Numerical Computation of Time-Fractional Fokker–Planck Equation Arising in Solid State Physics and Circuit Theory

Sunil Kumar
Department of Mathematics, National Institute of Technology, Jamshedpur, 801014, Jharkhand, India

Reprint requests to S. K.; E-mail: skumar.math@nitjsr.ac.in

Z. Naturforsch. **68a,** 777 – 784 (2013) / DOI: 10.5560/ZNA.2013-0057
Received May 10, 2013 / revised July 31, 2013 / published online October 30, 2013

The main aim of the present work is to propose a new and simple algorithm to obtain a quick and accurate analytical solution of the time fractional Fokker–Plank equation which arises in various fields in natural science, including solid-state physics, quantum optics, chemical physics, theoretical biology, and circuit theory. This new and simple algorithm is an innovative adjustment in Laplace transform algorithm which makes the calculations much simpler and applicable to several practical problems in science and engineering. The proposed scheme finds the solutions without any discretization or restrictive assumptions and is free from round-off errors and therefore reduces the numerical computations to a great extent. Furthermore, several numerical examples are presented to illustrate the accuracy and the stability of the method.

Key words: Fokker–Plank Equation; Mittag–Leffler Function; Analytical Solution; Homotopy Perturbation Method; Laplace Transform Method.

1. Introduction

The idea of fractional-order derivatives initially arose from a letter by Leibnitz to L'Hospital in 1695. Fractional calculus has gained considerable popularity and importance during the past three decades, mainly due to its applications in numerous fields of science and engineering. One of the main advantages of using fractional-order differential equations in mathematical modelling is their non-local property. It is a well-known fact that the integer-order differential operator is a local operator whereas the fractional-order differential operator is non-local in the sense that the next state of the system depends not only upon its current state but also upon all of its proceeding states. In the last few decades, many authors have made notable contributions to both theory and application of fractional differential equations in areas as diverse as finance [1], physics [2 – 5], control theory [6], and hydrology [7 – 9].

The Fokker–Planck equation (FPE) was first used by Fokker and Planck [10] to describe the Brownian motion of particles. A FPE describes the change of probability of a random function in space and time; hence it is naturally used to describe solute transport. Nonlinear FPE has important applications in various areas such as plasma physics, surface physics, population dynamic, biophysics, engineering, neurosciences, nonlinear hydrodynamics, polymer physics, laser physics, pattern formation, psychology, and marketing [11]. Recently, Yıldırım [12] has applied to obtain the solutions of the time fractional FPE by using the homotopy perturbation method (HPM). In this paper, we have solved the time fractional FPE by coupling of HPM and Laplace transform method (LTM) to the homotopy perturbation transform method (HPTM). In one variable case, the nonlinear FPE is written in the following form:

$$D_t u = \left[-D_x A(x,t,u) + D_x^2 B(x,t,u) \right] u(x,t) , \quad (1)$$

with the initial condition given by $u(x,0) = f(x)$, $x \in \mathbb{R}$, where $A(x,t,u)$ and $B(x,t,u)$ are drift and diffusion coefficient, respectively. The drift and diffusion coefficients may also depend on time. Equation (1) is a linear second-order partial differential equation of parabolic type.

In this paper, the HPTM basically illustrates how the Laplace transform can be used to approximate the solutions of the linear and nonlinear differential equations by manipulating the HPM. The perturbation methods which are generally used to solve nonlinear problems have some limitations, e. g. the approximate solution involves series of small parameters which poses a difficulty since the majority of nonlinear problems have no small parameters at all. Although appropriate choices of small parameters some times lead to ideal solutions but in most of the cases unsuitable choices lead to serious effects in the solutions. The HPM was introduced and applied by He [13–18]. Recently, many researchers [19–24] have obtained the series solution of the fractional differential equation by using HPM. The proposed method is a coupling of the Laplace transformation, the HPM, and He's polynomials mainly due to Ghorbani [25, 26]. In recent years, many authors have paid attention to studying the solutions of linear and nonlinear partial differential equations by using various methods which combined the Laplace transform. Among these are the Laplace decomposition methods [27, 28] and the homotopy perturbation transform method [29–33]. Newly, Faraz et al. [34–36] have applied to obtain the solutions of the fractional partial differential equation in physics by using fractional variational iteration method.

