Spinless Particles Subject to Unequal Scalar-Vector Nuclear Woods–Saxon Potentials in Arbitrary Dimensions

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We present analytical bound state solutions of the spin-zero particles in the Klein–Gordon (KG) equation in presence of an unequal mixture of scalar and vector Woods–Saxon potentials within the framework of the approximation scheme to the centrifugal potential term for any arbitrary l-state. The approximate energy eigenvalues and unnormalized wave functions are obtained in closed forms using a parametric Nikiforov–Uvarov (NU) method. Our numerical energy eigenvalues demonstrate the existence of inter-dimensional degeneracy amongst energy states of the KG-Woods–Saxon problem. The dependence of the energy levels on the dimension D is numerically discussed for spatial dimensions $D = 2 - 6$.

Key words: Klein–Gordon Equation; Woods–Saxon Potential; D-Dimensional Space; Parametric Nikiforov–Uvarov Method.
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1. Introduction

The Woods–Saxon potential is a mean field potential for the nucleons inside the atomic nucleus, which is used to approximately describe the forces applied on each nucleon, in the shell model for the structure of the nucleus [1]. It plays an essential role in microscopic physics since it can be used to describe the interaction of a nucleon with the heavy nucleus [2 – 7]. Although the Schrödinger equation with this potential has been solved for the ground state, and the single particle motion in atomic nuclei has also been explained quite well, the relativistic effects for a particle under the action of this potential are more important, especially for a strong-coupling system [8]. Badalov et al. presented the analytical solution of the radial Schrödinger equation for the Woods–Saxon potential and also obtained the radial part of the Klein–Gordon (KG) equation for the vector Woods–Saxon potential [9, 10]. Arda et al. obtained the scattering solutions of the one-dimensional Schrödinger equation for the Woods–Saxon potential within the position-dependent mass formalism [11]. Ikot and Akpan investigated the energy spectra and the wave function depending on the c-factor for a more general Woods–Saxon potential with an arbitrary l-state [12]. Very recently, Pahlavani and Alavi derived the analytical solutions of the radial Schrödinger equation for the central Woods–Saxon potential together with spin–orbit interaction and centrifugal terms by using the Nikiforov–Uvarov (NU) method [13].

The aim of the present work is to investigate the KG equation in an arbitrary dimension D [14 – 17] with an unequal mixture of scalar and vector Woods–Saxon potentials:

$$V(r) = -\frac{V_0}{1+\mathrm{e}^{(r-R_0)/a}}\,, \tag{1a}$$

$$S(r) = -\frac{S_0}{1+\mathrm{e}^{(r-R_0)/a}}\,, \tag{1b}$$

$$S(r) = \beta V(r)\,,\ -1 \le \beta \le 1\,, \tag{1c}$$

where V_0 represents the potential well depth, a is a length representing the surface thickness of the nu-

cleus, $R_0 = r_0 A^{1/3}$ is the nuclear radius where $r_0 = 1.285$ fm, and A is the mass number. Also, β is an arbitrary constant demonstrating the ratio of scalar potential to vector potential [17]. When $\beta = 1$, it represents the case of spinless particle in equal mixture potentials, i.e., $S(r) = V(r)$, which is being equivalent to the spin-1/2 fermion in the exact spin symmetric limit $\Delta(r) = S(r) - V(r) = 0$ (the potential difference is exactly zero). On the other hand, when $\beta = -1$, it represents the case of spinless anti-particle in equal magnitude potentials but opposite in sign, i.e., $S(r) = -V(r)$, which is being equivalent to spin-1/2 anti-fermion in the exact pseudo-spin (p-spin) symmetric limit $\Sigma(r) = S(r) + V(r) = 0$ (the potential sum is exactly zero). The strong singular centrifugal term is being approximated within the framework of an improved approximation scheme. Further, the spinless D-dimensional KG equation with the scalar and vector Woods–Saxon potentials is solved using the parametric generalization of the NU method [18–21] to obtain approximate analytical energy eigenvalues and corresponding wave functions for any l-state. The bound states of the Schrödinger equation for a second Pöschl–Teller-like potential [22] and a spherically harmonic oscillatory ring-shaped potential [23] were also obtained exactly using the NU method [18].

