Klein–Gordon Solutions for a Yukawa-like Potential

Sameer M. Ikhdair[a,b] and Majid Hamzavi[c]

[a] Department of Physics, Near East University, 922022 Nicosia, Northern Cyprus, Mersin 10, Turkey

[b] Department of Physics, Faculty of Science, an-Najah National University, Nablus, West Bank, Palestine

[c] Department of Science and Engineering, Abhar Branch, Islamic Azad University, Abhar, Iran

Reprint requests to S. M. I.; E-mail: sikhdair@neu.edu.tr and sikhdair@gmail.com

Z. Naturforsch. **68a,** 715 – 724 (2013) / DOI: 10.5560/ZNA.2013-0045
Received March 15, 2013 / revised June 24, 2013 / published online October 30, 2013

The Klein–Gordon equation for a recently proposed Yukawa-type potential is solved with any orbital quantum number l. In the equally mixed scalar-vector potential fields $S(r) = \pm V(r)$, the approximate energy eigenvalues and their wave functions for a particle and anti-particle are obtained by means of the parametric Nikiforov–Uvarov method. The non-relativistic solutions are also investigated. It is found that the present analytical results are in exact agreement with the previous ones.

Key words: Klein–Gordon Equation; Yukawa Potential; Parametric Nikiforov–Uvarov Method; Approximation Schemes.

PACS numbers: 03.65.Ge; 03.65.Fd; 03.65.Pm; 02.30.Gp

1. Introduction

Relativistic wave equations and particularly Dirac and Klein–Gordon (KG) equations have been of interest for theoretical physicists in many branches of physics [1, 2]. In recent years, there has been an increased interest in finding exact solutions to relativistic spinless KG particles with various vector and scalar potentials [3 – 8]. The most commonly used techniques to explore these wave equations are the Nikiforov–Uvarov (NU) method [9 – 12], the super-symmetric quantum mechanics method [13, 14], the point canonical transformation [15, 16], the asymptotic iteration method [17, 18], the proper quantization rule [19, 20], the shifted large $1/N$ expansion (SE) technique [21], and the ansatz approach [22].

Very recently, Hamzavi et al. [23, 24] presented exact spin-1/2 relativistic solutions for the Mie-type potential and approximate solutions for the Morse potential in the presence of a Coulomb-like tensor potential interaction. Moreover, Hamzavi et al. [25] introduced a novel potential, the so-called inversely quadratic Yukawa (IQY) potential. They solved it in the context of the Dirac equation with spin and pseudo-spin symmetric limits taking the Coulomb-like potential as a tensor interaction. Over the past few years, the non-relativistic and relativistic wave equations have been solved for various types of quantum potential models by different authors [26 – 47].

In this work, we introduce a novel potential in the form of a Yukawa potential very similar to the combinations of the IQY potential [25] and Yukawa potential [46] (i. e., IQY-plus-Yukawa potential) on the entire positive line range, $r \in (0,\infty)$, as shown in Figure 1. In short screening regime, it is an intermediate between $V_{\mathrm{IQY}}(r)$ and $V_{\mathrm{Y}}(r)$ as demonstrated in Figure 1. Further, in Figure 2, we also compare this potential with the Coulomb potential. The Yukawa-like potential has the form [48].

$$\begin{aligned} V(r) &= -V_0\left(1+\frac{1}{r}\,\mathrm{e}^{-\alpha r}\right)^2 \\ &= -\frac{A}{r^2}\,\mathrm{e}^{-2\alpha r}-\frac{B}{r}\,\mathrm{e}^{-\alpha r}-C\,, \\ A &= C = V_0\,,\ \ B = 2V_0\,, \end{aligned} \tag{1}$$

where α is the screening parameter, and V_0 is the coupling strength of the potential. It comes asymptotically to a finite value as $r \rightarrow \infty$ and goes to infinity at $r = 0$. A form of Yukawa potential has been earlier used by

Taseli [49] in obtaining a modified Laguerre basis for hydrogen-like systems. Also, Kermode et al. [50] have used different forms of the Yukawa potential to obtain the effective range functions.

The aim of the present work is to introduce a more general ansatz for the IQY potential studied in our previous work [25]. Further, we aim to investigate the bound states of a spin-0 particle in the field of Yukawa-type potential (1). We want to calculate the energy eigenvalues and the corresponding wave functions expressed in terms of the Jacobi polynomials.

This paper is organized as follows. In Section 2, we briefly introduce the Klein–Gordon equation with equal scalar and vector Yukawa-type potential for arbitrary orbital quantum number l. The parametric NU method is briefly introduced in Section 3. The energy eigenvalue equation and the corresponding eigenfunctions are obtained in Section 4. The Schrödinger and KG equations are studied for Yukawa-type, IQY, and Yukawa potentials in Section 5. We end with our conclusions in Section 6.

2. Klein–Gordon Equation for Equally Mixed Scalar-Vector Yukawa-Type Potential

In relativistic quantum mechanics, we usually use the KG equation for describing a scalar particle (i. e., the spin-0 particle dynamics). The discussion of the relativistic behaviour of spin-zero particles requires understanding the single particle spectrum and the exact solutions to the KG equation which are constructed by using the four-vector potential A_λ $(\lambda = 0,1,2,3)$ and the scalar potential $S(r)$. In order to simplify the analytical solution of the KG equation, the four-vector potential can be written as $A_\lambda = (A_0,0,0,0)$. The first component of the four-vector potential is represented by a vector potential $V(r)$, i. e., $A_0 = V(r)$. In this case, the motion of a relativistic spin-0 particle in a potential is always described by the KG equation with the potentials $V(r)$ and $S(r)$. For equally mixed scalar and vector potentials, $S(r) = \pm V(r)$ cases [42], the $(3+1)$-dimensional KG equation is reduced to a Schrödinger-type equation, and thereby the bound state solutions are easily obtained by using the well-known methods developed in non-relativistic quantum mechanics [51, 52].