The main aim of this article presents approximate analytical solutions of the time fractional FPE by using HPTM, which is a coupling of HPM and Laplace transform method. We discuss how to solve time fractional FPE by using HPTM. Probability density functions $u(x,t)$ for different fractional Brownian motions and also for the standard motion for various particular cases are derived successfully and presented graphically. The elegance of this article can be attributed to the simplistic approach in seeking the approximate analytical solutions of time fractional Fokker–Planck equation. Our concern in this work is to consider the numerical solution of the nonlinear FPE with time-fractional derivatives of the form

$$D_t^{\alpha} u = \big[-D_x A(x,t,u) + D_x^2 B(x,t,u)\big] u(x,t)\,, \quad 0 < \alpha \le 1\,, \tag{2}$$

where α is a parameter describing the order of the time fractional derivatives. The function $u(x,t)$ is assumed to be a causal function of time and space, i. e. vanishing for $t < 0$ and $x < 0$. The general response expression contains parameters describing the order of the fractional derivatives that can be varied to obtain various responses. In the case of $\alpha = 1$, the fractional equation reduces to the classical nonlinear FPE.

Definition 1. The Laplace transform of the function $f(t)$ is defined by

$$F(s) = L\big[f(t)\big] = \int_0^{\infty} \mathrm{e}^{-st} f(t)\,\mathrm{d}t\,. \tag{3}$$

Definition 2. The Laplace transform $L[u(x,t)]$ of the Riemann–Liouville fractional integral is defined as [37]

$$L\big[I_t^{\alpha} u(x,t)\big] = s^{-\alpha} L\big[u(x,t)\big]\,. \tag{4}$$

Definition 3. The Laplace transform $L[u(x,t)]$ of the Caputo fractional derivative is defined as [37]

$$L\big[D_t^{n\alpha} u(x,t)\big] = s^{n\alpha} L\big[u(x,t)\big] - \sum_{k=0}^{n-1} s^{(n\alpha-k-1)} u^{(k)}(x,0)\,, \quad n-1 < n\alpha \le n\,. \tag{5}$$

2. Basic Idea of Fractional Homotopy Perturbation Transform Method

In order to elucidate the solution procedure of the fractional Laplace homotopy perturbation method, we consider the following nonlinear fractional differential equation:

$$D_t^{n\alpha} u(x,t) + R[x]u(x,t) + N[x]u(x,t) = q(x,t)\,, \quad t > 0\,,\ x \in R\,,\ n-1 < n\alpha \le n\,,\ u(x,0) = h(x)\,, \tag{6}$$

where $D_t^{n\alpha} = \frac{\partial^{n\alpha}}{\partial t^{n\alpha}}$. $\mathbb{R}[x]$ is the linear operator in x, $N[x]$ is the general nonlinear operator in x, and $q(x,t)$ are continuous functions. Now, the methodology consists of applying the Laplace transform first on both sides of (6). So we get

$$L\big[D_t^{n\alpha} u(x,t)\big] + L\big[R[x]u(x,t) + N[x]u(x,t)\big] = L\big[q(x,t)\big]\,. \tag{7}$$

Now, using the differentiation property of the Laplace transform, we have

$$L\big[u(x,t)\big] = s^{-1}h(x) - s^{-n\alpha} L\big[q(x,t)\big] + s^{-n\alpha} L\big[R[x]u(x,t) + N[x]u(x,t)\big]\,. \tag{8}$$

Operating the inverse Laplace transform on both sides in (8), we get

$$u(x,t) = G(x,t) - L^{-1}\Big(s^{-n\alpha} L\big[R[x]u(x,t) + N[x]u(x,t)\big]\Big)\,, \tag{9}$$

where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Further, applying the classical perturbation technique, we can assume that the solution can be expressed as a power series in p as given below:

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t)\,, \tag{10}$$

where the homotopy parameter p is considered as a small parameter ($p \in [0,1]$). The nonlinear term can be decomposed as

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u)\,, \tag{11}$$

where H_n are He's polynomials [25, 26] of $u_0, u_1, u_2, \ldots, u_n$ and it can be calculated by the following formula:

$$H_n(u_0,u_1,u_2,\ldots,u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right]_{p=0}. \quad n = 0,1,2,3,\ldots$$

