^*}\frac{\partial^2 u^*}{\partial y^{*2}}\right)\right]
\end{aligned} \tag{10}$$

$$\frac{\partial p^*}{\partial y^*} = 0 \tag{11}$$

from which dependence of pressure on y^* is eliminated. In the calculations, $\lambda_1 \sim \mathrm{O}(1)$, $\lambda_2 \sim \mathrm{O}(1)$, and

$\mu \sim \mathrm{O}(\delta^2)$ are assumed where δ is the boundary layer thickness. Dimensionless variables and parameters are defined as follows:

$$x=\frac{x^*}{L}\,,\quad y=\frac{y^*}{\delta}\,,\quad u=\frac{u^*}{V}\,,\quad v=\frac{v^*L}{V\delta}\,,\quad p=\frac{p^*}{\rho V^2}\,,$$
$$\varepsilon=\frac{\mu L}{\rho V\delta^2}\,,\quad \varepsilon_1=\frac{\lambda_1 V}{L}\,,\quad \varepsilon_2=\frac{\lambda_2 V}{L}\,, \tag{12}$$

where L is a characteristic length, and V is a reference velocity. Expressing the pressure in terms of outer velocity, the final dimensionless boundary layer equations including the mass conservation become

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,, \tag{13}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\varepsilon_1\left[u^2\frac{\partial^2 u}{\partial x^2}+2uv\frac{\partial^2 u}{\partial x\partial y}+v^2\frac{\partial^2 u}{\partial y^2}\right]= U\frac{\partial U}{\partial x}+\varepsilon\left[\frac{\partial^2 u}{\partial y^2}+\varepsilon_2\left(u\frac{\partial^3 u}{\partial x\partial y^2}+v\frac{\partial^3 u}{\partial y^3}-\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y^2}\right)\right]\,, \tag{14}$$

where U is the usual outer dimensionless velocity substituted for pressure gradient. $\varepsilon = 1/\mathrm{Re}$, ε_1, and ε_2 can be defined as the dimensionless relaxation and retardation parameters. For $\varepsilon_1 = 0$, the equations reduce to those of a second-grade fluid, and for $\varepsilon_2 = 0$, they reduce to those of an upper convected Maxwell fluid. For $\varepsilon_1 = 0$ and $\varepsilon_2 = 0$, the fluid is Newtonian, and the dimensional boundary layer equations can be found in [28] for example.

3. Lie Group Theory and Symmetry Analysis

The Lie Group theory is employed in search of symmetries of the equations. Details of the theory can be found in Bluman and Kumei [29] and Stephani [30]. The infinitesimal generator for the problem is

$$X=\xi_1(x,y,u,v)\frac{\partial}{\partial x}+\xi_2(x,y,u,v)\frac{\partial}{\partial y}+\eta_1(x,y,u,v)\frac{\partial}{\partial u}+\eta_2(x,y,u,v)\frac{\partial}{\partial v}\,. \tag{15}$$

A straightforward calculation yields

$$\xi_1=ax+b\,,\quad \xi_2=cx+d\,,\quad \eta_1=au\,,\quad \eta_2=cu\,. \tag{16}$$

The classifying relation for the outer velocity is

$$(ax+b)\frac{\mathrm{d}}{\mathrm{d}x}(UU')-a(UU')=0\,. \tag{17}$$

Some of the symbolic packages developed to calculate symmetries fail to produce the above results due to the arbitrary outer velocity function and some others require user intervention during the calculation process. There are four finite parameter Lie Group symmetries represented by parameters a, b, c, and d. Parameters b and d represent translational symmetries in x and y coordinates, respectively. Parameter a is a restricted scaling symmetry. There is an additional finite parameter Lie Group symmetry represented by parameter c. This type of symmetry is uncommon in boundary layers of non-Newtonian fluids. Usually fluid problems inherit some type of scaling symmetry due to the Buckingham Pi theorem. For the relationship of the Buckingham Pi theorem with the scaling symmetries, see Bluman and Kumei [29].

4. Symmetry Reductions for the Stretching Sheet Problem

For the stretching sheet problem, the outer velocity U is zero, and the boundary conditions can be written as

$$u(x,0)=\alpha x\,,\quad v(x,0)=0\,,\qquad u(x,\infty)=0\,,\quad \frac{\partial u}{\partial y}(x,\infty)=0\,. \tag{18}$$

Usually boundary conditions put much restriction on the symmetries which may lead to removal of all the symmetries. In our case however, some of the symmetries remain stable after imposing the boundary conditions. For nonlinear equations, the generators should be applied to the boundaries and boundary conditions also [29]. Applying the generator to the boundary $y = 0$ yields $c = 0$ and $d = 0$. Applying the generator to the first boundary condition yields $b = 0$, and the remaining boundary conditions do not impose further restrictions and hence only one of the symmetries survive after the application of boundary conditions

$$\xi_1=ax\,,\quad \xi_2=0\,,\quad \eta_1=au\,,\quad \eta_2=0\,. \tag{19}$$

Note that since the outer velocity $U = 0$, the classifying relation (17) is satisfied identically.