Let us now consider the $(3+1)$-dimensional time-independent KG equation describing a scalar particle with Lorentz scalar $S(r)$ and Lorentz vector $V(r)$ potentials which takes the form [53, 54]

$$\begin{aligned}&\Big[c^2P_{\mathrm{op}}^2-(V(r)-E_{\mathrm{R}})^2\\&+\big(S(r)+Mc^2\big)^2\Big]\psi_{\mathrm{KG}}(\mathbf{r})=0,\end{aligned}\tag{2}$$

where M and E_{R} denote the reduced mass and relativistic binding energy of two interacting particles, respectively, with $\mathbf{P}_{\mathrm{op}} = -\mathrm{i}\hbar\nabla$ is the momentum operator. It would be natural to scale the potential terms in (2) so that in the non-relativistic limit the interaction potential becomes $V(r)$, not $2V(r)$. We will follow Alhaidari et al. [52] to reduce the above equation to the form [52–57]

$$\begin{aligned}&\left\{\nabla^2+\frac{1}{\hbar^2c^2}\left[\left(E_{\mathrm{R}}-\frac{1}{2}V(r)\right)^2\right.\right.\\&\left.\left.-\left(Mc^2+\frac{1}{2}S(r)\right)^2\right]\right\}\psi_{\mathrm{KG}}(\mathbf{r})=0.\end{aligned}\tag{3}$$

Now making use of the equal scalar and vector Yukawa-type functions $S(r)=\pm V(r)$, (3) recasts to

$$\begin{aligned}&\left\{\nabla^2-\frac{1}{\hbar^2c^2}\Big[M^2c^4-E_{\mathrm{R}}^2\pm V(r)\right.\\&\left.\cdot\left(Mc^2\pm E_{\mathrm{R}}\right)\Big]\right\}\psi_{\mathrm{KG}}(\mathbf{r})=0\end{aligned}\tag{4a}$$

and

$$\begin{aligned}\nabla^2=&\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right.\\&\left.+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right],\quad r^2=\sum_{j=1}^{3}x_j^2.\end{aligned}\tag{4b}$$

In addition, we take the interaction potential as in (1) and decompose the total wave function $\psi_{\mathrm{KG}}(\mathbf{r})$, with a given angular momentum l, as a product of a radial wave function $R_l(r)=g(r)/r$, and the angular dependent spherical harmonic functions $Y_l^m(\widehat{r})$ [54]

$$\psi_{\mathrm{KG}}(\mathbf{r})=\frac{g(r)}{r}Y_l^m(\widehat{r}),\tag{5}$$

with angular momentum quantum numbers being l and m. This reduces (4a) into the simple form

$$\begin{aligned}&\frac{\mathrm{d}^2g(r)}{\mathrm{d}r^2}-\frac{1}{\hbar^2c^2}\left[M^2c^4-E_{\mathrm{R}}^2\pm V(r)\left(Mc^2\pm E_{\mathrm{R}}\right)\right.\\&\left.+\frac{l(l+1)\hbar^2c^2}{r^2}\right]g(r)=0,\end{aligned}\tag{6}$$