Substituting (10) and (11) in (9) and using HPM [13 – 18], we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) - p\left(L^{-1}\left[s^{-n\alpha} \cdot L\left[R\sum_{n=0}^{\infty} p^n u_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right). \tag{12}$$

This is coupling of the Laplace transform and homotopy perturbation method using He's polynomials. Now, equating the coefficient of corresponding power of p on both sides, the following approximations are obtained:

$$p^0 : u_0(x,t) = G(x,t)\,, \tag{13}$$
$$p^n : u_n(x,t) = L^{-1}\left(s^{-n\alpha}L\left[R[x]u_{n-1}(x,t) + H_{n-1}(u)\right]\right), \quad n \geq 1\,.$$

Proceeding in this same manner, the rest of the components $u_n(x,t)$, $n \geq 1$, can be completely obtained, and the series solutions are thus entirely determined.

Finally, we approximate the analytical solution $u(x,t)$ by the truncated series

$$u(x,t) = \lim_{N\to\infty}\sum_{n=1}^{N} u_n(x,t)\,. \tag{14}$$

The convergence of series (14) has been proved by He in his paper [18]. It is worth to note that the major advantage of He's homotopy perturbation method is that the perturbation equation can be freely constructed in many ways (therefore is problem dependent) by homotopy in topology and the initial approximation can also be freely selected.

3. Theoretical and Numerical Experiments

In this section, we shall illustrate the fractional homotopy perturbation transform technique by several examples. These examples are somewhat artificial in the sense that the exact answer, for the special case $\alpha = 1$ is known in advance and the initial and boundary conditions are directly taken from this answer. Nonetheless, such an approach is needed to evaluate the accuracy of the analytical techniques and to examine the effect of varying the order of the time-fractional derivatives on the behaviour of the solution. All the results are calculated by using the symbolic calculus software Mathematica 7.

Example 1. We consider the following linear time fractional Fokker–Plank equation [12]:

$$D_t^{\alpha} u = -D_x(xu) + D_x^2\left(\frac{x^2 u}{2}\right), \quad x,t > 0,\; 0 < \alpha \leq 1\,, \tag{15}$$

with the initial condition $u(x,0) = x$. The exact solution of the problem is given by $u(x,t) = x\mathrm{e}^t$ for $\alpha = 1$.

First of all applying the Laplace transform [29, 30] on both sides in (15) and using the differentiation property of Laplace transform, we get

$$L[u(x,t)] = s^{-1}x + s^{-\alpha}L\left[-D_x(xu) + D_x^2\left(\frac{x^2 u}{2}\right)\right]. \tag{16}$$

The inverse Laplace transform on both sides implies that

$$u(x,t) = x + L^{-1}\left(s^{-\alpha}L\left[-D_x(xu) + D_x^2\left(\frac{x^2 u}{2}\right)\right]\right). \tag{17}$$

Now, we apply the homotopy perturbation method [14 – 19] and get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = x + pL^{-1}\left(s^{-\alpha}L\left[-D_x\left(x\sum_{n=0}^{\infty} p^n u_n(x,t)\right) + D_x^2\left(\frac{x^2}{2}\sum_{n=0}^{\infty} p^n u_n(x,t)\right)\right]\right). \tag{18}$$

Then, equating the coefficient of corresponding power of p on both sides in (18) results in

$$p^0 : u_0(x,t) = x,$$
$$p^{n+1} : u_{n+1}(x,t) = L^{-1}\left(s^{-\alpha}L\left[-D_x(xu_n) + D_x^2\left(\frac{x^2u_n}{2}\right)\right]\right) = x\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \quad n = 1,2,3\ldots,$$

and so on, in this manner the rest of components of the homotopy perturbation solution can be obtained. Using the above terms, the solution $u(x,t)$ obtained by the present method is given as

$$u(x,t) = x\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots\right) = x\sum_{k=0}^{\infty}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)} = xE_\alpha(t^\alpha),$$

where $E_\alpha(t) = \sum_{k=0}^{\infty}\frac{t^k}{\Gamma(k\alpha+1)}$, $\alpha > 0$, is the Mittag–Leffler function in one parameter. As $\alpha = 1$, this series has the closed form $x\mathrm{e}^t$, which is an exact solution of the standard Fokker–Plank equation. It can be seen that the solution obtained by the present method is nearly identical with the exact solution for standard Fokker–Plank equation, i. e. for $\alpha = 1$.

