Effects of Side Walls on the Motion Induced by an Infinite Plate that Applies Shear Stresses to an Oldroyd-B Fluid

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Unsteady motions of Oldroyd-B fluids between two parallel walls perpendicular to a plate that applies two types of shears to the fluid are studied using integral transforms. Exact solutions are obtained both for velocity and non-trivial shear stresses. They are presented in simple forms as sums of steady-state and transient solutions and can easily be particularized to give the similar solutions for Maxwell, second-grade and Newtonian fluids. Known solutions for the motion over an infinite plate, applying the same shears to the fluid, are recovered as limiting cases of general solutions. Finally, the influence of side walls on the fluid motion, the distance between walls for which their presence can be neglected, and the required time to reach the steady-state are graphically determined.

Key words: Non-Newtonian Fluids; Shear Stress; Side Walls; Exact Solutions.

1. Introduction

The non-Newtonian fluids are important due to their applications in various branches of industry and technology, and their study presents a special challenge to engineers, physicists, and mathematicians. Motions of these fluids due to an oscillating plate have been extensively studied in the literature [1 – 6]. However, the first closed-form expressions for starting solutions corresponding to the motion of Newtonian fluids caused by cosine or sine oscillations of the plate have been late enough obtained [7]. These solutions have been extended to non-Newtonian fluids by different authors [8 – 10]. Furthermore, the problem has been extended to fluid motions between two side walls perpendicular to an oscillating plate [11].

Over the last decade, the interest of researchers in problems with shear stress boundary conditions (instead of velocity boundary conditions) has significantly increased. This is very important as in some problems what is specified is the force applied on the boundary. Further, the ‘no slip’ boundary condition may not be necessarily applicable to flows of polymeric fluids that can slip or slide on the boundary. In general, the slip velocity depends on the shear and mostly the slip conditions are developed under the assumption that they depend on the shear stress. Thus, shear stress boundary conditions are particularly meaningful [12, 13]. The first exact solutions for motion of second-grade fluids over an infinite plate that applies a constant shear stress to the fluid seem to be those of Bandelli et al. [14] and Erdogan [15]. In [16], Fetecau et al. studied the walls effect on the motion of a viscous fluid induced by the bottom plate, applying the oscillating shears to the fluid. Further the motions of the fluids between two parallel walls perpendicular to a plate that applies a constant or an oscillating shear to the fluid can be seen in [17 – 27]. Recently, Shahid et al. [28] found some exact solutions for motions of Oldroyd-B fluids over an infinite plate that applies oscillating shear stresses to the fluid.

The aim of this work is to extend the last results to motions between parallel walls. More exactly, we establish exact solutions for the motion of an Oldroyd-B fluid between two side walls perpendicular to an infinite plate that applies two types of shears to the fluid. These solutions, presented as a sum between steady-state and transient solutions, satisfy all imposed initial and boundary conditions and reduce as limiting cases to previous solutions. Moreover, they describe the motion of the fluid some time after its initiation. After that time, when the transients disappear, the fluid flows according to the steady-state solutions that are periodic in time and independent of the initial conditions. The influence of the walls on the fluid motion and the required time to reach the steady-state are graphically determined.

2. Governing Equations

The Cauchy stress tensor $\mathbf{T}$ for an incompressible Oldroyd-B fluid is related to the fluid motion in the following manner [29]:

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \tag{1a}$$

$$\mathbf{S} + \lambda(\dot{\mathbf{S}} - \mathbf{LS} - \mathbf{SL}^{\mathrm{T}}) = \mu\left(\mathbf{A} + \lambda_r(\dot{\mathbf{A}} - \mathbf{LA} - \mathbf{AL}^{\mathrm{T}})\right), \tag{1b}$$

where $-p\mathbf{I}$ is the indeterminate part of the stress due to the constraint of incompressibility, $\mathbf{S}$ is the extra-stress tensor, λ and λ_r are the relaxation and retardation times, $\mathbf{L}$ is the velocity gradient, μ is the dynamic viscosity, $\mathbf{A} = \mathbf{L} + \mathbf{L}^{\mathrm{T}}$ is the first Rivlin–Ericksen tensor, and the superposed dot denotes the material time derivative. In the following, we shall seek a velocity field $\mathbf{v}$ and an extra-stress $\mathbf{S}$ of the form

$$\mathbf{v} = \mathbf{v}(y,z,t) = u(y,z,t)\mathbf{i}, \tag{2a}$$

$$\mathbf{S} = \mathbf{S}(y,z,t), \tag{2b}$$

where i is the unit vector along the x-direction of the Cartesian coordinate system x, y, and z. For such flows, the constraint of incompressibility is automatically satisfied. If the fluid is at rest at the moment $t = 0$, then

$$\mathbf{v}(y,z,0) = \mathbf{0}, \quad \mathbf{S}(y,z,0) = \mathbf{0}, \tag{3}$$

and the constitutive equation (1b) lead to the meaningful relations

$$\begin{aligned} \left(1+\lambda\frac{\partial}{\partial t}\right)\tau_1(y,z,t) &= \mu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\frac{\partial u(y,z,t)}{\partial y}, \\ \left(1+\lambda\frac{\partial}{\partial t}\right)\tau_2(y,z,t) &= \mu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\frac{\partial u(y,z,t)}{\partial z}, \end{aligned} \tag{4}$$

where $\tau_1(y,z,t) = S_{xy}(y,z,t)$ and $\tau_2(y,z,t) = S_{xz}(y,z,t)$ are the non-trivial shear stresses. In the absence of a pressure gradient along the flow direction and neglecting body forces, (4) together with the motion equations leads to the governing equation for velocity [30], i.e.

$$\lambda\frac{\partial^2 u(y,z,t)}{\partial t^2} + \frac{\partial u(y,z,t)}{\partial t} = \nu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\left[\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]u(y,z,t); \; t > 0, \tag{5}$$

where $\nu = \mu/\rho$ is the kinematic viscosity and ρ is the constant density of the fluid. In the following, the governing equations (4) and (5) together with appropriate initial and boundary conditions will be solved using the Fourier and Laplace transforms.