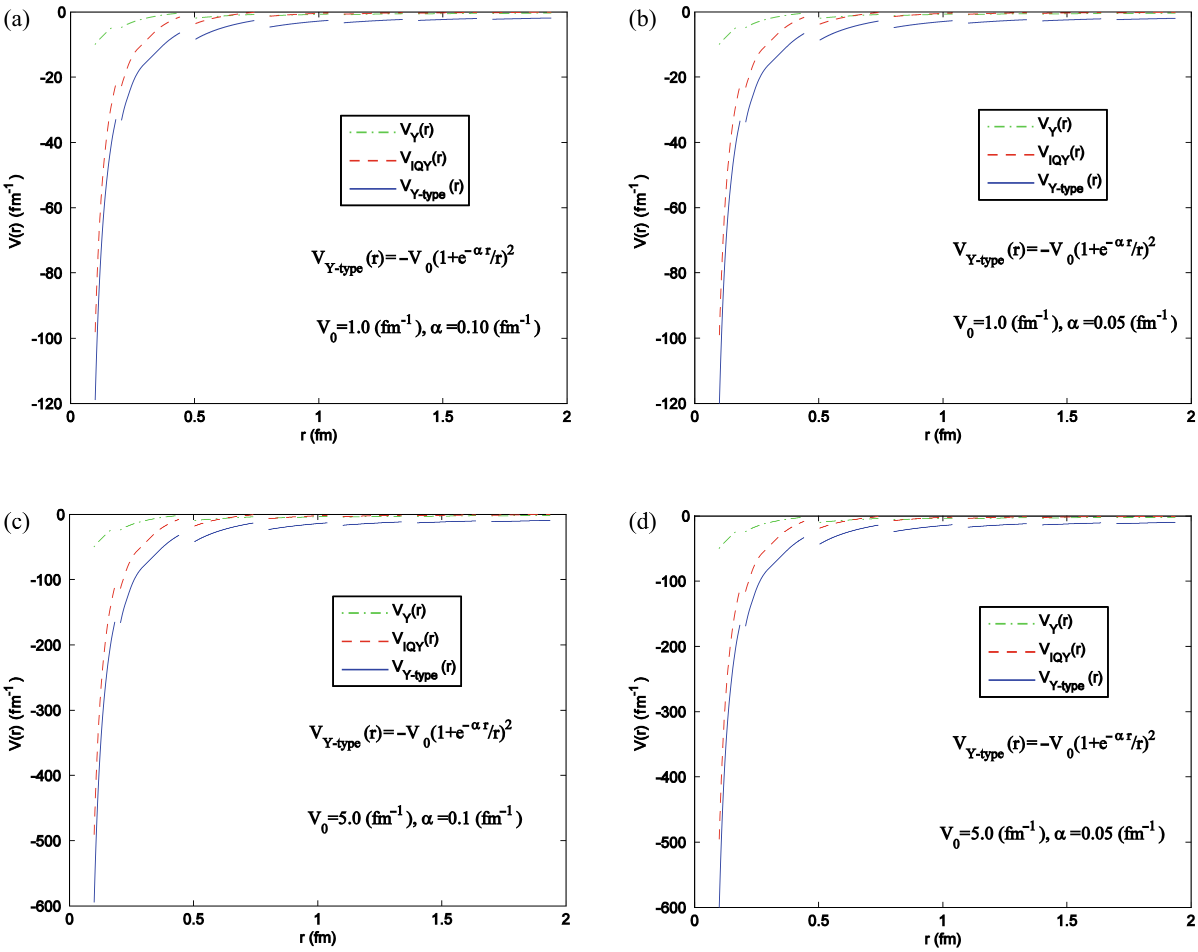

Fig. 1 (colour online). Behaviour of the Yukawa-type potential compared with the Yukawa potential [46] and the IQY potential [25] for screening parameter values (a) $V_0 = 1.0\,\mathrm{fm}^{-1}$, $\alpha = 0.10\,\mathrm{fm}^{-1}$, (b) $V_0 = 1.0\,\mathrm{fm}^{-1}$, $\alpha = 0.05\,\mathrm{fm}^{-1}$, (c) $V_0 = 5.0\,\mathrm{fm}^{-1}$, $\alpha = 0.10\,\mathrm{fm}^{-1}$, and (d) $V_0 = 5.0\,\mathrm{fm}^{-1}$, $\alpha = 0.05\,\mathrm{fm}^{-1}$.

where $l(l+1)r^{-2}$ is the centrifugal potential and the boundary conditions $g(0) = 0$ and $g(\infty) \to 0$ satisfying the asymptotic behaviours as we are dealing with bound state solutions. Since the KG equation with the Yukawa-type potential has no exact solution, we resort to an approximation if l is not sufficiently large, the case of the vibrations of small amplitude about the minimum, for the centrifugal term as [58–61].

$$\frac{1}{r^2} = \lim_{\alpha\to 0}\left[4\alpha^2 \frac{\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-2\alpha r})^2}\right]. \tag{7}$$

In accordance with the above approximation, the novel Yukawa-type potential (1), turns out to become

$$V(r) = -4\alpha^2 A \frac{\mathrm{e}^{-4\alpha r}}{(1-\mathrm{e}^{-2\alpha r})^2} - 2\alpha B \frac{\mathrm{e}^{-2\alpha r}}{1-\mathrm{e}^{-2\alpha r}} - C. \tag{8}$$

Finally, the solution of (6) with the approximations (7) and (8) can be found for $S(r) = +V(r)$ and $S(r) = -V(r)$ corresponding to the particle and anti-particle cases, respectively, by employing the parametric NU method which is briefly introduced in the following section.

3. Parametric Nikiforov–Uvarov Method

This mathematical tool is used to solve second-order differential equations in the form [62–65]

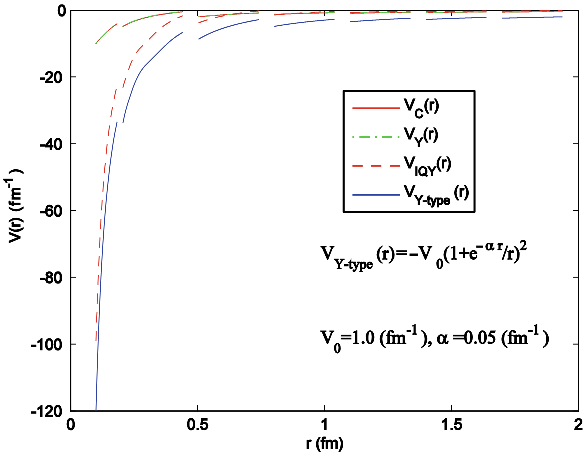

Fig. 2 (colour online). Behaviour of the Yukawa-type potential compared with the Yukawa potential [46], the IQY potential [25], and the Coulomb potential for screening parameter values $V_0 = 1.0\,\mathrm{fm}^{-1}$, $\alpha = 0.10\,\mathrm{fm}^{-1}$, and $\alpha = 0.05\,\mathrm{fm}^{-1}$.

$$\psi_n''(s)+\frac{\tilde\tau(s)}{\sigma(s)}\psi_n'(s)+\frac{\tilde\sigma(s)}{\sigma^2(s)}\psi_n(s)=0, \tag{9}$$

where $\sigma(s)$ and $\tilde\sigma(s)$ are polynomials with at most of second degree, and $\tilde\tau(s)$ is a first degree polynomial. To make the application of the NU method simpler and direct without need to check the validity of the solution. We present a simple shortcut for the method. We begin by writing a more general form of the above Schrödinger-like equation (9) as [63–65]

$$\begin{aligned}&\psi_n''(s)+\left(\frac{c_1-c_2s}{s(1-c_3s)}\right)\psi_n'(s)\\&+\left(\frac{-p_2s^2+p_1s-p_0}{s^2(1-c_3s)^2}\right)\psi_n(s)=0,\end{aligned} \tag{10}$$

satisfying the wave functions

$$\psi_n(s)=\varphi(s)y_n(s). \tag{11}$$

When comparing (10) with its counterpart (9), we can obtain the following identifications:

$$\begin{aligned}&\tilde\tau(s)=c_1-c_2s,\ \ \sigma(s)=s(1-c_3s),\\&\tilde\sigma(s)=-p_2s^2+p_1s-p_0,\end{aligned} \tag{12}$$

and following the NU method [62], we obtain the energy equation [63–65]

$$\begin{aligned}&c_2n-(2n+1)c_5+(2n+1)\left(\sqrt{c_9}-c_3\sqrt{c_8}\right)\\&+n(n-1)c_3+c_7+2c_3c_8-2\sqrt{c_8c_9}=0.\end{aligned} \tag{13}$$