Figures 1a and b show the comparison between well-known exact solution and approximate analytical solution which is obtained by the present method. It can be seen from Figure 1a and b that the solution obtained by the present method is nearly identical with the exact solution for standard Fokker–Plank equation, i. e. for $\alpha = 1$. It is evident that the efficiency of this approach can be dramatically enhanced by computing further terms of $u(x,t)$ when the homotopy perturbation transform method is used. The above result is in complete agreement with Yıldırım [12]. It is clear that no linearization or perturbation was used and a closed form solution is obtainable by adding more terms to the homotopy perturbation series.

Example 2. In this example, we consider the linear time fractional Fokker–Plank equation [12] as follows:

$$D_t^\alpha u = -D_x\left(\frac{xu}{6}\right) + D_x^2\left(\frac{x^2u}{12}\right), \quad x,t > 0, \; 0 < \alpha \le 1, \tag{19}$$

with the initial condition $u(x,0) = x^2$. The exact solution of the problem is given by $u(x,t) = x^2\mathrm{e}^{\frac{t}{2}}$ for $\alpha = 1$.

Now, by applying the aforesaid homotopy perturbation method [14–19], we have

$$\sum_{n=0}^{\infty}p^nu_n(x,t) = x^2 + pL^{-1}\left(s^{-\alpha}L\left[-D_x\left(\frac{x}{6}\sum_{n=0}^{\infty}p^nu_n(x,t)\right) + D_x^2\left(\frac{x^2}{12}\sum_{n=0}^{\infty}p^nu_n(x,t)\right)\right]\right). \tag{20}$$

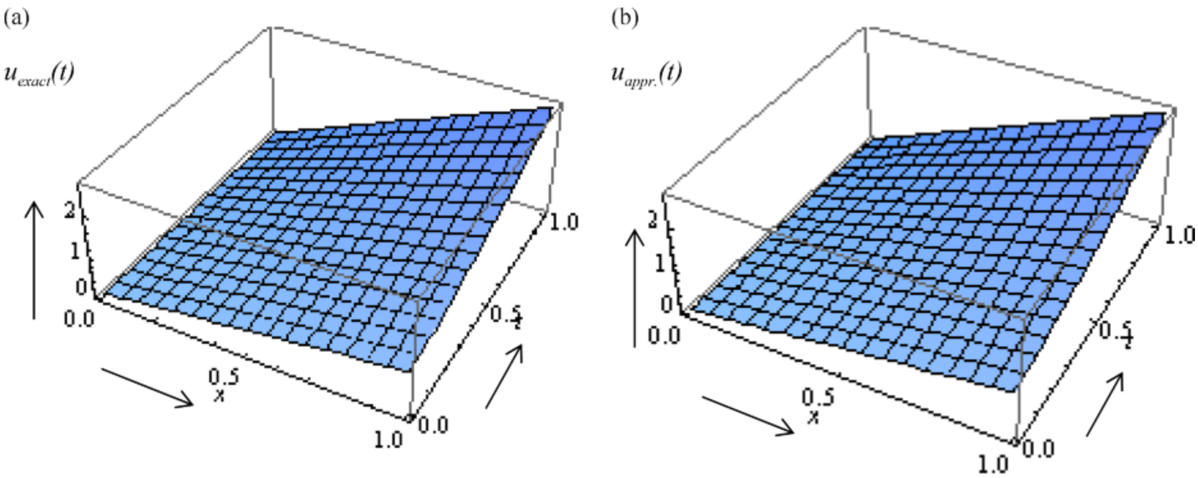

Fig. 1 (colour online). Graphs (a) and (b) show the comparison between the exact solution and approximate analytical solution at $\alpha = 1$ for Example 1.