The present work is organized as follows. In Section 2, the parametric NU method is briefly introduced. In Section 3, we give a review to the KG equation in D-dimensional space and then obtain the bound state solutions of the hyperradial KG equation with unequal mixture of scalar and vector Woods–Saxon potentials by using a shortcut of the NU method. The approximate numerical energy levels for arbitrary values of orbital l and principal n quantum numbers are also obtained. Finally, we end with our concluding remarks in Section 4.

2. Parametric Nikiforov–Uvarov Method

This powerful mathematical tool is used to solve the second-order differential equations. It begins by considering the following differential equation [18–21]:

$$\psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0\,, \tag{2}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials with at most of second degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. To make the application of the NU method simpler and straightforward without need to check the validity of solution, we present a recipe for the method. At first we write the general form of the Schrödinger-like equation (2) in a more general parametric form:

$$\begin{aligned}&\psi_n''(s) + \left(\frac{c_1 - c_2 s}{s(1-c_3 s)}\right)\psi_n'(s)\\ &+ \left(\frac{-p_2 s^2 + p_1 s - p_0}{s^2(1-c_3 s)^2}\right)\psi_n(s) = 0\,,\end{aligned} \tag{3}$$

satisfying the wave functions

$$\psi_n(s) = \varphi(s) y_n(s)\,. \tag{4}$$

Now, comparing (3) with its counterpart (2), one can obtain the following identifications:

$$\begin{aligned}&\tilde{\tau}(s) = c_1 - c_2 s\,,\ \ \sigma(s) = s(1-c_3 s)\,,\\ &\tilde{\sigma}(s) = -p_2 s^2 + p_1 s - p_0\,.\end{aligned} \tag{5}$$

Further, following the NU method [15], the bound state energy equation can be found as [19–21]

$$\begin{aligned}&c_2 n - (2n+1)c_5 + (2n+1)\left(\sqrt{c_9} - c_3\sqrt{c_8}\right)\\ &+ n(n-1)c_3 + c_7 + 2c_3 c_8 - 2\sqrt{c_8 c_9} = 0\,.\end{aligned} \tag{6}$$

On the other hand, we can also find the functions

$$\begin{aligned}&\rho(s) = s^{c_{10}}(1-c_3 s)^{c_{11}}\,,\\ &\varphi(s) = s^{c_{12}}(1-c_3 s)^{c_{13}}\,,\\ &c_{12} > 0\,,\ \ c_{13} > 0\,,\\ &y_n(s) = P_n^{(c_{10},c_{11})}(1-2c_3 s)\,,\\ &c_{10} > -1\,,\ \ c_{11} > -1\,,\end{aligned} \tag{7a}$$

which are used in calculating the wave functions

$$\begin{aligned}&\psi_{nl}(s) = N_{nl} s^{c_{12}}(1-c_3 s)^{c_{13}} P_n^{(c_{10},c_{11})}\\ &\cdot(1-2c_3 s)\,,\end{aligned} \tag{7b}$$

where $P_n^{(\mu,\nu)}(x)$, $\mu > -1$, $\nu > -1$, $x \in [-1,1]$ are Jacobi polynomials with constant parameters [19, 20]

$$\begin{aligned}&c_4 = \frac{1}{2}(1-c_1)\,,\ \ c_5 = \frac{1}{2}(c_2 - 2c_3)\,,\\ &c_6 = c_5^2 + p_2\,,\ \ c_7 = 2c_4 c_5 - p_1\,,\\ &c_8 = c_4^2 + p_0\,,\ \ c_9 = c_3(c_7 + c_3 c_8) + c_6\,,\\ &c_{10} = c_1 + 2c_4 - 2\sqrt{c_8} - 1 > -1\,,\end{aligned}$$