On the other hand, the wave function can be found as

$$\begin{aligned}&\rho(s)=s^{c_{10}}(1-c_3s)^{c_{11}},\ \ \varphi(s)=s^{c_{12}}(1-c_3s)^{c_{13}},\\&c_{12}>0,\ \ c_{13}>0,\\&y_n(s)=P_n^{(c_{10},c_{11})}(1-2c_3s),\ \ c_{10}>-1,\ \ c_{11}>-1,\\&\psi_{n\kappa}(s)=N_{n\kappa}s^{c_{12}}(1-c_3s)^{c_{13}}P_n^{(c_{10},c_{11})}(1-2c_3s),\end{aligned} \tag{14}$$

where $P_n^{(\mu,\nu)}(x)$, $\mu>-1$, $\nu>-1$, and $x\in[-1,1]$ are Jacobi polynomials with the parametric constants

$$\begin{aligned}&c_4=\frac{1}{2}(1-c_1),\ \ c_5=\frac{1}{2}(c_2-2c_3),\ \ c_6=c_5^2+p_2;\\&c_7=2c_4c_5-p_1,\ \ c_8=c_4^2+p_0,\ \ c_9=c_3(c_7+c_3c_8)+c_6,\\&c_{10}=c_1+2c_4-2\sqrt{c_8}-1>-1,\\&c_{11}=1-c_1-2c_4+\frac{2}{c_3}\sqrt{c_9}>-1,\ \ c_3\neq0,\\&c_{12}=c_4-\sqrt{c_8}>0,\ \ c_{13}=-c_4+\frac{1}{c_3}\left(\sqrt{c_9}-c_5\right)>0,\\&c_3\neq0,\end{aligned} \tag{15}$$

where $c_{12}>0$, $c_{13}>0$, and $s\in[0,1/c_3]$, $c_3\neq0$.

In a rather more special case where $c_3=0$, the wave function (11) turns out to become

$$\begin{aligned}&\lim_{c_3\to0}P_n^{(c_{10},c_{11})}(1-2c_3s)=L_n^{c_{10}}(c_{11}s),\\&\lim_{c_3\to0}(1-c_3s)^{c_{13}}=\mathrm{e}^{c_{13}s},\\&\psi(s)=Ns^{c_{12}}\,\mathrm{e}^{c_{13}s}L_n^{c_{10}}(c_{11}s).\end{aligned} \tag{16}$$

where $L_n^{\beta}(x)$ is the Laguerre polynomials.

4. Bound State Solutions of the KG Equation

Now, we seek to solve the KG equation with our novel potential in the framework of the above parametric NU method. We consider the particle and anti-particle cases.

4.1. Equal Mixture of $S(r)=V(r)$ Case

The KG equation (6) for equally mixed scalar $S(r)$ and vector $V(r)$ potentials, i. e., $S(r)=+V(r)$, becomes

$$\begin{aligned}&\frac{\mathrm{d}^2g(r)}{\mathrm{d}r^2}-\frac{1}{\hbar^2c^2}\left[\alpha_2^2\left(\alpha_1^2+V(r)\right)\right.\\&\left.+\frac{l(l+1)\hbar^2c^2}{r^2}\right]g(r)=0,\\&\alpha_1^2=Mc^2-E_\mathrm{R},\ \ \alpha_2^2=Mc^2+E_\mathrm{R},\end{aligned} \tag{17}$$

and using the change of variables $s = e^{-2\alpha r}$, we can re-write the above equation in the form

$$\frac{d^2 g(s)}{ds^2} + \frac{(1-s)}{s(1-s)}\frac{dg(s)}{ds} + \frac{1}{s^2(1-s)^2} \cdot \left(-\nu s^2 + \delta s - \eta^2\right) g(s) = 0, \tag{18a}$$

$$\nu = \frac{\alpha_2^2}{\hbar^2 c^2}\left(\frac{B}{2\alpha} + \frac{\alpha_1^2 - C}{4\alpha^2} - A\right), \tag{18b}$$

$$\delta = \frac{\alpha_2^2}{2\alpha\hbar^2 c^2}\left(B + \frac{\alpha_1^2 - C}{\alpha}\right) - l(l+1), \tag{18c}$$

$$\eta^2 = \frac{\alpha_2^2}{4\alpha^2\hbar^2 c^2}\left(\alpha_1^2 - C\right). \tag{18d}$$

Note that the solution of the (3 + 1)-dimensional KG equation can be reduced to the solution of the Schrödinger equation with an appropriate choice of parameters: $\alpha_1^2 \to -E_{\mathrm{NR}}$, and $\alpha_2^2/\hbar^2 c^2 \to 2\mu/\hbar^2$, where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced atomic mass for a two interacting particle system.

In the non-relativistic limit, (18) becomes

$$\begin{aligned} \nu &= \frac{2\mu}{\hbar^2}\left[\frac{B}{2\alpha} - \frac{(E_{\mathrm{nl}} + C)}{4\alpha^2} - A\right], \\ \delta &= \frac{\mu}{\alpha\hbar^2}\left[B - \frac{(E_{\mathrm{nl}} + C)}{\alpha}\right] - l(l+1), \\ \eta^2 &= -\frac{\mu}{2\alpha^2\hbar^2}\left(E_{\mathrm{nl}} + C\right). \end{aligned} \tag{19}$$

Comparing (17) with (10), we obtain

$$\begin{aligned} &c_1 = 1, \; c_2 = 1, \; c_3 = 1, \\ &p_1 = \delta, \; p_2 = \nu, \; p_0 = \eta^2, \end{aligned} \tag{20}$$

and by the use of (15), we can obtain

$$\begin{aligned} &c_4 = 0, \; c_5 = -\frac{1}{2}, \; c_6 = \frac{1}{4} + \nu, \; c_7 = -\delta, \\ &c_8 = \eta^2, \; c_9 = (l+1/2)^2 - \frac{\alpha_2^2}{\hbar^2 c^2}A, \\ &c_{10} = -2\eta, \; c_{11} = 2\sqrt{(l+1/2)^2 - \frac{\alpha_2^2}{\hbar^2 c^2}A}, \\ &c_{12} = -\eta, \; c_{13} = \frac{1}{2} + \sqrt{(l+1/2)^2 - \frac{\alpha_2^2}{\hbar^2 c^2}A}. \end{aligned} \tag{21}$$

Further, the energy equation can be found by using (13), (20), and (21) to get

$$\left(n + \frac{1}{2} + \sqrt{(l+1/2)^2 - \frac{\alpha_2^2}{\hbar^2 c^2}A} - \frac{1}{2\alpha\hbar c} \cdot \sqrt{\alpha_2^2\left(\alpha_1^2 - C\right)}\right)^2 = \frac{\alpha_2^2}{\hbar^2 c^2}\left(\frac{\alpha_1^2 - C}{4\alpha^2} + \frac{B}{2\alpha} - A\right), \tag{22}$$

or it can be explicitly expressed in terms of the energy as (in units $\hbar = c = 1$)

$$\begin{aligned} &\left(n + \frac{1}{2} + \sqrt{(l+1/2)^2 - A(M + E_{\mathrm{nl}})} + \frac{\sqrt{(M + E_{\mathrm{nl}})(M - E_{\mathrm{nl}} - C)}}{2\alpha}\right)^2 \\ &= (M + E_{\mathrm{nl}})\left[\frac{(M - E_{\mathrm{nl}} - C)}{4\alpha^2} + \frac{B}{2\alpha} - A\right]. \end{aligned} \tag{23}$$

Note that once $B = C = 0$, (23) becomes identical to [25, (30) therein] with same energy spectra already found as in [25, Table 2 therein] if one sets $C_s = 0$ and $H = 0$.