Then, equating the coefficient of corresponding power of p on both sides in (20), we get

$$p^0 : u_0(x,t) = x^2\,,$$
$$p^{n+1} : u_1(x,t) = L^{-1}\left(s^{-\alpha}L\left[-D_x\left(\frac{xu_n}{6}\right) + D_x^2\left(\frac{x^2u_n}{12}\right)\right]\right) = \frac{x^2}{\Gamma(\alpha+1)}\left(\frac{t^\alpha}{2}\right)^n\,,\quad n=1,2,3,\dots\,,$$

and so on, in this manner the rest of components of the homotopy perturbation series can be obtained. Thus, we have

$$u(x,t) = x^2\left(1+\frac{(t^\alpha)}{2\Gamma(\alpha+1)}+\frac{(t^\alpha)^2}{2^2\Gamma(2\alpha+1)} + \frac{(t^\alpha)^3}{2^3\Gamma(3\alpha+1)}+\dots\right) = x^2\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\frac{t^\alpha}{2}\right)^k = x^2E_\alpha\left(\frac{t^\alpha}{2}\right).$$

For the standard case, i. e. for $\alpha = 1$, this series has the closed form of the solution $u(x,t) = x^2\,\mathrm{e}^{\frac{t}{2}}$ which is an exact solution of the given standard Fokker–Plank equation (19) for $\alpha = 1$. The above result is in complete agreement with Yıldırım [12].

It can be seen from Figure 2a and b that the solution obtained by the present method is nearly identical with the exact solution for standard Fokker–Plank equation, i. e. for $\alpha = 1$.

Example 3. In this example, we consider the linear time fractional Fokker–Plank equation [12]

$$D_t^\alpha u = -D_x\left(\frac{4u^2}{x}-\frac{xu}{3}\right)+D_x^2(u^2)\,,\quad x,t>0,\ 0<\alpha\le 1\,, \tag{21}$$

with the initial condition $u(x,0) = x^2$. The exact solution of the problem is given by $u(x,t) = x^2\,\mathrm{e}^t$ for $\alpha = 1$.

By applying the homotopy perturbation method [13–18], we have

$$\sum_{n=0}^{\infty}p^nu_n(x,t) = x^2 + pL^{-1}\left(s^{-\alpha}L\left[-D_x \cdot\left\{\frac{4}{x}\left(\sum_{n=0}^{\infty}p^nu_n(x,t)\right)^2-\frac{x}{3}\left(\sum_{n=0}^{\infty}p^nu_n(x,t)\right)\right\} + D_x^2\left(x\sum_{n=0}^{\infty}p^nu_n(x,t)\right)^2\right]\right). \tag{22}$$

Equating the coefficient of corresponding power of p on both sides in (22), we get

$$p^0 : u_0(x,t) = x^2\,,$$
$$p^1 : u_1(x,t) = L^{-1}\left(s^{-\alpha}L\left[-D_x\left(\frac{4u_0^2}{x}-\frac{xu_0}{3}\right) + D_x^2(u_0^2)\right]\right) = x^2\frac{t^\alpha}{\Gamma(\alpha+1)}\,,$$
$$p^2 : u_2(x,t) = L^{-1}\left(s^{-\alpha}L\left[-D_x\left(\frac{8u_0u_1}{x}-\frac{xu_1}{3}\right)\right.\right.$$