3. Formulation and Solution of the Problem

Let us consider an incompressible Oldroyd-B fluid at rest over an infinite flat plate situated in the (x,z)-plane, and between two side walls situated in the planes $z = 0$ and $z = d$. After time $t = 0^+$, the bottom plate is set into motion so that the shear stress in its plane is given by

$$\tau_1(0,z,t) = f\frac{\lambda\omega}{1+\lambda^2\omega^2} \cdot \left\{\frac{1}{\lambda\omega}\sin(\omega t) - \cos(\omega t) + \mathrm{e}^{\frac{-t}{\lambda}}\right\} \tag{6}$$

or

$$\tau_1(0,z,t) = f\frac{\lambda\omega}{1+\lambda^2\omega^2} \cdot \left\{\sin(\omega t) + \frac{1}{\lambda\omega}\cos(\omega t) - \frac{1}{\lambda\omega}\mathrm{e}^{\frac{-t}{\lambda}}\right\}, \tag{7}$$

here f and $\omega > 0$ are constants, ω being the frequency of the oscillations. Owing to the shear, the fluid is gradually moved. Its velocity is of the form (2a), the governing equations are given by (4) and (5) while the appropriate initial and boundary conditions are given by

$$u(y,z,0) = \left.\frac{\partial u(y,z,t)}{\partial t}\right|_{t=0} = 0, \quad \tau_1(y,z,0) = 0, \quad \tau_2(y,z,0) = 0; \; y > 0, \; z \in [0,d], \tag{8}$$

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\tau_1(0,z,t) = \mu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\left.\frac{\partial u(y,z,t)}{\partial y}\right|_{y=0} = f\sin(\omega t) \text{ or } f\cos(\omega t); \; z \in (0,d), \; t > 0, \tag{9}$$

$$u(y,0,t) = u(y,d,t) = 0; \; y > 0, \; t \geq 0, \tag{10}$$

$$u(y,z,t), \frac{\partial u(y,z,t)}{\partial y} \to 0 \text{ as } y \to \infty; \; z \in [0,d], \; t \geq 0. \tag{11}$$

Of course, the expressions of $\tau_1(0,z,t)$ given by (6) and (7) are just the solutions of the partial differential equation (9). For $\lambda \to 0$, (6) and (7) take the simplified forms

$$\tau_1(0,z,t) = f\sin(\omega t) \text{ or } \tau_1(0,z,t) = f\cos(\omega t), \tag{12}$$

they are boundary conditions corresponding to the motion of a Newtonian or second-grade fluid between two side walls perpendicular to a plate that applies an oscillating shear stress $f\sin(\omega t)$ or $f\cos(\omega t)$ to the fluid [16, 24].

In the following, let us consider the complex fields

$$\begin{aligned} V(y,z,t) &= u_c(y,z,t) + \mathrm{i}u_s(y,z,t)\,, \\ T_1(y,z,t) &= \tau_{1c}(y,z,t) + \mathrm{i}\tau_{1s}(y,z,t)\,, \\ T_2(y,z,t) &= \tau_{2c}(y,z,t) + \mathrm{i}\tau_{2s}(y,z,t)\,, \end{aligned} \tag{13}$$

where $u_s(y,z,t)$, $\tau_{1s}(y,z,t)$, $\tau_{2s}(y,z,t)$, and $u_c(y,z,t)$, $\tau_{1c}(y,z,t)$, $\tau_{2c}(y,z,t)$ are the solutions of our problem corresponding to the boundary conditions (6), respectively (7), and i is the imaginary unit. In the following, for simplicity, we shall refer to them as the solutions corresponding to the sinusoidal or co-sinusoidal oscillations of the shear stress on the boundary.

In view of the above notations, we obtain the following initial-boundary value problem:

$$\begin{aligned} &\lambda\frac{\partial^2 V(y,z,t)}{\partial t^2} + \frac{\partial V(y,z,t)}{\partial t} = \nu\left(1+\lambda_r\frac{\partial}{\partial t}\right) \\ &\cdot\left[\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]V(y,z,t)\,;\;\; y,t>0\,,\;\; z\in(0,d)\,, \end{aligned} \tag{14}$$

$$\begin{aligned} &\left(1+\lambda\frac{\partial}{\partial t}\right)T_1(y,z,t) = \mu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\frac{\partial V(y,z,t)}{\partial y}\,, \\ &\left(1+\lambda\frac{\partial}{\partial t}\right)T_2(y,z,t) = \mu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\frac{\partial V(y,z,t)}{\partial z}\,; \\ &y,t>0\,,\;\; z\in(0,d)\,, \end{aligned} \tag{15}$$

$$\begin{aligned} &V(y,z,0) = \left.\frac{\partial V(y,z,t)}{\partial t}\right|_{t=0} = 0\,, \\ &T_1(y,z,0) = T_2(y,z,0) = 0\,;\;\; y>0\,,\;\; z\in[0,d]\,, \end{aligned} \tag{16}$$

$$\begin{aligned} &\left(1+\lambda\frac{\partial}{\partial t}\right)T_1(0,z,t) = \mu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\left.\frac{\partial V(y,z,t)}{\partial y}\right|_{y=0} \\ &= f\,\mathrm{e}^{\mathrm{i}\omega t}\,;\;\; z\in(0,d)\,,\;\; t>0\,, \end{aligned} \tag{17}$$

$$V(y,0,t) = V(y,d,t) = 0\,;\;\; y>0\,,\;\; t\ge 0\,, \tag{18}$$

$$\begin{aligned} &V(y,z,t),\, \frac{\partial V(y,z,t)}{\partial y} \to 0 \;\text{ as }\; y\to\infty\,; \\ &z\in[0,d]\,,\;\; t\ge 0\,. \end{aligned} \tag{19}$$

3.1. Calculations of Velocity Field

In order to determine the solution of problem (14)–(19), we use the Fourier and Laplace transforms [31–33]. Multiplying both sides of (14) by $\sqrt{2/\pi}\cos(y\xi)\sin(\alpha_k z)$, where $\alpha_k = k\pi/d$, integrating with respect to y and z from 0 to ∞ and 0 to d, respectively, and using the corresponding boundary conditions, we find that

$$\begin{aligned} &\lambda\frac{\partial^2 V_k(\xi,t)}{\partial t^2} + \left[1+\alpha(\xi^2+\alpha_k^2)\right]\frac{\partial V_k(\xi,t)}{\partial t} \\ &+\nu(\xi^2+\alpha_k^2)V_k(\xi,t) = \frac{f}{\rho}\sqrt{\frac{2}{\pi}}\left[\frac{(-1)^k-1}{\alpha_k}\right]\mathrm{e}^{\mathrm{i}\omega t}\,, \end{aligned} \tag{20}$$

where $\alpha = \nu\lambda_r$ and the double Fourier cosine and sine transforms

$$\begin{aligned} &V_k(\xi,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty\int_0^d V(y,z,t)\cos(y\xi)\sin(\alpha_k z)\,\mathrm{d}z\,\mathrm{d}y\,, \\ &k = 1,2,3,\ldots\,, \end{aligned} \tag{21}$$

of the function $V(y,z,t)$ satisfy the initial conditions

$$V_k(\xi,0) = \left.\frac{\partial V_k(\xi,t)}{\partial t}\right|_{t=0} = 0\,,\;\; k=1,2,3,\ldots\,. \tag{22}$$