$$
\begin{aligned}
&c_{11} = 1 - c_1 - 2c_4 + \frac{2}{c_3}\sqrt{c_9} > -1\,,\ \ c_3 \neq 0\,,\\
&c_{12} = c_4 - \sqrt{c_8} > 0\,,\\
&c_{13} = -c_4 + \frac{1}{c_3}(\sqrt{c_9} - c_5) > 0,\ \ c_3 \neq 0\,,
\end{aligned}
\tag{8}
$$

where $c_{12} > 0$, $c_{13} > 0$ and $s \in [0, 1/c_3], c_3 \neq 0$.

When considering a more special case of $c_3 = 0$, the wave function (7b) becomes

$$
\begin{aligned}
&\lim_{c_3 \to 0} P_n^{(c_{10},c_{11})}(1 - 2c_3 s) = L_n^{c_{10}}(c_{11}s)\,,\\
&\lim_{c_3 \to 0} (1 - c_3 s)^{c_{13}} = \mathrm{e}^{c_{13}s}\,,\\
&\psi(s) = N s^{c_{12}}\,\mathrm{e}^{c_{13}s} L_n^{c_{10}}(c_{11}s)\,,
\end{aligned}
\tag{9}
$$

where $L_n^{\alpha}(x)$ are the associated Laguerre polynomials.

3. Hyperradial Part of the Klein–Gordon Equation in D-Dimensional Space

In spherical coordinates, the KG equation with vector $V(r)$ and scalar $S(r)$ potentials can be written (in units $\hbar = c = 1$) as [14–17, 24]

$$
\begin{aligned}
&\left[\Delta_D + (E_{nl} - V(r))^2 - (M + S(r))^2\right]\\
&\cdot \psi_{nlm}(r, \Omega_D) = 0
\end{aligned}
\tag{10}
$$

with

$$
\begin{aligned}
&\Delta_D = \nabla_D^2 = \frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial}{\partial r}\right)\\
&- \frac{\Lambda_D^2(\Omega_D)}{r^2},
\end{aligned}
\tag{11}
$$

where E_{nl}, $\Lambda_D^2(\Omega_D)/r^2$, and Ω_D are the energy eigenvalues, generation of the centrifugal barrier for D-dimensional space, and the angular coordinates, respectively [14]. The eigenvalues of $\Lambda_D^2(\Omega_D)$ are given by [14–17]

$$
\begin{aligned}
&\Lambda_D^2(\Omega_D) Y_l^m(\Omega_D) = \frac{(D + 2l - 2)^2 - 1}{4} Y_l^m(\Omega_D)\,,\\
&D > 1\,,
\end{aligned}
\tag{12}
$$

where $Y_l^m(\Omega_D)$ is the hyperspherical harmonics. For $D = 2$, we have $\Lambda_{D=2}^2(\Omega_{D=2}) Y_l^m(\Omega_{D=2}) = \left(m^2 - 1/4\right) Y_l^m(\Omega_{D=2})$ and for $D = 3$, we have a familiar form $\Lambda_{D=3}^2(\Omega_{D=3}) Y_l^m(\Omega_{D=3}) = l(l+1) Y_l^m(\Omega_{D=3})$. Using the procedure of separation of variables by means of the wave function $\psi_{nlm}(r, \Omega_D) = r^{-(D-1)/2} R_{nl}(r) Y_l^m(\Omega_D)$, (10) with the substitution of (11) and (12) can be rewritten as [15–17]

$$
\begin{aligned}
&\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \left(M^2 - E_{nl}^2\right) - 2\left(E_{nl} V(r) + M S(r)\right)\right.\\
&\left. + V^2(r) - S^2(r) - \frac{(D + 2l - 2)^2 - 1}{4r^2}\right] R_{nl}(r) = 0\,.
\end{aligned}
\tag{13}
$$