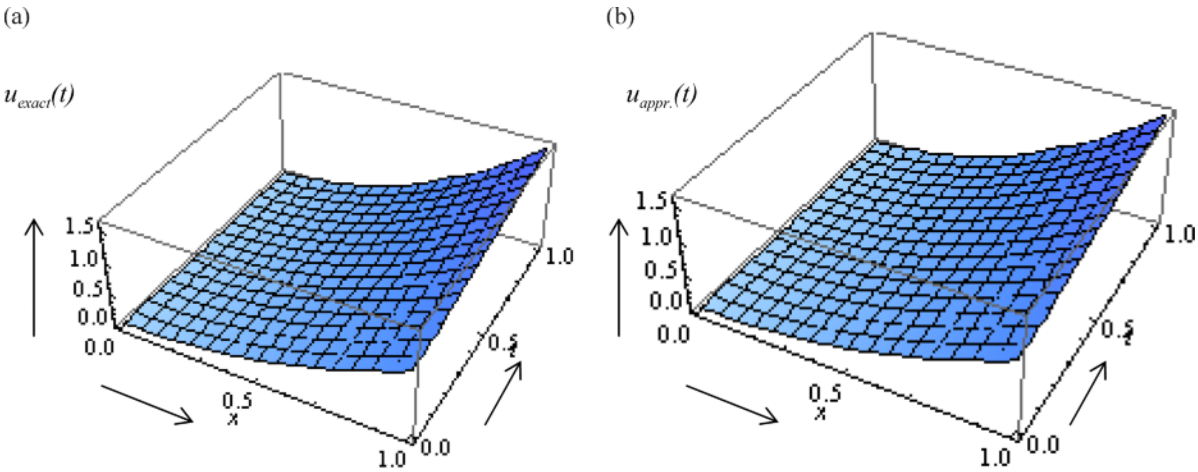

Fig. 2 (colour online). Graphs (a) and (b) show the comparison between the exact solution and approximate analytical solution at $\alpha = 1$ for Example 2.

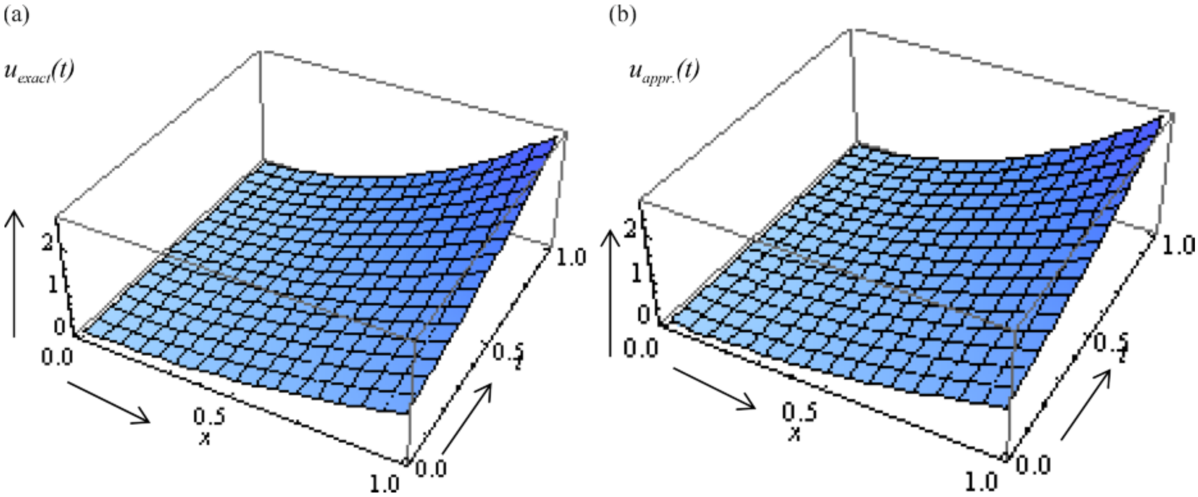

Fig. 3 (colour online). Graphs (a) and (b) show the comparison between the exact solution and approximate analytical solution at $\alpha = 1$ for Example 3.

$$+D_x^2(2u_0u_1)\Big]\Big) = x^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$p^3: u_3(x,t) = L^{-1}\Big(s^{-\alpha}L\Big[-D_x\Big(\frac{4(u_1^2+2u_0u_2)}{x}-\frac{xu_2}{3}\Big)$$
$$+D_x^2(u_1^2+2u_0u_2)\Big]\Big) = x^2\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},\ldots.$$

In this manner, the rest of components of the homotopy perturbation transform solution can be obtained. Thus the solution $u(x,t)$ is given as

$$u(x,t) = x^2\Big(1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$
$$+\ldots\Big) = x^2\sum_{k=0}^{\infty}\frac{(t^{\alpha})^k}{\Gamma(k\alpha+1)} = x^2E_{\alpha}(t^{\alpha}).$$

Now for the standard case, i. e. for $\alpha = 1$, this series has the closed form of the solution $u(x,t) = x^2\,\mathrm{e}^t$, which is an exact solution of the given standard Fokker–Plank equation (21) for $\alpha = 1$.