In the limiting case when the screening parameter $\alpha \to 0$ (low screening regime), the potential approximates as $V_{\mathrm{IQY}}(r) = -V_0 \lim_{\alpha \to 0}(1 + \frac{1}{r}e^{-\alpha r})^2 \simeq \frac{a}{r^2} - \frac{b}{r} + c$, where the potential parameters are defined as $a = -V_0$, $b = -2(1-\alpha)V_0$, $c = -(1-\alpha)^2 V_0$. This potential is well known as Mie-type potential [23, 32]. The energy eigenvalue equation for this potential has recently been found in [32] as

$$\sqrt{M^2 - E_{\mathrm{nl}}^2 + \left(E_{\mathrm{nl}} + M\right)c} = \frac{\left(E_{\mathrm{nl}} + M\right)b}{1 + 2n + 2\sqrt{(l+1/2)^2 + \left(E_{\mathrm{nl}} + M\right)a}}. \tag{24}$$

In the special case when $a = c = 0$, it gives the energy formula for the Coulomb-like potential as [6, 32]

$$E_{\mathrm{nl}} = M\frac{4(n+l+1)^2 - b^2}{4(n+l+1)^2 + b^2}. \tag{25}$$

For the case when $n \to \infty$, one obtains $E_{\mathrm{nl}} = M$ (continuum states), that is, it shows that when n goes to infinity the energy solution of (25) becomes finite.

On the other hand, to find the corresponding wave functions, referring to (14), we find the functions

$$\rho(s) = s^{2\sqrt{\frac{\alpha_2^2}{4\alpha^2\hbar^2 c^2}\left(\alpha_1^2 - C\right)}}$$

$$\cdot(1-s)^{2\sqrt{(l+1/2)^2-\frac{\alpha_2^2}{\hbar^2c^2}A}},$$
$$\varphi(s)=s^{\sqrt{\frac{\alpha_2^2}{4\alpha^2\hbar^2c^2}(\alpha_1^2-C)}}\cdot(1-s)^{\frac{1}{2}+\sqrt{(l+1/2)^2-\frac{\alpha_2^2}{\hbar^2c^2}A}}. \tag{26}$$

Hence, (19) with the help of the weight function $\rho(s)$ in (26) gives

$$y_n(s)=P_n^{\left(2\sqrt{\frac{\alpha_2^2}{4\alpha^2\hbar^2c^2}(\alpha_1^2-C)},2\sqrt{(l+\frac{1}{2})^2-\frac{\alpha_2^2}{\hbar^2c^2}A}\right)}\cdot(1-2s). \tag{27}$$

Further, using $R_{\mathrm{nl}}(s)=\varphi(s)y_n(s)$, we get the wave function for the spinless KG particle as

$$R_{\mathrm{nl}}(r)=A_{\mathrm{nl}}\left(\mathrm{e}^{-2\alpha r}\right)^{\sqrt{\frac{\alpha_2^2}{4\alpha^2\hbar^2c^2}\left(\alpha_1^2-C\right)}}\cdot\left(1-\mathrm{e}^{-2\alpha r}\right)^{\frac{1}{2}+\sqrt{(l+1/2)^2-\frac{\alpha_2^2}{\hbar^2c^2}A}}\cdot P_n^{\left(2\sqrt{\frac{\alpha_2^2}{4\alpha^2\hbar^2c^2}(\alpha_1^2-C)},2\sqrt{(l+1/2)^2-\frac{\alpha_2^2}{\hbar^2c^2}A}\right)}\cdot\left(1-2\,\mathrm{e}^{-2\alpha r}\right) \tag{28}$$

with A_{nl} is the normalization constant. The above wave function can be expressed in terms of the energy as

$$R_{\mathrm{nl}}(r)=A_{\mathrm{nl}}\,\mathrm{e}^{-\sqrt{(M+E_{\mathrm{nl}})(M-E_{\mathrm{nl}}-C)}r}\cdot\left(1-\mathrm{e}^{-2\alpha r}\right)^{\frac{1}{2}+\sqrt{(l+1/2)^2-A(M+E_{\mathrm{nl}})}}\cdot P_n^{\left(\frac{\sqrt{(M+E_{\mathrm{nl}})(M-E_{\mathrm{nl}}-C)}}{\alpha},2\sqrt{(l+1/2)^2-A(M+E_{\mathrm{nl}})}\right)}\cdot\left(1-2\,\mathrm{e}^{-2\alpha r}\right), \tag{29}$$

satisfying the asymptotic behaviours at $r=0$ and $r\to\infty$. Note that the energy eigenvalue equation (22) admits two solutions (negative and positive), however, we are forced to choose the positive energy solution for this case.