Applying the Laplace transform to (20) and using (22), we get

$$\begin{aligned} \bar{V}_k(\xi,q) = &\frac{f}{\rho}\sqrt{\frac{2}{\pi}}\left[\frac{(-1)^k-1}{\alpha_k}\right]\Bigg((q-\mathrm{i}\omega) \\ &\cdot\left[\lambda q^2+\left[1+\alpha(\xi^2+\alpha_k^2)\right]q+\nu(\xi^2+\alpha_k^2)\right]\Bigg)^{-1}\,, \end{aligned} \tag{23}$$

where $\bar{V}_k(\xi,q)$ is the Laplace transform of $V_k(\xi,t)$. Now, applying the inverse Laplace transform to (23) and inverting the result by means of Fourier sine and cosine formulae [32, 33], we obtain a first form of the complex velocity $V(y,z,t)$, namely

$$\begin{aligned} V(y,z,t) = &-\frac{8f}{\rho\pi d}\sum_{k=1}^\infty\frac{\sin(\alpha_m z)}{\alpha_m} \\ &\cdot\int_0^\infty f_m(\xi,t)\cos(y\xi)\,\mathrm{d}\xi\,, \end{aligned} \tag{24}$$

where $m = 2k-1$ and $f_m(\xi,t)$ is the inverse Laplace transform of the function

$$\begin{aligned} F_m(\xi,q) = &\Bigg((q-\mathrm{i}\omega)\left[\lambda q^2+\left[1+\alpha(\xi^2+\alpha_m^2)\right]q\right. \\ &\left.+\nu(\xi^2+\alpha_m^2)\right]\Bigg)^{-1}\,. \end{aligned}$$

By setting $d = 2h$ and changing the origin of the coordinate system at the middle of the channel (taking

$z = z^* + h$ and dropping out the $*$ notation), the complex velocity can be written in the more suitable form

$$V(y,z,t) = \frac{4f}{\rho\pi h}\sum_{k=1}^{\infty}\frac{(-1)^k\cos(\gamma_m z)}{\gamma_m} \cdot \int_0^{\infty} f_m(\xi,t)\cos(y\xi)\,\mathrm{d}\xi\,, \tag{25}$$

where $\gamma_m = \frac{(2k-1)\pi}{2h}$.

In order to determine the function $f_m(\xi,t)$, we firstly write $F_m(\xi,q)$ in the form

$$F_m(\xi,q) = \frac{2}{b_m}\frac{1}{q-\mathrm{i}\omega}\frac{\frac{b_m}{2\lambda}}{\left(q+\frac{a_m}{2\lambda}\right)^2-\frac{b_m^2}{4\lambda^2}}\,, \tag{26}$$

where $a_m = a_m(\xi) = 1+\alpha(\xi^2+\gamma_m^2)$ and

$$b_m = b_m(\xi) = \sqrt{[1+\alpha(\xi^2+\gamma_m^2)]^2-4\nu\lambda(\xi^2+\gamma_m^2)}\,.$$

Applying the inverse Laplace transform to (26), it results that

$$f_m(\xi,t) = \frac{2}{b_m(\xi)}\int_0^t \mathrm{e}^{\mathrm{i}\omega(t-s)}\,\mathrm{sh}\left(\frac{b_m(\xi)}{2\lambda}s\right) \cdot \exp\left(-\frac{a_m(\xi)}{2\lambda}s\right)\mathrm{d}s\,. \tag{27}$$

In view of (A1) from the Appendix, lengthy but straightforward computations allow us to write $f_m(\xi,t)$ in the form

$$\begin{aligned} f_m(\xi,t) &= \frac{\omega a_m\sin(\omega t)+\left[\nu(\xi^2+\gamma_m^2)-\lambda\omega^2\right]\cos(\omega t)}{\left[\nu(\xi^2+\gamma_m^2)-\lambda\omega^2\right]^2+\omega^2a_m^2} \\ &+\left[\frac{\lambda\omega^2-\nu(\xi^2+\gamma_m^2)}{\left[\nu(\xi^2+\gamma_m^2)-\lambda\omega^2\right]^2+\omega^2a_m^2}\,\mathrm{ch}\left(\frac{b_mt}{2\lambda}\right)\right. \\ &\left.+\frac{a_m\left[\lambda\omega^2-\nu(\xi^2+\gamma_m^2)\right]-2\lambda\omega^2a_m}{b_m\left[\nu(\xi^2+\gamma_m^2)-\lambda\omega^2\right]^2+\omega^2a_m^2}\,\mathrm{sh}\left(\frac{b_mt}{2\lambda}\right)\right]\mathrm{e}^{-\frac{a_m}{2\lambda}t} \\ &+\mathrm{i}\left\{\frac{\left[\nu(\xi^2+\gamma_m^2)-\lambda\omega^2\right]\sin(\omega t)-\omega a_m\cos(\omega t)}{\left[\nu(\xi^2+\gamma_m^2)-\lambda\omega^2\right]^2+\omega^2a_m^2}\right. \\ &+\left[\frac{2\lambda\omega\left[\lambda\omega^2-\nu(\xi^2+\gamma_m^2)\right]+\omega a_m^2}{b_m\left[\nu(\xi^2+\gamma_m^2)-\lambda\omega^2\right]^2+\omega^2a_m^2}\,\mathrm{sh}\left(\frac{b_mt}{2\lambda}\right)\right. \\ &\left.\left.+\frac{\omega a_m}{\left[\nu(\xi^2+\gamma_m^2)-\lambda\omega^2\right]^2+\omega^2a_m^2}\,\mathrm{ch}\left(\frac{b_mt}{2\lambda}\right)\right]\mathrm{e}^{-\frac{a_m}{2\lambda}t}\right\}. \end{aligned} \tag{28}$$