From (19), the associated equations which define similarity variable and functions are

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}u}{u} = \frac{\mathrm{d}v}{0} \,. \tag{20}$$

Solving the system yields the similarity functions

$$u = xf(y)\,, \quad v = g(y)\,, \tag{21}$$

where y is the similarity variable. Substituting all into the boundary layer equations yields the ordinary differential system

$$f + g' = 0\,, \tag{22}$$

$$f^2 + gf' + \varepsilon_1(2gff' + g^2 f'') = \varepsilon[f'' + \varepsilon_2(ff'' + gf''' - f'^2 - g'f'')]\,. \tag{23}$$

The boundary conditions also transform as follows:

$$f(0) = \alpha\,, \; g(0) = 0\,, \quad f(\infty) = 0\,, \; f'(\infty) = 0\,. \tag{24}$$

5. Numerical Results

Equations (22) and (23) are numerically integrated using a finite difference scheme subject to the boundary conditions (24). In Figure 1, f function and in Figure 2, g function related to the x and y components of the velocities are drawn for various dimensionless relaxation ε_1 parameters. Boundary layer becomes thicker for the lower relaxation parameters. Decrease of the relaxation parameter implies a transformation of Oldroy-B character of the fluid into a second-grade

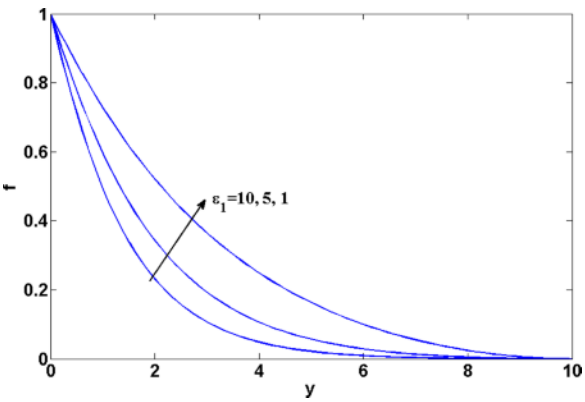

Fig. 1 (colour online). Effect of relaxation parameter ε_1 on the similarity function f related to the x component of velocity ($\varepsilon_2 = 1$, $\varepsilon = 10$).

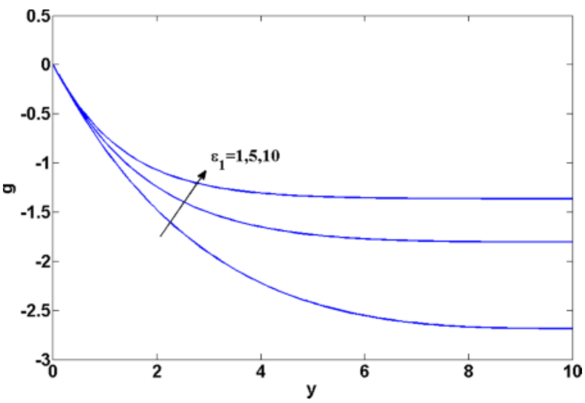

Fig. 2 (colour online). Effect of relaxation parameter ε_1 on the similarity function g related to the y component of velocity ($\varepsilon_2 = 1$, $\varepsilon = 10$).

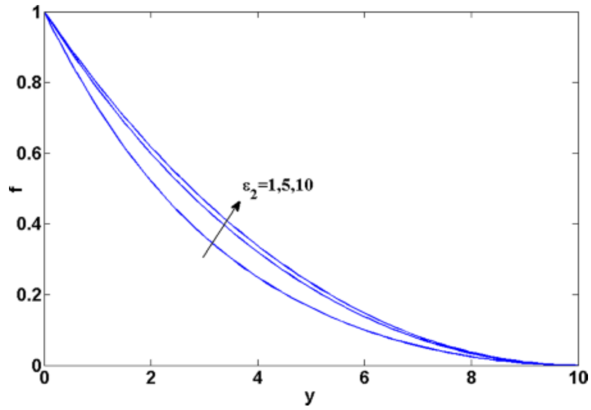

Fig. 3 (colour online). Effect of retardation parameter ε_2 on the similarity function f related to the x component of velocity ($\varepsilon_1 = 1$, $\varepsilon = 10$).