4.2. Equal Mixture of $S(r)=-V(r)$ Case

Here, we want to solve the following KG equation:

$$\frac{\mathrm{d}^2g(r)}{\mathrm{d}r^2}-\frac{1}{\hbar^2c^2}\left[\alpha_1^2\left(\alpha_2^2-V(r)\right)+\frac{l(l+1)\hbar^2c^2}{r^2}\right]g(r)=0. \tag{30}$$

To avoid repetition in the solution of (30), it is necessary to apply the following appropriate transformations [55]:

$$\alpha_1^2\to\alpha_2^2,\ \ \alpha_2^2\to\alpha_1^2\ \ (\text{i.e., } E\to-E),\quad \text{and } V(r)\to-V(r), \tag{31}$$

on (22) and (28) to obtain the energy eigenvalue equation and wave function. We just write the final forms for energy equation as

$$\left(n+\frac{1}{2}+\sqrt{(l+1/2)^2+\frac{\alpha_1^2}{\hbar^2c^2}A}+\frac{1}{2\alpha\hbar c}\cdot\sqrt{\alpha_1^2\left(\alpha_2^2+C\right)}\right)^2=\frac{\alpha_1^2}{\hbar^2c^2}\left(\frac{\alpha_2^2+C}{4\alpha^2}-\frac{B}{2\alpha}+A\right), \tag{32}$$

or equivalently

$$\left(n+\frac{1}{2}+\sqrt{(l+1/2)^2+A(M-E_{\mathrm{nl}})}-\frac{\sqrt{(M-E_{\mathrm{nl}})(M+E_{\mathrm{nl}}+C)}}{2\alpha}\right)^2=\left(M-E_{\mathrm{nl}}\right)\left[\frac{(M+E_{\mathrm{nl}}+C)}{4\alpha^2}-\frac{B}{2\alpha}+A\right]. \tag{33}$$

We choose the negative energy solution to (32), i. e. $E_{\mathrm{KG}}^{(-)}$. Further, the wave functions become

$$R_{\mathrm{nl}}(r)=A_{\mathrm{nl}}\,\mathrm{e}^{-\sqrt{\frac{\alpha_1^2}{\hbar^2c^2}\left(\alpha_2^2+C\right)}r}\cdot\left(1-\mathrm{e}^{-2\alpha r}\right)^{\frac{1}{2}+\sqrt{(l+1/2)^2+\frac{\alpha_1^2}{\hbar^2c^2}A}}\cdot P_n^{\left(2\sqrt{\frac{\alpha_1^2}{4\alpha^2\hbar^2c^2}\left(\alpha_2^2+C\right)},2\sqrt{(l+1/2)^2+\frac{\alpha_1^2}{\hbar^2c^2}A}\right)}\cdot\left(1-2\,\mathrm{e}^{-2\alpha r}\right), \tag{34}$$

or equivalently

$$R_{\mathrm{nl}}(r) = A_{\mathrm{nl}}\,\mathrm{e}^{-\sqrt{(M-E_{n\kappa})(M+E_{n\kappa}+C)}r} \cdot \left(1-\mathrm{e}^{-2\alpha r}\right)^{1/2+\sqrt{(l+1/2)^2+A(M-E_{\mathrm{nl}})}} \cdot P_n^{\left(\frac{\sqrt{(M-E_{\mathrm{nl}})(M+E_{\mathrm{nl}}+C)}}{\alpha},2\sqrt{(l+1/2)^2+A(M-E_{\mathrm{nl}})}\right)} \cdot \left(1-2\,\mathrm{e}^{-2\alpha r}\right). \tag{35}$$

We remark that (33) and (35) are corresponding to the energy equation and wave function for the spinless anti-particles.

4.3. Numerical Results

We present some numerical results in Table 1 with parameters values $M = 5.0\,\mathrm{fm}^{-1}$, $\alpha = 0.015\,\mathrm{fm}^{-1}$, and $V_0 = 1.0$. We calculate the energy eigenvalues of KG–Yukawa-like problem by taking various values of n and l quantum numbers in the equal mixture of scalar and vector Yukawa-like potentials, i. e., $S(r) = \pm V(r)$.

5. Some Special Cases

Here, we study the solutions of Schrödinger and KG wave equations for some special cases of much interest.

5.1. The Schrödinger Solution

In the non-relativistic limit, the Schrödinger solution can be obtained from the $S(r) = V(r)$ case by means of (23). Applying the transformations $E_{\mathrm{nl}}+M \approx 2\mu/\hbar^2$, $E_{\mathrm{nl}}-M \approx E_{\mathrm{nl}}$, one obtains the energy formula

$$E_{\mathrm{nl}} = -C - \frac{2\mu}{\hbar^2} \cdot \left[\frac{\alpha\left(n+\frac{1}{2}+\sqrt{(l+1/2)^2-2\mu A/\hbar^2}\right)\hbar^2}{2\mu} + \frac{(2\alpha A - B)}{2\left(n+\frac{1}{2}+\sqrt{(l+1/2)^2-2\mu A/\hbar^2}\right)}\right]^2 \tag{36}$$

and the radial wave function

$$R_{\mathrm{nl}}(r) = D_{\mathrm{nl}}\,\mathrm{e}^{-\sqrt{-2\mu(E_{\mathrm{nl}}+C)/\hbar^2}r} \cdot \left(1-\mathrm{e}^{-2\alpha r}\right)^{\frac{1}{2}+\sqrt{(l+1/2)^2-2\mu A/\hbar^2}} \cdot P_n^{\left(\frac{\sqrt{-2\mu(E_{\mathrm{nl}}+C)/\hbar^2}}{\alpha},2\sqrt{(l+1/2)^2-2\mu A/\hbar^2}\right)} \cdot \left(1-2\,\mathrm{e}^{-2\alpha r}\right). \tag{37}$$

When $B = C = 0$, we obtain the non-relativistic energy levels of the IQY potential:

$$E_{\mathrm{nl}} = -\frac{2\mu}{\hbar^2} \cdot \left[\frac{\alpha\left(n+\frac{1}{2}+\sqrt{(l+1/2)^2-2\mu A/\hbar^2}\right)\hbar^2}{2\mu} + \frac{2\alpha A}{2\left(n+\frac{1}{2}+\sqrt{(l+1/2)^2-2\mu A/\hbar^2}\right)}\right]^2 \tag{38}$$

and the wave function

Table 1. Energy levels for spinless KG particle with new Yukawa-type potential for various values of n and l quantum numbers.