This expression can be further processed to give the more suitable form

$$\begin{aligned} f_m(\xi,t) &= \frac{\lambda_r\omega(\xi^2+d_m^2)+c}{\beta\left[(\xi^2+d_m^2)^2+c^2\right]}\sin(\omega t) \\ &+\frac{(\xi^2+d_m^2)-\lambda_r\omega c}{\beta\left[(\xi^2+d_m^2)^2+c^2\right]}\cos(\omega t) \\ &+\left[M_m\,\mathrm{ch}\left(\frac{b_mt}{2\lambda}\right)+N_m\,\mathrm{sh}\left(\frac{b_mt}{2\lambda}\right)\right]\exp\left(-\frac{a_mt}{2\lambda}\right) \\ &+\mathrm{i}\left\{\frac{(\xi^2+d_m^2)-\lambda_r\omega c}{\beta\left[(\xi^2+d_m^2)^2+c^2\right]}\sin(\omega t)\right. \\ &-\frac{\lambda_r\omega(\xi^2+d_m^2)+c}{\beta\left[(\xi^2+d_m^2)^2+c^2\right]}\cos(\omega t) \\ &\left.+\left[P_m\,\mathrm{ch}\left(\frac{b_mt}{2\lambda}\right)+Q_m\,\mathrm{sh}\left(\frac{b_mt}{2\lambda}\right)\right]\exp\left(-\frac{a_mt}{2\lambda}\right)\right\}, \end{aligned} \tag{29}$$

where

$$\beta = \nu(1+\lambda_r^2\omega^2)\,,$$

$$c = \frac{(1+\lambda\lambda_r\omega^2)\omega}{\beta}\,,\quad d_m^2 = \gamma_m^2+\frac{(\lambda_r-\lambda)\omega^2}{\beta}\,,$$

$$M_m = M_m(\xi) = \frac{\lambda_r\omega c-(\xi^2+d_m^2)}{\beta\left[(\xi^2+d_m^2)^2+c^2\right]}\,,$$

$$N_m = N_m(\xi) = \frac{\left[\frac{c}{\omega}+\lambda_r(\xi^2+d_m^2)\right]\left[\nu a-\lambda\omega^2-\nu(\xi^2+d_m^2)\right]}{\beta b_m\left[(\xi^2+d_m^2)^2+c^2\right]}\,,$$

$$P_m = P_m(\xi) = \frac{\lambda_r\omega(\xi^2+d_m^2)+c}{\beta\left[(\xi^2+d_m^2)^2+c^2\right]} \quad \text{and}$$

$$Q_m = Q_m(\xi) = \left(\nu\omega\lambda_r^2(\xi^2+d_m^2)^2+2(\nu\lambda_rc-\lambda\omega)(\xi^2+d_m^2)+\frac{\nu c^2}{\omega}+2\lambda\lambda_r\omega^2c\right)\left(\beta b_m\left[(\xi^2+d_m^2)^2+c^2\right]\right)^{-1}.$$

Of course, the velocities $u_c(y,z,t)$ and $u_s(y,z,t)$ corresponding to cosine or sine type oscillations on the boundary are obtained by introducing (29) into (25) and taking the real part, respectively the imaginary part of the function $f_m(\xi,t)$. Their expressions can be written as a sum of steady-state and transient solutions, namely

$$u_c(y,z,t) = u_c^s(y,z,t) + u_c^t(y,z,t)\,, \tag{30a}$$
$$u_s(y,z,t) = u_s^s(y,z,t) + u_s^t(y,z,t)\,. \tag{30b}$$

It is worth pointing out that in view of (A2) and (A3) from the Appendix, the steady-state solutions can be written in the simple forms

$$u_c^s(y,z,t) = \frac{2f}{\mu h}\frac{1}{\sqrt{1+\lambda_r^2\omega^2}} \cdot \sum_{k=1}^{\infty}\frac{(-1)^k\cos(\gamma_m z)}{\gamma_m}\frac{\mathrm{e}^{-B_m y}}{\sqrt{A_m^2+B_m^2}}\cos(\omega t - A_m y - \phi_m)\,, \tag{31}$$

$$u_s^s(y,z,t) = \frac{2f}{\mu h}\frac{1}{\sqrt{1+\lambda_r^2\omega^2}} \cdot \sum_{k=1}^{\infty}\frac{(-1)^k\cos(\gamma_m z)}{\gamma_m}\frac{\mathrm{e}^{-B_m y}}{\sqrt{A_m^2+B_m^2}}\sin(\omega t - A_m y - \phi_m)\,, \tag{32}$$

where $2A_m^2 = \sqrt{d_m^4+c^2} - d_m^2$, $2B_m^2 = \sqrt{d_m^4+c^2} + d_m^2$, and $\tan\phi_m = (A_m + \lambda_r\omega B_m)(B_m - \lambda_r\omega A_m)^{-1}$.

As expected, they differ by a phase shift. This is not true for transient components. Therefore we separately present here the starting solutions for both types of oscillations. As regards, the transient solutions from (29) and (25), it results that

$$u_c^t(y,z,t) = \frac{4f}{\rho\pi h}\sum_{k=1}^{\infty}\frac{(-1)^k\cos(\gamma_m z)}{\gamma_m} \cdot \int_0^{\infty}\left[M_m(\xi)\,\mathrm{ch}\left(\frac{b_m t}{2\lambda}\right) + N_m(\xi)\,\mathrm{sh}\left(\frac{b_m t}{2\lambda}\right)\right] \cdot \exp\left(-\frac{a_m t}{2\lambda}\right)\cos(y\xi)\,\mathrm{d}\xi\,, \tag{33}$$

$$u_s^t(y,z,t) = \frac{4f}{\rho\pi h}\sum_{k=1}^{\infty}\frac{(-1)^k\cos(\gamma_m z)}{\gamma_m} \cdot \int_0^{\infty}\left[P_m(\xi)\,\mathrm{ch}\left(\frac{b_m t}{2\lambda}\right) + Q_m(\xi)\,\mathrm{sh}\left(\frac{b_m t}{2\lambda}\right)\right] \exp\left(-\frac{a_m t}{2\lambda}\right)\cos(y\xi)\,\mathrm{d}\xi\,. \tag{34}$$

The starting solutions are usually important for those who want to eliminate the transients from their experiments. They describe the motion of the fluid some time after its initiation. After that time, they tend to steady-state solutions that are periodic in time and independent of the initial conditions. However, they satisfy the governing equations and boundary conditions. As a check of our results, by letting $\lambda \to 0$, $\lambda_r \to 0$, or $\lambda, \lambda_r \to 0$ into general solutions (30), we attain to the known solutions for second-grade [24], Maxwell [26] and, respectively, Newtonian fluids [16]. Equation (32), for instance, reduces to [24, (29) with $\varphi = 0$] if $\lambda \to 0$ while (30b), for instance reduces to [26, (26)] if $\lambda_r \to 0$. Also (30b) reduces to [16, (14)] if $\lambda, \lambda_r \to 0$ and corresponds to an oscillating shear on the boundary of the form $f\sin\omega t$. Finally, the required time to reach the steady state will be determined by graphical illustrations.