The unequal scalar and vector potentials in the framework of KG was first discussed by Greiner in 2000 [25] and then by Dong and Lozado-Cassou [26]. The exact solutions of the KG equation (13) with unequal scalar and vector potentials in D dimensions were exactly found for the energy eigenvalues and wave functions [26, 27]. By the substitution of the scalar and vector Woods–Saxon potentials (1a)–(1c) into (13), one obtains

$$
\begin{aligned}
&\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \varepsilon^2 + \frac{2(E_{nl} V_0 + M S_0)}{1 + \mathrm{e}^{(r - R_0)/a}} + \frac{(V_0^2 - S_0^2)}{\left(1 + \mathrm{e}^{(r - R_0)/a}\right)^2}\right.\\
&\left. - \frac{(D + 2l - 2)^2 - 1}{4r^2}\right] R_{nl}(r) = 0\,,
\end{aligned}
\tag{14}
$$

where $\varepsilon^2 = M^2 - E_{nl}^2$. It is obvious that (14) does not admit an exact solution due to the presence of the singular centrifugal term $1/r^2$. Therefore, we need to make the Pekeris-type approximation to deal with this centrifugal term [28]. In this approximation, we expand the centrifugal potential around $r = R_0$ $(x = 0)$ in a series of powers of $x = (r - R_0)/R_0$ as

$$
\begin{aligned}
V_l(x) &= \frac{(D + 2l - 2)^2 - 1}{4r^2} = \frac{(D + 2l - 2)^2 - 1}{4R_0^2(1 + x)^2}\\
&= \frac{(D + 2l - 2)^2 - 1}{4R_0^2}(1 - 2x + 3x^2 - 4x^3 + \ldots)\,.
\end{aligned}
\tag{15}
$$

It is sufficient to keep the expansion terms only up to the second order. The following form of the potential can be used instead of the centrifugal potential in the Pekeris-type approximation scheme:

$$
\begin{aligned}
\tilde{V}_l(x) &= \frac{(D + 2l - 2)^2 - 1}{4R_0^2}\\
&\cdot \left(D_0 + \frac{D_1}{1 + \exp(\nu x)} + \frac{D_2}{(1 + \exp(\nu x))^2}\right),
\end{aligned}
\tag{16}
$$

where $\nu = R_0/a$ and D_i is the parameter of coefficients $(i = 0, 1, 2)$. After expanding (16) up to the terms of order x^2, followed by making some arrangements and

Table 1. Parametric constants used in calculating the energy eigenvalues and wave functions.

Analytical value	Constant
c_4	0
c_5	0
c_6	$a^2\left(S_0^2-V_0^2+\gamma_0 D_2\right),\ \gamma_0=\frac{(D+2l-2)^2-1}{4R_0^2}$
c_7	$-a^2\left(2\left(E_{nl}V_0+MS_0\right)-\gamma_0 D_1\right)$
c_8	$a^2\left(\varepsilon^2+\gamma_0 D_0\right)$
c_9	$a^2\left(\varepsilon^2+S_0^2-V_0^2-2\left(E_{nl}V_0+MS_0\right)+\gamma_0\left(D_0+D_1+D_2\right)\right)$
c_{10}	$-2a\sqrt{\varepsilon^2+\gamma_0 D_0}$
c_{11}	$2a\sqrt{\varepsilon^2+S_0^2-V_0^2-2\left(E_{nl}V_0+MS_0\right)+\gamma_0\left(D_0+D_1+D_2\right)}$
c_{12}	$-a\sqrt{\varepsilon^2+\gamma_0 D_0}$
c_{13}	$a\sqrt{\varepsilon^2+S_0^2-V_0^2-2\left(E_{nl}V_0+MS_0\right)+\gamma_0\left(D_0+D_1+D_2\right)}$

further combining equal powers in x with (15), we obtain relations between the coefficient parameters and the potential parameter ν as follows [29–31]:

$$D_0=1-\frac{4}{\nu}+\frac{12}{\nu^2},\ D_1=\frac{8}{\nu}-\frac{48}{\nu^2},\qquad D_2=\frac{48}{\nu^2}. \tag{17}$$

The use of the Pekeris-type approximation to the centrifugal term was also mentioned by Wei and Dong [32, 33].