n	l	$S(r)=V(r)$	$S(r)=-V(r)$
0	0	−4.999975083	4.999979490
	1	−4.999900329	4.999917959
	2	−4.999775737	4.999815405
	3	−4.999601302	4.999671828
1	0	−4.999900335	4.999917954
	1	−4.999775741	4.999815402
	2	−4.999601306	4.999671825
	3	−4.999377023	4.999487219
2	0	−4.999775763	4.999815388
	1	−4.999601317	4.999671818
	2	−4.999377032	4.999487213
	3	−4.999102892	4.999261576
3	0	−4.999601369	4.999671782
	1	−4.999377052	4.999487199
	2	−4.999102907	4.999261566
	3	−4.998778901	4.998994893
4	0	−4.999377155	4.999487130
	1	−4.999102943	4.999261542
	2	−4.998778925	4.998994876
	3	−4.998405041	4.998687162
5	0	−4.999103121	4.999261421
	1	−4.998778982	4.998994838
	2	−4.998405077	4.998687138
	3	−4.997981299	4.998338377

$$R_{\mathrm{nl}}(r)=D_{\mathrm{nl}}\,\mathrm{e}^{-\sqrt{-2\mu E_{\mathrm{nl}}/\hbar^2}r}\cdot\left(1-\mathrm{e}^{-2\alpha r}\right)^{\frac{1}{2}+\sqrt{(l+1/2)^2-2\mu A/\hbar^2}}\cdot P_n^{\left(\frac{\sqrt{-2\mu E_{\mathrm{nl}}/\hbar^2}}{\alpha},2\sqrt{(l+1/2)^2-2\mu A/\hbar^2}\right)}\cdot\left(1-2\,\mathrm{e}^{-2\alpha r}\right). \tag{39}$$

If $A=C=0$ and $B\to A$, it turns to the Yukawa potential solution, and the energy levels of (36) become

$$E_{\mathrm{nl}}=-\frac{2\mu}{\hbar^2}\left[\frac{A}{2(n+l+1)}-\frac{\hbar^2(n+l+1)}{2\mu}\alpha\right]^2, \tag{40}$$

which is identical to [46, (52) thereof], and the wave function is

$$R_{\mathrm{nl}}(r)=D_{\mathrm{nl}}\,\mathrm{e}^{-\sqrt{-2\mu E_{\mathrm{nl}}/\hbar^2}r}\left(1-\mathrm{e}^{-2\alpha r}\right)^{l+1}\cdot P_n^{\left(\frac{\sqrt{-2\mu E_{\mathrm{nl}}/\hbar^2}}{\alpha},2l+1\right)}\left(1-2\,\mathrm{e}^{-2\alpha r}\right). \tag{41}$$

Finally, when we set $A\to -A$, $C\to -C$, and $\alpha=0$, (1) turns to become the Kratzer-type potential [66] with non-relativistic energy eigenvalues obtained via (36) as

$$E_{\mathrm{nl}}=C-\frac{2\mu B^2/\hbar^2}{\left(2n+1+\sqrt{(2l+1)^2+8\mu A/\hbar^2}\right)^2}, \tag{42}$$

which is identical to [66, (27) thereof].

5.2. Spinless KG Solution of IQY Potential

When $B=C=0$, we obtain the energy equation and wave functions for the KG-IQY problem for the $S(r)=V(r)$ case

$$\left(n+\frac{1}{2}+\sqrt{(l+1/2)^2-A(M+E_{\mathrm{nl}})}+\frac{\sqrt{M^2-E_{\mathrm{nl}}^2}}{2\alpha}\right)^2=\frac{(M+E_{\mathrm{nl}})(M-E_{\mathrm{nl}}-4\alpha^2A)}{4\alpha^2}, \tag{43}$$

$$R_{\mathrm{nl}}(r)=B_{\mathrm{nl}}\,\mathrm{e}^{-\sqrt{M^2-E_{\mathrm{nl}}^2}r}\cdot\left(1-\mathrm{e}^{-2\alpha r}\right)^{1/2+\sqrt{(l+1/2)^2-A(M+E_{\mathrm{nl}})}}\cdot P_n^{\left(\frac{\sqrt{M^2-E_{\mathrm{nl}}^2}}{\alpha},2\sqrt{(l+1/2)^2-A(M+E_{\mathrm{nl}})}\right)}\cdot\left(1-2\,\mathrm{e}^{-2\alpha r}\right). \tag{44}$$

When $B=C=0$, we obtain the energy equation and wave function for the KG-IQY problem for the $S(r)=-V(r)$ case

$$\left(n+\frac{1}{2}+\sqrt{(l+1/2)^2+A(M-E_{\mathrm{nl}})}+\frac{\sqrt{M^2-E_{\mathrm{nl}}^2}}{2\alpha}\right)^2=\frac{(M-E_{\mathrm{nl}})\left(M+E_{\mathrm{nl}}+4\alpha^2A\right)}{4\alpha^2}, \tag{45}$$

and

$$R_{\mathrm{nl}}(r)=A_{\mathrm{nl}}\,\mathrm{e}^{-\sqrt{M^2-E_{\mathrm{nl}}^2}r}\cdot\left(1-\mathrm{e}^{-2\alpha r}\right)^{\frac{1}{2}+\sqrt{(l+1/2)^2+A(M-E_{\mathrm{nl}})}}\cdot P_n^{\left(\frac{\sqrt{M^2-E_{\mathrm{nl}}^2}}{\alpha},2\sqrt{(l+1/2)^2+A(M-E_{\mathrm{nl}})}\right)}\cdot\left(1-2\,\mathrm{e}^{-2\alpha r}\right). \tag{46}$$

When we set $\alpha=0$, $A\to -A$, and $C\to -C$, (1) becomes the Kratzer-type potential $V(r)=D_e\left(r_e^2/r^2-2r_e/r+1\right)$ (see [66, (13) and references therein]). The KG solution can be simply obtained following similar procedures in pNU method where we apply the case $c_3=0$. The energy eigenvalue equation becomes

$$\sqrt{M^2c^4-E_{\mathrm{R}}^2+\left(E_{\mathrm{R}}+Mc^2\right)C}=\frac{(Mc^2+E_{\mathrm{R}})B}{\hbar c\left(1+2n+\frac{2}{\hbar c}\sqrt{(Mc^2+E_{\mathrm{R}})A+(l+1/2)^2\hbar^2c^2}\right)}, \tag{47}$$

and the wave function is

$$g_{\mathrm{nl}}(r)=A_{\mathrm{nl}}\,\mathrm{e}^{-\sqrt{M^2c^4-E_{\mathrm{R}}^2+(E_{\mathrm{R}}+Mc^2)C}r}\cdot r^{\frac{1}{2}+\frac{1}{\hbar c}\sqrt{(l+1/2)^2\hbar^2c^2+(Mc^2+E_{\mathrm{R}})A}}\cdot L_n^{\left(\frac{2}{\hbar c}\sqrt{(l+1/2)^2\hbar^2c^2+(Mc^2+E_{\mathrm{R}})A}\right)}\cdot\left(2\sqrt{M^2c^4-E_{\mathrm{R}}^2+\left(E_{\mathrm{R}}+Mc^2\right)C}\,r\right). \tag{48}$$