3.2. Calculation of Shear Stress

In order to be able to find the shear stresses in planes parallel to the bottom wall or on the side walls, we need the general expressions of $\tau_1(y,z,t)$ and $\tau_2(y,z,t)$. To do that, we introduce $V(y,z,t)$, given by (25), into (15), apply the Laplace transform and use the corresponding initial conditions (16). In order to avoid repetition, we here present the final results for $\tau_1(y,z,t)$ only:

$$\tau_{1c}(y,z,t) = \tau_{1c}^s(y,z,t) + \tau_{1c}^t(y,z,t)\,, \tag{35}$$
$$\tau_{1s}(y,z,t) = \tau_{1s}^s(y,z,t) + \tau_{1s}^t(y,z,t)\,. \tag{36}$$

Equations (35) and (36) clearly show that shear stresses are also presented as sum of steady-state and transient solutions. In view of (A4) and (A5) from the Appendix, the steady-state solutions can be written in simple forms as

$$\tau_{1c}^s(y,z,t) = \frac{2f}{h\sqrt{1+\lambda^2\omega^2}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\cos(\gamma_m z)}{\gamma_m} \cdot \mathrm{e}^{-B_m y}\cos(\omega t - A_m y - \psi)\,, \tag{37}$$

$$\tau_{1s}^s(y,z,t) = \frac{2f}{h\sqrt{1+\lambda^2\omega^2}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\cos(\gamma_m z)}{\gamma_m} \cdot \mathrm{e}^{-B_m y}\sin(\omega t - A_m y - \psi)\,, \tag{38}$$

where $\tan\psi = \lambda\omega$. As expected, they also differ by a phase shift.

The transient solution components are

$$\begin{aligned}\tau_{1c}^t(y,z,t) &= \frac{2f}{h(1+\lambda^2\omega^2)}\sum_{k=1}^{\infty}\frac{(-1)^k\cos(\gamma_m z)}{\gamma_m}\mathrm{e}^{-y\gamma_m - \frac{t}{\lambda}} \\ &+ \frac{4f}{\pi h\lambda}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\cos(\gamma_m z)}{\gamma_m}\int_0^{\infty}\left\{\left(\lambda_r\nu M_m + \frac{(2-a_m)M_m - b_m N_m}{2(\xi^2+\gamma_m^2)}\right)\mathrm{ch}\left(\frac{b_m t}{2\lambda}\right)\right. \\ &+ \left.\left(\lambda_r\nu N_m + \frac{(2-a_m)N_m - b_m M_m}{2(\xi^2+\gamma_m^2)}\right)\mathrm{sh}\left(\frac{b_m t}{2\lambda}\right)\right\} \\ &\cdot \exp\left(-\frac{a_m t}{2\lambda}\right)\xi\sin(y\xi)\,\mathrm{d}\xi\,,\end{aligned} \tag{39}$$

$$\tau_{1s}^{t}(y,z,t) = \frac{2f\lambda\omega}{h(1+\lambda^2\omega^2)}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\cos(\gamma_m z)}{\gamma_m} \cdot \mathrm{e}^{-y\gamma_m-\frac{t}{\lambda}} + \frac{4f}{\pi h\lambda}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\cos(\gamma_m z)}{\gamma_m} \cdot \int_0^{\infty}\left\{\left(\lambda_r\nu P_m + \frac{(2-a_m)P_m - b_m Q_m}{2(\xi^2+\gamma_m^2)}\right)\mathrm{ch}\left(\frac{b_m t}{2\lambda}\right) + \left(\lambda_r\nu Q_m + \frac{(2-a_m)Q_m - b_m P_m}{2(\xi^2+\gamma_m^2)}\right)\mathrm{sh}\left(\frac{b_m t}{2\lambda}\right)\right\} \cdot \exp\left(-\frac{a_m t}{2\lambda}\right)\xi\sin(y\xi)\,\mathrm{d}\xi\,. \quad (40)$$

As a check of general results, by letting $\lambda \to 0$, $\lambda_r \to 0$, or $\lambda, \lambda_r \to 0$ into general solutions (35) and (36), for instance, the known solutions for second-grade [24, (30) and (31)], Maxwell [26, (28)], and Newtonian fluids [16, (17) and (19)], respectively, can be recovered.

4. Limiting Case $h \to \infty$ (Flow Over an Infinite Plate)

In order to determine the influence of the side walls on the fluid motion, we need the similar solutions corresponding to the motion over an infinite plate that applies shear of the same form (6) or (7) to the fluid. By making $h \to \infty$ into general solutions (30a), (30b) and bearing in mind (A6) from the Appendix and the fact that

$$\gamma_m \to 0\,,\quad d_m^2 \to d^2 = \frac{(\lambda_r-\lambda)\omega^2}{\nu(1+\lambda_r^2\omega^2)}\,,$$

$$2A_m^2 \to 2A^2 = \sqrt{d^4+c^2} - d^2\,,$$

$$2B_m^2 \to 2B^2 = \sqrt{d^4+c^2} + d^2\,,$$

$$\tan\phi_m \to \tan\phi = \frac{A+\lambda_r\omega B}{B-\lambda_r\omega B} = \frac{\sqrt{1+\lambda^2\omega^2}+\lambda\omega\sqrt{1+\lambda_r^2\omega^2}}{\sqrt{1+\lambda_r^2\omega^2}-\lambda_r\omega\sqrt{1+\lambda^2\omega^2}}\,,$$

we find for the velocities $\mathbf{v}_c(y,t)$ and $\mathbf{v}_s(y,t)$ the expressions

$$\mathbf{v}_c(y,t) = -\frac{f}{\mu}\sqrt{\frac{\nu}{\omega}}\frac{\mathrm{e}^{-By}}{\sqrt[4]{(1+\lambda^2\omega^2)(1+\lambda_r^2\omega^2)}} \cdot \cos(\omega t - Ay - \phi) + \frac{2f}{\mu\nu\pi}\frac{1}{1+\lambda_r^2\omega^2}\int_0^{\infty}\frac{\cos(y\xi)}{(\xi^2+d^2)^2+c^2} \cdot \left[(\nu\xi^2-\lambda\omega^2)\,\mathrm{ch}\left(\frac{b(\xi)}{2\lambda}t\right) + \frac{a(\xi)}{b(\xi)}(\nu\xi^2+\lambda\omega^2) \cdot \mathrm{sh}\left(\frac{b(\xi)}{2\lambda}t\right)\right]\exp\left(-\frac{a(\xi)}{2\lambda}t\right)\mathrm{d}\xi\,, \quad (41)$$

$$\mathbf{v}_s(y,t) = -\frac{f}{\mu}\sqrt{\frac{\nu}{\omega}}\frac{\mathrm{e}^{-By}}{\sqrt[4]{(1+\lambda^2\omega^2)(1+\lambda_r^2\omega^2)}} \cdot \sin(\omega t - Ay - \phi) - \frac{2f}{\mu\nu\pi}\frac{\omega}{1+\lambda_r^2\omega^2}\int_0^{\infty}\frac{\cos(y\xi)}{(\xi^2+d^2)^2+c^2} \cdot \left[a(\xi)\,\mathrm{ch}\left(\frac{b(\xi)}{2\lambda}t\right) + \frac{a^2(\xi)-2\lambda(\nu\xi^2-\lambda\omega^2)}{b(\xi)} \cdot \mathrm{sh}\left(\frac{b(\xi)}{2\lambda}t\right)\right]\exp\left(-\frac{a(\xi)}{2\lambda}t\right)\mathrm{d}\xi\,. \quad (42)$$

As expected, these solutions are identical to those obtained in [28, (27) and (28)]. In order to show this for steady-state solutions, it is enough to observe that between the two angles there exists the relation $\varphi = \frac{\pi}{2} - \phi$.