Figures 3a and b show the comparison between well-known exact solution and approximate analytical solution which is obtained by the present method. It can be seen from Figure 3a and b that the solution obtained by the present method is nearly identical with the exact solution for standard Fokker–Plank equation, i. e. for $\alpha = 1$. The above result is in complete agreement with Yıldırım [12].

4. Numerical Result and Discussion

In this section, Figures 4–6 show the evaluation results of the approximate analytical solution for Example 1 to 3, respectively, and show the behaviour of the approximate solution obtained by the HPTM for different fractional Brownian motions $\alpha = 0.7, 0.8, 0.9$ and for standard motions, i. e. for $\alpha = 1$, at the value of $x = 1$.

From Figures 4–6, it is seen that the approximate analytical solution obtained by the present method (HPTM) increases very rapidly with the increases in t at the value of $x = 1$ for all Examples 1–3. The behaviour of the solution obtained by the present method for all examples is identical.

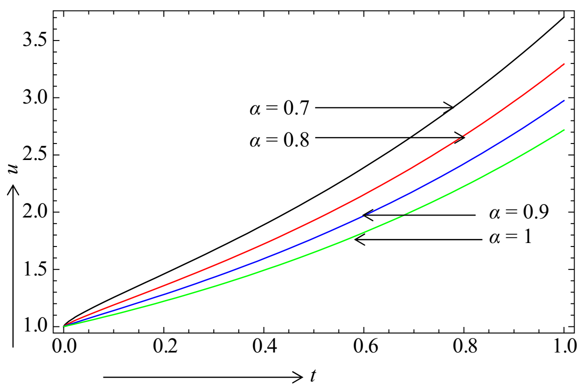

Fig. 4 (colour online). Plot of $u(x,t)$ vs. time t at $x = 1$ and different values of α for Example 1.

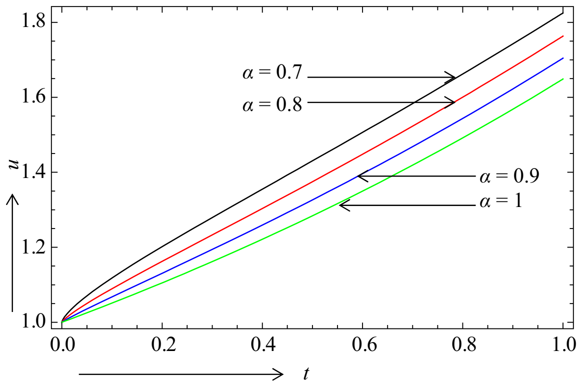

Fig. 5 (colour online). Plot of $u(x,t)$ vs. time t at $x = 1$ and different values of α for Example 2.

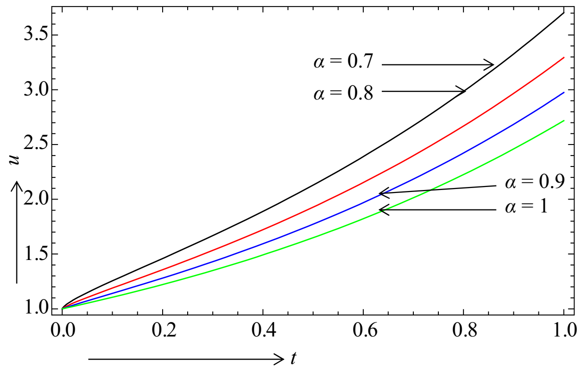

Fig. 6 (colour online). Plot of $u(x,t)$ vs. time t at $x = 1$ and different values of α for Example 3.

5. Conclusion

This paper develops an effective modification of the homotopy perturbation method, which is a coupling of Laplace transform and homotopy perturbation method, and studied its validity in a wide range with three examples of linear and nonlinear time fractional Fokker–Planck equation. It provides the solutions in terms of convergent series with easily computable components in a direct way without using linearization, perturbation or restrictive assumptions. It is clear that the homotopy perturbation transform method yields very accurate approximate solutions using only a few iterates. Thus, it can be concluded that the HPTM methodology is very powerful and efficient in finding approximate solutions as well as numerical solutions.

Acknowledgement

The author is very grateful to the referees for carefully reading the paper and for their comments and suggestions which have improved the paper.