Taking the potential $\tilde{V}_l$ (16) instead of the centrifugal potential (15), (14) reduces to the following form:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\varepsilon^2+\frac{2(E_{nl}V_0+MS_0)}{1+\exp(\nu x)}+\frac{(V_0^2-S_0^2)}{(1+\exp(\nu x))^2}-\frac{(D+2l-2)^2-1}{4R_0^2}\left(D_0+\frac{D_1}{1+\exp(\nu x)}+\frac{D_2}{(1+\exp(\nu x))^2}\right)\right]R_{nl}(r)=0. \tag{18}$$

Further making a suitable change of variables as $z=\frac{1}{1+\exp(\nu x)}$, we can thereby recast (18) as follows:

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\frac{1-2z}{z(1-z)}\frac{\mathrm{d}}{\mathrm{d}z}+\frac{a^2}{z^2(1-z)^2}\cdot\left[-\left(\varepsilon^2+\gamma_0 D_0\right)+\left(2E_{nl}V_0+2MS_0-\gamma_0 D_1\right)z-\left(S_0^2-V_0^2-\gamma_0 D_2\right)z^2\right]\right\}R_{nl}(z)=0, \tag{19}$$

where $\gamma_0=\frac{(D+2l-2)^2-1}{4R_0^2}$. Hence, (19) can be solved by comparing it with (3) to get

$$\begin{aligned} &c_1=1,\ p_0=a^2\left(\varepsilon^2+\gamma_0 D_0\right),\\ &c_2=2,\ p_1=a^2\left(2E_{nl}V_0+2MS_0-\gamma_0 D_1\right),\\ &c_3=1,\ p_2=a^2\left(S_0^2-V_0^2+\gamma_0 D_2\right). \end{aligned} \tag{20}$$

In addition, the values of constant parametric coefficients c_i $(i=4,5,\ldots,13)$ are found from (8) and displayed in Table 1. Thus, the energy eigenvalue equation can be obtained via (6) as follows:

$$\left[n+\frac{1}{2}+a\sqrt{M^2-E_{nl}^2+S_0^2-V_0^2-2\left(E_{nl}V_0+MS_0\right)+\frac{(D+2l-2)^2-1}{4R_0^2}\left(D_0+D_1+D_2\right)}-a\sqrt{M^2-E_{nl}^2+\frac{(D+2l-2)^2-1}{4R_0^2}D_0}\right]^2=a^2\left(S_0^2-V_0^2+\frac{(D+2l-2)^2-1}{4R_0^2}D_2\right)+\frac{1}{4}, \tag{21}$$

where $-M<E_{nl}<M$. Some approximated numerical energy states are calculated in Tables 2 to 4. The empirical potential parameters are taken as $r_0=1.285$ fm, $V_0=(40.5+0.13A)\,\mathrm{MeV}=47.78\,\mathrm{MeV}$, $R_0=r_0A^{1/3}=4.9162$ fm, $a=0.65$ fm for the atomic mass number of target nucleus $A=56$ and a pion with mass $M_0c^2=139.570\,\mathrm{MeV}$ [9, 10, 34]. In Table 2, we calculated energy states $E(n,l,D)$ for various values of dimensions D and quantum numbers n and l when the vector potential is different from the scalar one,

Table 2. Bound state energy levels of the KG particles in the field of scalar and vector Woods–Saxon potentials for various D, n, and l values with $V_0 \neq S_0$.