Also, by letting $\lambda_r \to 0$ into (41) and (42), we obtain the known results for Maxwell fluids [27, (23) and (22)].

Similarly, we find for the shear stresses $\tau_c(y,t)$ and $\tau_s(y,t)$ the expressions

$$\tau_c(y,t) = \frac{f}{\sqrt{1+\lambda^2\omega^2}}\mathrm{e}^{-By}\cos(\omega t - Ay - \psi) - \frac{f}{1+\lambda^2\omega^2}\mathrm{e}^{-\frac{t}{\lambda}} + \frac{2f}{\pi}\frac{1}{\nu^2(1+\lambda_r^2\omega^2)} \cdot \int_0^{\infty}\frac{\sin(y\xi)}{\xi((\xi^2+d^2)^2+c^2)}\left\{\omega^2 a(\xi)\,\mathrm{ch}\left(\frac{b(\xi)t}{2\lambda}\right) - \frac{a^2(\xi)\omega^2+2\nu\xi^2(\nu\xi^2-\lambda\omega^2)}{b(\xi)}\mathrm{sh}\left(\frac{b(\xi)t}{2\lambda}\right)\right\} \cdot \exp\left(-\frac{a(\xi)t}{2\lambda}\right)\mathrm{d}\xi\,, \quad (43)$$

$$\tau_s(y,t) = \frac{f}{\sqrt{1+\lambda^2\omega^2}}\mathrm{e}^{-By}\sin(\omega t - Ay - \psi) + \frac{f\lambda\omega}{1+\lambda^2\omega^2}\mathrm{e}^{-\frac{t}{\lambda}} + \frac{2f}{\pi}\frac{\omega}{\nu^2(1+\lambda_r^2\omega^2)} \cdot \int_0^{\infty}\frac{\sin(y\xi)}{\xi((\xi^2+d^2)^2+c^2)}\left\{(\nu\xi^2-\lambda\omega^2)\,\mathrm{ch}\left(\frac{b(\xi)t}{2\lambda}\right) + \frac{a(\xi)(\nu\xi^2+\lambda\omega^2)}{b(\xi)}\mathrm{sh}\left(\frac{b(\xi)t}{2\lambda}\right)\right\}\exp\left(-\frac{a(\xi)t}{2\lambda}\right)\mathrm{d}\xi\,. \quad (44)$$

They are identical to those obtained in [28, (42) and (43)]. Again, by letting $\lambda_r \to 0$ in (41) and (42),

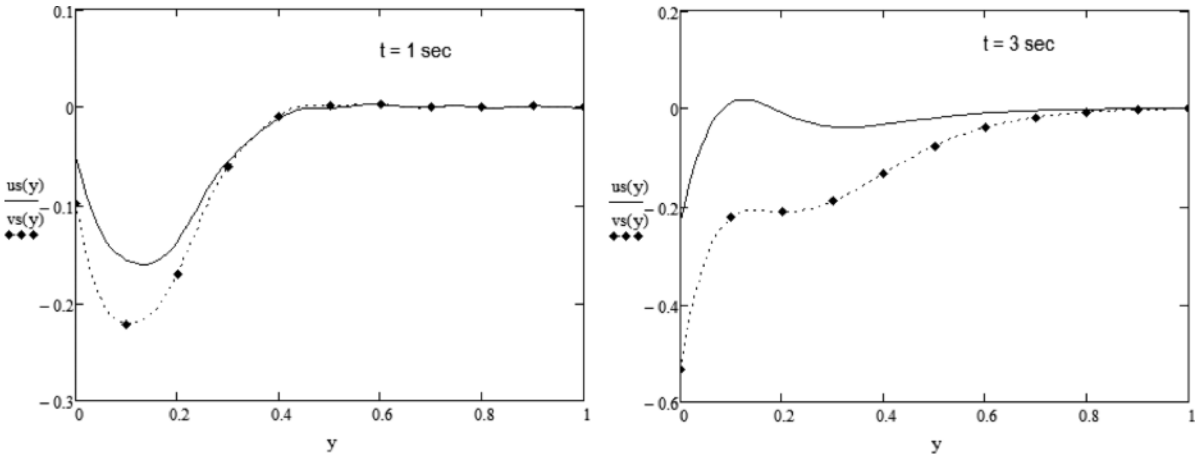

Fig. 1. Profile of velocities $u_s(y,0,t)$ given by (30b)-curve $u_s(y)$ and $\mathbf{v}_s(y,t)$ given by (42)-curve $\mathbf{v}_s(y)$ with $f = 50$, $\nu = 0.024$, $\mu = 2.28$, $\lambda = 1.2$, $\lambda_r = 0.8$, $\omega = 4$, and $h = 0.25$ for sine oscillations of the shear.

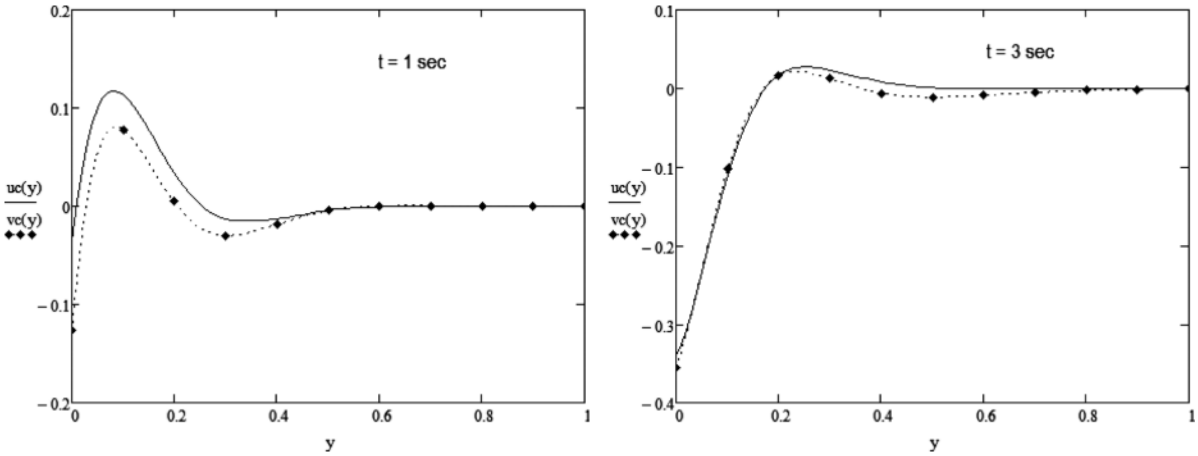

Fig. 2. Profile of velocities $u_c(y,0,t)$ given by (30a)-curve $u_c(y)$ and $\mathbf{v}_c(y,t)$ given by (41)-curve $\mathbf{v}_c(y)$ with $f = 50$, $\nu = 0.024$, $\mu = 2.28$, $\lambda = 1.2$, $\lambda_r = 0.8$, $\omega = 4$, and $h = 0.25$ for cosine oscillations of the shear.

the known results for shear stress of Maxwell fluids model [27, (36) and (35)] can be recovered.