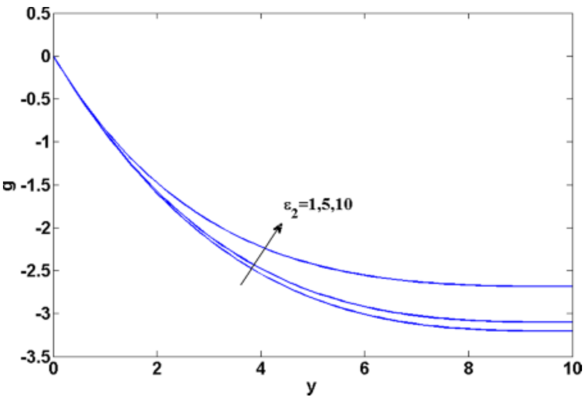

Fig. 4 (colour online). Effect of retardation parameter ε_2 on the similarity function g related to the y component of velocity ($\varepsilon_1 = 1$, $\varepsilon = 10$).

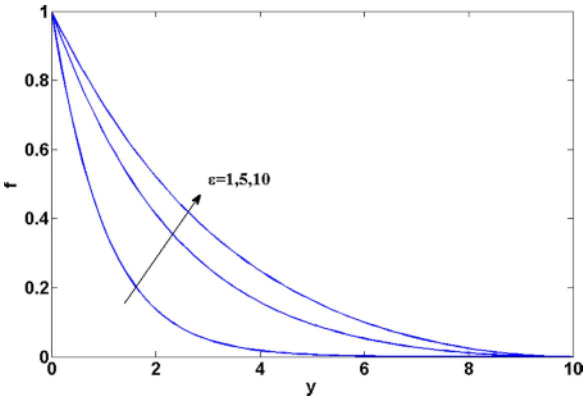

Fig. 5 (colour online). Effect of viscosity parameter ε on the similarity function f related to the x component of velocity ($\varepsilon_1 = 1$, $\varepsilon_2 = 1$).

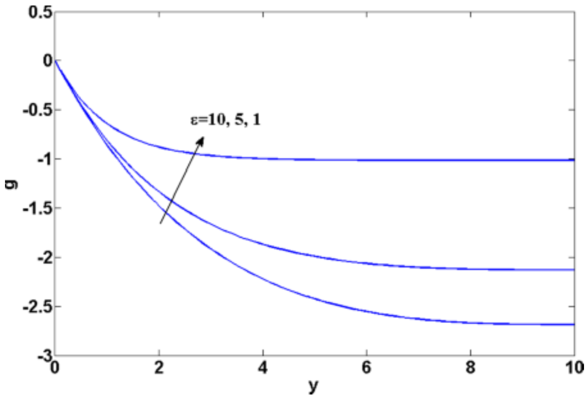

Fig. 6 (colour online). Effect of viscosity parameter ε on the similarity function g related to the y component of velocity ($\varepsilon_1 = 1$, $\varepsilon_2 = 1$).

fluid character. Therefore second-grade fluids possess thicker boundary layers compared to Oldroy-B fluids if the remaining non-Newtonian parameters are taken identical in both models. There is an absolute increase in the y component of the velocity which is negative over the whole domain for a decrease in the relaxation parameter ε_1 as can be seen from Figure 2.

In comparison, increase of the retardation parameter ε_2 results in a thicker boundary layer (Fig. 3). This means Oldroy-B fluids possess thicker boundary layers compared to the upper convected Maxwell fluids with similar parameters. Velocity in the normal direction, which is negative, increases absolutely for the decreasing retardation parameter (Fig. 4). Finally the effect of parameter ε which is inverse of Reynolds number and associated with the viscous effects is investigated in Figures 5 and 6. As viscous effects increase, boundary layers thicken as expected.

6. Concluding Remarks

Boundary layer equations of an Oldroy-B fluid are derived for the first time. The Lie group theory is applied to the equations. Equations admit four finite parameter Lie Group transformations. For stretching sheet boundary layer conditions, only one of the symmetries remains stable. Using this symmetry, the partial differential system is transferred into an ordinary differential system via a similarity transformation. The resulting ordinary differential system is solved numerically using a finite difference scheme. Effects of the viscosity, the relaxation and retardation parameters on the boundary layers are depicted in figures. Special to this type of motion, it is found that a gradual transformation from an Oldroy-B fluid character to a second-grade fluid character thickens the boundary layer. In comparison, a gradual transformation from an Oldroy-B fluid character to an upper-convected Maxwell fluid character narrows the boundary layers.

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