	$E_{n,l}$ (MeV)					
n,l	1,0	2,0	2,1	3,0	3,1	3,2
$D=2$	−96.73917303	−88.51608375	−88.51104985	−81.03713418	−81.03255004	−81.01879952
3	−96.73771236	−88.51482354	−88.50475852	−81.03598655	−81.02682066	−81.00848741
4	−96.73333834	−88.51104985	−88.49595103	−81.03255004	−81.01879952	−80.99588728
5	−96.72604676	−88.50475852	−88.48462860	−81.02682066	−81.00848741	−80.98099889
6	−96.71584009	−88.49595103	−88.47079486	−81.01879952	−80.99588728	−80.96382416

Table 3. Bound state energy levels of the KG particle subjects to scalar and vector Woods–Saxon potentials for various D, n, and l values with $V_0 = S_0$.

	$E_{n,l}$ (MeV)					
n,l	1,0	2,0	2,1	3,0	3,1	3,2
$D=2$	−138.6285863	−137.4598557	−137.4383488	−135.8431759	−135.8152852	−135.7353424
3	−138.6248328	−137.4544218	−137.4121295	−135.8361250	−135.7813395	−135.6785084
4	−138.6137138	−137.4383488	−137.3765274	−135.8152852	−135.7353424	−135.6120883
5	−138.5955263	−137.4121295	−137.3324213	−135.7813395	−135.6785084	−135.5372268
6	−138.5707336	−137.3765274	−137.2807174	−135.7353424	−135.6785084	−135.4549553

Table 4. Bound state energy levels of the KG particle in the field of scalar and vector Woods–Saxon potentials for various D, n, and l values with $V_0 = -S_0$.

	$E_{n,l}$ (MeV)					
n,l	1,0	2,0	2,1	3,0	3,1	3,2
$D=2$	138.6285863	137.4598557	137.4383488	135.8431759	135.8152852	135.7353424
3	138.6248328	137.4544218	137.4121295	135.8361250	135.7813395	135.6785084
4	138.6137138	137.4383488	137.3765274	135.8152852	135.7353424	135.6120883
5	138.5955263	137.4121295	137.3324213	135.7813395	135.6785084	135.5372268
6	138.5707336	137.3765274	137.2807174	135.7353424	135.6785084	135.4549553

i.e. $V(r) \neq S(r)$. In Table 3, we obtained energy levels for various values of dimensions D and quantum numbers n and l when the vector potential is equal to the scalar one, i.e. $V(r) = S(r)$ which represents the energy spectrum for a particle [16]. Finally, in Table 4, we listed bound state energy eigenvalues for various values of spatial dimension D and quantum numbers n and l when the vector potential is opposite to the scalar one, i.e. $V(r) = -S(r)$ which represents the energy spectrum for an anti-particle [16]. It is obvious from Tables 2–4 that the inter-dimensional degeneracy of eigenvalues of upper/lower dimensional system forms the well-known eigenvalues of a lower/upper dimensional system by means of the transformation $(n,l,D) \rightarrow (n,l \pm 1, D \mp 2)$ [35]. We observe from Tables 2 to 4 the following degenerate energy states: $E(2,1,2) = E(2,0,4)$; $E(2,1,3) = E(2,0,5)$; $E(2,1,4) = E(2,0,6)$.

Let us now calculate the corresponding wave functions; for this purpose, we use the relations in (7) to obtain the necessary functions

$$\begin{aligned} \rho(z) &= z^{2\eta_l}(1-z)^{2\delta_l}, \\ \varphi(z) &= z^{\eta_l}(1-z)^{\delta_l}, \\ y_n(z) &= P_n^{(2\eta_l,2\delta_l)}(1-2z), \\ R_{nl}(z) &= N_{nl} z^{\eta_l}(1-z)^{\delta_l} P_n^{(2\eta_l,2\delta_l)}(1-2z), \end{aligned} \tag{22}$$

or in a more convenient form in terms of the potential parameters as

$$\begin{aligned} R_{nl}(r) = N_{nl} &\left(\frac{1}{1+\mathrm{e}^{(r-R_0)/a}}\right)^{\eta_l} \\ &\left(\frac{\mathrm{e}^{(r-R_0)/a}}{1+\mathrm{e}^{(r-R_0)/a}}\right)^{\delta_l} P_n^{(2\eta_l,2\delta_l)}\left(\frac{\mathrm{e}^{(r-R_0)/a}-1}{\mathrm{e}^{(r-R_0)/a}+1}\right) \end{aligned} \tag{23}$$