5. Numerical Results and Conclusions

In this paper, unsteady motions of Oldroyd-B fluids between two parallel walls perpendicular to a plate that applies two types of shears to the fluid are studied using integral transforms. Exact solutions are obtained both for velocity and non-trivial shear stresses. They are presented in simple forms as a sum of steady-state and transient solutions and can easily be particularized to give the similar solutions for Maxwell, second-grade, and Newtonian fluids. Known solutions for the motion over an infinite plate, applying the same shears to the fluid, are recovered as limiting cases of general solutions. To analyse the influence of side walls on the motion of the fluid and to explore some of the relevant physical aspects of the obtained results, the diagrams of velocities $u_s(y,0,t)$ and $u_c(y,0,t)$ in the middle of the channel as well as those of $\mathbf{v}_s(y,t)$ and $\mathbf{v}_c(y,t)$, corresponding to the motion over an infinite plate, have been drawn against y for the same values of t and of the material constants. As it results from Figures 1 and 2, there is a significant effect of walls. At low values of t, the influence of the side walls on the fluid motion near the bottom plate is stronger for sine oscillations as compare to cosine oscillations. This is obvious be-

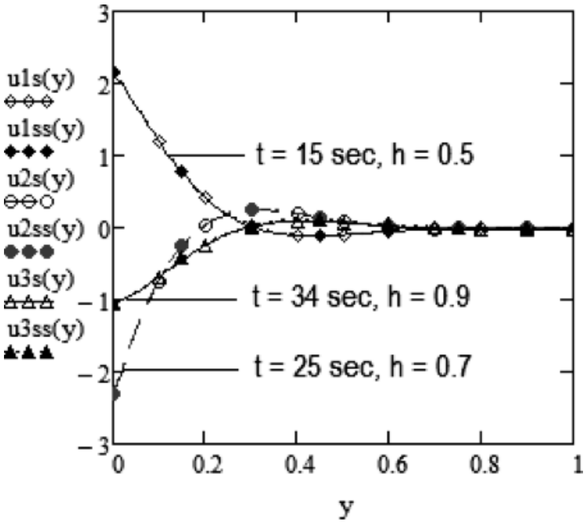

Fig. 3. Required time to reach the steady state for the sine oscillations of the shear stress for $f = 50$, $\nu = 0.024$, $\mu = 2.28$, $\lambda = 1.2$, $\lambda_r = 0.8$, $\omega = 4$, and different values of h.

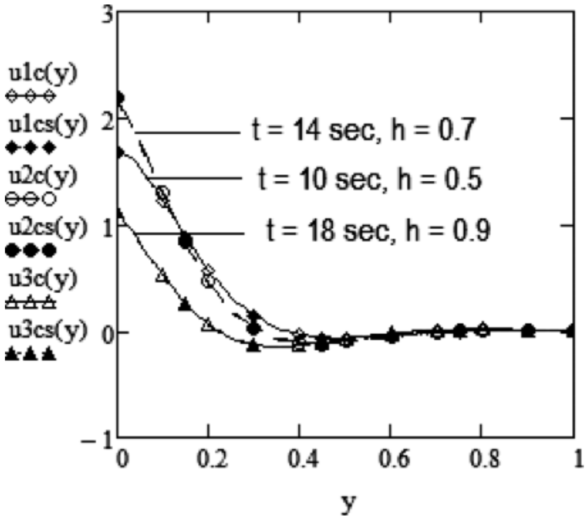

Fig. 4. Required time to reach the steady state for the cosine oscillations of the shear stress for $f = 50$, $\nu = 0.024$, $\mu = 2.28$, $\lambda = 1.2$, $\lambda_r = 0.8$, $\omega = 4$, and different values of h.

cause at $t = 0$ the shear stress on the boundary is zero for sine oscillations.

In practice, it is also important to know the required time to reach the steady state. This time has been determined in Figures 3 and 4 for different values of h. As expected, the required time to reach the steady state increases if the distance between the side walls increases. Furthuremore, this time value is greater for the sine oscillations as compare to the cosine oscillations of the shear stress on the boundary. As it can be seen in Figures 5 and 6, this time increases if the frequency ω of the oscillation decreases.

To determine the distance between the side walls for which the measured value of the velocity in the middle of the channel is unaffected by the presence of the side walls (approximately it is equal to the velocity corresponding to the motion over an infinite plate), Figures 7 and 8 have been prepared.

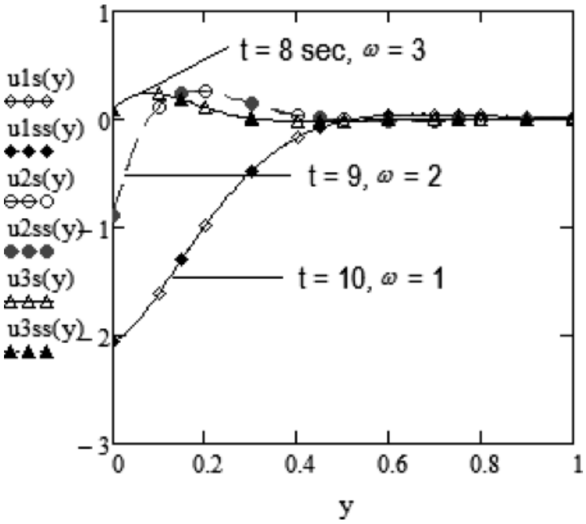

Fig. 5. Required time to reach the steady state for the sine oscillations of the shear stress for $f = 50$, $\nu = 0.024$, $\mu = 2.28$, $\lambda = 1.2$, $\lambda_r = 0.8$, $h = 0.35$, and different values of ω.