[1] L. Sabatelli, S. Keating, J. Dudley, and P. Richmond, Eur. Phys. J. B **27**, 273 (2002).
[2] M. Meerschaaert, D. Benson, H. P. Scheffler, and B. Baeumer, Phys. Rev. E **65**, 1103 (2002).
[3] B. West, M. Bologna, and P. Grigolini, Physics of Fractal Operators, Springer, New York 2003.
[4] G. Zaslavsky, Hamiltonian Choas and Fractional Dynamics, Oxford University Press, Oxford 2005.
[5] R. Hilfer, Application of Fractional Calculus in Physics, World Scientific, Singapore 2000.
[6] J. T. Mechado, Frac. Cal. Appl. Anal. **4**, 47 (2001).
[7] B. Baeumer, M. M. Meerschaert, D. A. Benson, and S. W. Wheatcraft, Wat. Res. **37**, 1543 (2001).
[8] R. Schumer, D, A. Benson, M. M. Meerschaert, B. Baeumer, and S. W. Wheatcraft, J. Cont. Hydrol. **48**, 69 (2001).
[9] R. Schumer, D. A. Benson, M. M. Meerschaert, and B. Baeumer, Wat. Res. **39**, 1022 (2003).
[10] H. Risken, The Fokker–Planck Equation: Method of Solution and Applications, Springer, Berlin 1989.
[11] T. D. Frank, Physi. A **331**, 391 (2004).
[12] A. Yıldırım, J. King Saud Uni. **22**, 257 (2010).
[13] J. H. He, Comput. Meth. Appl. Mech. Eng. **178**, 257 (1999).
[14] J. H. He, Chaos Solit. Fract. **26**, 695 (2005).
[15] J. H. He, Int. J. Nonlin. Mech. **35**, 37 (2000).
[16] J. H. He, Int. J. Mod. Phys. B **20**, 1141 (2006).
[17] J. H. He, J. Sou. Vibr. **229**, 1257 (2000).
[18] J. H. He, Appl. Math. Comp. **151**, 287 (2004).
[19] S. Kumar and O. P. Singh, Z. Naturforsch. **65a**, 677 (2010).
[20] S. Kumar, Y. Khan, and A. Yıldırım, Asia Pacific J. Chem. Eng. **7**, 835 (2012).
[21] A. Yıldırım and Gulkanat, Commun. Theor. Phys. **53**, 1005 (2010).
[22] N. A. Khan, N. U. Khan, M. Ayaz, and A. Mahmood, Comput. Math Appl. **61**, 2182 (2011).
[23] A. Golbabai and K. Sayevand, Comput. Math. Appl. **61**, 2227 (2011).
[24] A. Golbabai and K. Sayevand, Comput. Math. Appl. **6**, 1003 (2011).
[25] A. Ghorbani and J. S. Nadjafi, Int. J. Nonlin. Sci. Numer. Simul. **8**, 229 (2007).
[26] A. Ghorbani, Chaos Solit. Fract. **39**, 1486 (2009).
[27] M. Khan and M. Hussain, Numer. Algor. **56**, 211 (2011).
[28] M. Khan, M. A. Gondal, and S. Kumar, Math. Comput. Modell. **55**, 1892 (2012).

[29] S. Kumar, H. Kocak, and A. Yıldırım, Z. Naturforsch. **67a**, 389 (2012).
[30] Y. Khan, N. Faraz, S. Kumar, and A. Yıldırım, U. P. B. Sci. Bull. Series A **74**, 1 (2012).
[31] S. Kumar, A. Yıldırım, Y. Khan, and W. Leilei, Sci. Iran. B **19**, 1117 (2012).
[32] J. Singh, D. Kumar, and Shushila, J. Appl. Statis. Infor. **8**, 55 (2012).
[33] J. Singh, D. Kumar, and S. Kumar, Sci. Iran. F. **20**, 1059 (2013).
[34] N. Faraz, Y. Khan, H. Jafari, A. Yıldırım, and M. Madani, J. King Saud Univ. Sci. **23**, 413 (2011).
[35] N. Faraz and Y. Khan, Ain Shams Eng. J. **2**, 221 (2011).
[36] N. Faraz, Comput. Math. Appl. **61**, 1991 (2011).
[37] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, New York 1999.