with

$$\eta_l = -a\sqrt{\varepsilon^2 + \frac{(D+2l-2)^2-1}{4R_0^2}D_0}\,, \tag{24a}$$

$$\delta_l = a\sqrt{\varepsilon^2 + S_0^2 - V_0^2 - 2\left(E_{nl}V_0 + MS_0\right) + \frac{(D+2l-2)^2-1}{4R_0^2}\left(D_0+D_1+D_2\right)}\,, \tag{24b}$$

where N_{nl} is the normalization constant.

From (18), we obtain the energy eigenvalues of the KG particles in two-dimensional (2D) and three-dimensional (3D) spaces as

$$\left[n+\frac{1}{2}+a\sqrt{M^2-E_{nl}^2+S_0^2-V_0^2-2\left(E_{nl}V_0+MS_0\right)+\frac{m^2-1/4}{R_0^2}\left(D_0+D_1+D_2\right)}\right.$$
$$\left.-a\sqrt{M^2-E_{nl}^2+\frac{m^2-1/4}{R_0^2}D_0}\right]^2 = a^2\left(S_0^2-V_0^2+\frac{m^2-1/4}{R_0^2}D_2\right)+\frac{1}{4} \tag{25}$$

and

$$\left[n+\frac{1}{2}+a\sqrt{M^2-E_{nl}^2+S_0^2-V_0^2-2\left(E_{nl}V_0+MS_0\right)+\frac{l(l+1)}{R_0^2}\left(D_0+D_1+D_2\right)}\right.$$
$$\left.-a\sqrt{M^2-E_{nl}^2+\frac{l(l+1)}{R_0^2}D_0}\right]^2 = a^2\left(S_0^2-V_0^2+\frac{l(l+1)}{R_0^2}D_2\right)+\frac{1}{4}\,, \tag{26}$$

respectively, where, in (18), we inserted $l \to m-1/2$ in two dimensions.

When we set $V(r)\to V(r)/2$, $S(r)\to S(r)/2$, $E_{nl}+M\to 2\mu/\hbar^2$, and $E_{nl}-M\to E_{nl}$ [36], we can obtain the solution in the non-relativistic limit of the KG-Woods–Saxon problem. Here $\mu = m_1m_2/(m_1+m_2)$ is the reduced mass. Under these conditions, one can obtain the non-relativistic energy eigenvalues of the Woods–Saxon potential in closed form as

$$\left[n+\frac{1}{2}\right.$$
$$\left.+a\sqrt{-\frac{2\mu E_{nl}}{\hbar^2}-\frac{\mu V_0}{\hbar^2}+\frac{l(l+1)}{R_0^2}\left(D_0+D_1+D_2\right)}\right.$$
$$\left.-a\sqrt{-\frac{2\mu E_{nl}}{\hbar^2}+\frac{l(l+1)}{R_0^2}D_0}\right]^2 = \frac{a^2l(l+1)}{R_0^2}D_2+\frac{1}{4}\,. \tag{27}$$

4. Concluding Remarks

To sum up, we have used a parametric NU method as well as an appropriate approximation for the singular centrifugal term to obtain approximate analytical bound state solutions including energy eigenvalues and wave functions of the D-dimensional KG equation for scalar and vector Woods–Saxon potentials. Numerical tests using energy calculations show the existence of inter-dimensional degeneracy of energy states prevailing to the following transformation: $(n,l,D)\to(n,l\pm1,D\mp2)$ as shown in Tables 2–4.

In the present solution, we have not faced any cumbersome and time-consuming procedures in obtaining the numerical results for the energy eigenvalues KG-Woods–Saxon problem by the present simple methodology. The present analytical expressions can be used in nuclear physics.

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