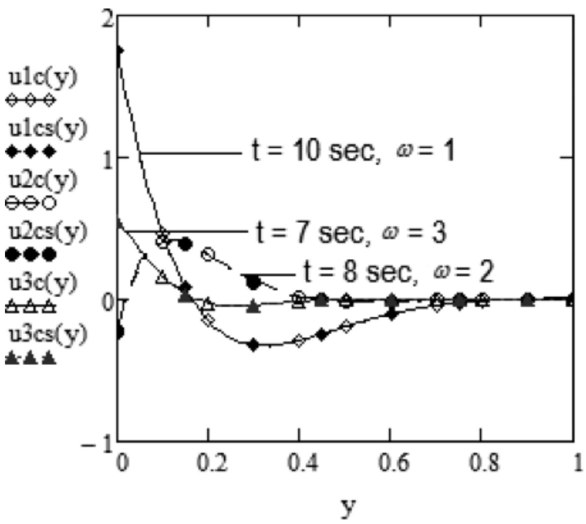

Fig. 6. Required time to reach the steady state for the cosine oscillations of the shear stress for $f = 50$, $\nu = 0.024$, $\mu = 2.28$, $\lambda = 1.2$, $\lambda_r = 0.8$, $h = 0.35$, and different values of ω.

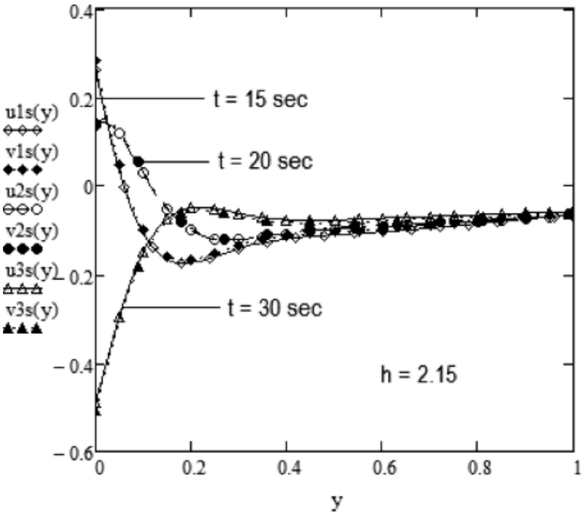

Fig. 7. Profile of velocities $u_s(y,0,t)$ given by (30b)-curve $u_{1s}(y)$, $u_{2s}(y)$, $u_{3s}(y)$, and $\mathbf{v}_s(y,t)$ given by (42)-curve $\mathbf{v}_{1s}(y)$, $\mathbf{v}_{2s}(y)$, $\mathbf{v}_{3s}(y)$ for $f = 50$, $\nu = 0.024$, $\mu = 2.28$, $\lambda = 1.2$, $\lambda_r = 0.8$, $\omega = 4$, and different values of t.

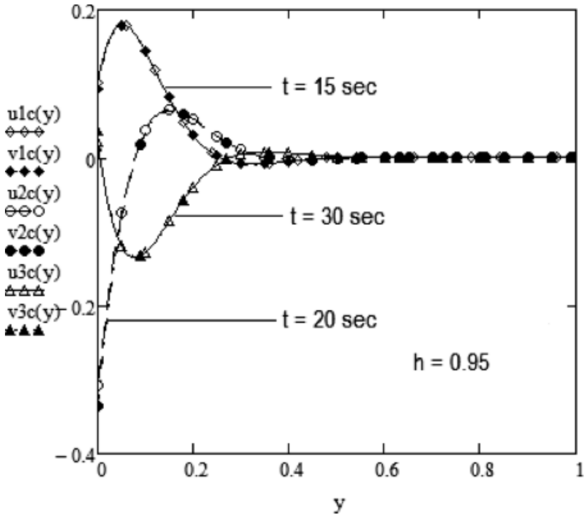

Fig. 8. Profile of velocities $u_c(y,0,t)$ given by (30a)-curve $u_{1c}(y)$, $u_{2c}(y)$, $u_{3c}(y)$, and $\mathbf{v}_c(y,t)$ given by (41)-curve $\mathbf{v}_{1c}(y)$, $\mathbf{v}_{2c}(y)$, $\mathbf{v}_{3c}(y)$ for $f = 50$, $\nu = 0.024$, $\mu = 2.28$, $\lambda = 1.2$, $\lambda_r = 0.8$, $\omega = 4$, and different values of t.

This distance as it results from graphs, is $h_s = 2.15$ for sine oscillations and $h_c = 0.95$ for the cosine oscillations of the shear stress. The units of the material constants in Figures 1–8 are SI units.

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Appendix

$$\int_0^t \mathrm{e}^{-(\mathrm{i}\omega+\frac{a_m}{2\lambda})s}\,\mathrm{sh}\left(\frac{b_m}{2\lambda}s\right)\mathrm{d}s = \frac{1}{2\left[\lambda\omega^2-\nu(\xi^2+\gamma_m^2)-\mathrm{i}\omega a_m\right]}\cdot\left\{\left[b_m\,\mathrm{ch}\left(\frac{b_m}{2\lambda}t\right) + 2\lambda\left(\mathrm{i}\omega+\frac{a_m}{2\lambda}\right)\mathrm{sh}\left(\frac{b_m}{2\lambda}t\right)\right]\mathrm{e}^{-(\mathrm{i}\omega+\frac{a_m}{2\lambda})t} - b_m\right\}, \tag{A1}$$

$$\int_0^\infty \frac{(\xi^2+d^2)\cos(y\xi)}{(\xi^2+d^2)^2+c^2}\,\mathrm{d}\xi = \frac{\pi\,\mathrm{e}^{-yB}}{2(A^2+B^2)}\left[B\cos(yA)-A\sin(yA)\right], \tag{A2}$$

$$\int_0^\infty \frac{\cos(y\xi)}{(\xi^2+d^2)^2+c^2}\,\mathrm{d}\xi = \frac{\pi\,\mathrm{e}^{-yB}}{2c(A^2+B^2)}\left[A\cos(yA)+B\sin(yA)\right], \tag{A3}$$

$$\int_0^\infty \frac{(\xi^2+d^2)\xi\sin(y\xi)}{(\xi^2+d^2)^2+c^2}\,\mathrm{d}\xi = \frac{\pi}{2}\,\mathrm{e}^{-yB}\cos(yA), \tag{A4}$$

$$\int_0^\infty \frac{\xi\sin(y\xi)}{(\xi^2+d^2)^2+c^2}\,\mathrm{d}\xi = \frac{\pi}{2c}\,\mathrm{e}^{-yB}\sin(yA), \tag{A5}$$

where $2A^2 = \sqrt{d^4+c^2}-d^2$, $2B^2 = \sqrt{d^4+c^2}+d^2$.

$$\lim_{h\to\infty}\frac{2}{h}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos(\gamma_m z)}{\gamma_m} = 1. \tag{A6}$$

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