6. Conclusion

To sum up, in this paper, we have obtained the bound state solution of the KG wave equation for a novel potential which is an intermediate between the Yukawa potential [46] and the inversely quadratic

Yukawa (IQY) potential [25] that milds the strong singularity of $1/r^2$ in the IQY potential and the soft singularity $1/r$ in the Yukawa potential. The energy levels of this potential are very close to energy levels of the other two potentials studied in our previous works [25, 46]. The present solution is a more generalized one and can be easily reduced into the other two potential forms [25, 46]. We have obtained the approximate bound states of a spinless KG particle confined to the field of equal mixture of scalar and vector Yukawa-type interactions. We used a parametric version of the powerful NU method [62]. Some numerical values of the energy levels are calculated in Table 1 in view of the $S(r) = V(r)$ and $S(r) = -V(r)$ case, respectively. We find out that the spectra in Tables 1 and 2 of the present potential when the potential parameters $B = C = 0$ are identical to those ones obtained in [25] when the spin and pseudospin symmetries are exact, i. e., $C_s = C_{ps} = 0$. Further, it has a continuum spectrum once $n \to \infty$. The relativistic solution reduces into the Schrödinger solution for the Yukawa potential [46] and the IQY potential [25] under appropriate transformations of the parameters. Moreover, the Klein–Gordon positive energy solution with mixed scalar and vector potential $S(r) = V(r)$ describes the energy for the particle, whereas the Klein–Gordon negative energy solution for the $S(r) = -V(r)$ case gives the energy for the anti-particle.

It is worth mentioning that the potential (1) when $\alpha = 0$ turns to become the Kratzer-type potential with non-relativistic solution given by (42). However, it is not possible to obtain the spinless KG solution of (1) with $\alpha = 0$ since this problem admits an exact solution ($c_3 = 0$ special case, see Section 3) and can be easily solved without need to make an approximation to the centrifugal term. It is necessary to state that we can also obtain the N-dimensional bound state solutions in any arbitrary dimension $N \geq 2$ by making the following replacement $l \to l + (N-3)/2$ throughout this work [66].

It is also notable that solving such a flexible form of the Yukawa-like potential allows one to generate some other potentials used in molecular [66] and particle [67] physics. In the low screening region where $\alpha \ll 1$, the potential turns to become the Cornell potential or Killingbeck (harmonic oscillator plus Cornell) potential (see [67, and references therein]). These potentials are usually used to study the mesons and baryons. If the screening parameter $\alpha \to 0$, the potential turns to become the modified Kratzer potential [66] which is used in molecular physics to study the diatomic molecules [67–70].

Acknowledgements

The authors thank the referees for their enlightening comments which have greatly improved this manuscript. S. M. Ikhdair acknowledges the partial support of the Scientific and Technological Research Council of Turkey.

[1] A. W. Thomas and W. Weise, Structure of the Nucleon, Wiley, Berlin 2001.
[2] B. Thaller, The Dirac Equation, Springer, New York 1992.
[3] G. Chen, Z. D. Chen, and X. L. Peng, Phys. Lett. A **352**, 317 (2006).
[4] A. de Souza Dutra and G. Chen, Phys. Lett. A **349**, 297 (2006),
[5] A. S. de Castro, Phys. Lett. A **338**, 81 (2005).
[6] S. M. Ikhdair, Eur. Phys. J. A **40**, 143 (2009).
[7] Z. Q. Ma, S. H. Dong, X. Y. Gu, and J. Yu, Int. J. Mod. Phys. E **13**, 597 (2004).
[8] W. C. Qiang, R. S. Zhou, and Y. Gao, Phys. Lett. A **371**, 201 (2007).
[9] A. Arda, R. Sever, and C. Tezcan, Phys. Scr. **79**, 015006 (2009).
[10] S. M. Ikhdair and R. Sever, Phys. Scr. **79**, 035002 (2009).
[11] M. G. Miranda, G. H. Sun, and S. H. Dong, Int. J. Mod. Phys. E **19**, 123 (2010).
[12] M. C. Zhang, G. H. Sun, and S. H. Dong, Phys. Lett. A **374**, 704 (2010).
[13] F. Cooper, A. Khare, and U. Sukhatme, Phys. Rep. **251**, 267 (1995).
[14] B. Bagchi, A. Banerjee, C. Quesne, and V. M. Tkachuk, J. Phys. A: Math. Gen. **38**, 2929 (2005).
[15] R. De, R. Dutt, and U. Sukhatme, J. Phys. A: Math. Gen. **25**, L843 (1992).
[16] N. Kandirmaz and R. Sever, Phys. Scr. **81**, 035302 (2010).
[17] H. Ciftci, R. L. Hall, and N. Saad, J. Phys. A: Math. Gen. **36**, 11807 (2003).
[18] H. Ciftci, R. L. Hall, and N. Saad, Phys. Lett. A **340**, 338 (2005).
[19] Z. Q. Ma and B. W. Xu, Europhys. Lett. **69**, 685 (2005).

[20] W. C. Qiang, G. H. Sun, and S. H. Dong, Ann. Phys. **524**, 360 (2012).
[21] M. Bag, M. M. Panja, R. Dutt, and Y. P. Varshni, Phys. Rev. A **46**, 6059 (1992).
[22] S. M. Ikhdair and R. Sever, Int. J. Mod. Phys. A **21**, 6465 (2006).
[23] M. Hamzavi, A. A. Rajabi, and H. Hassanabadi, Few-Body Syst. **48**, 171 (2010).
[24] M. Hamzavi, A. A. Rajabi, and H. Hassanabadi, Few-Body Syst. **52**, 19 (2012).
[25] M. Hamzavi, S. M. Ikhdair, and B. I. Ita, Phys. Scr. **85**, 045009 (2012).
[26] R. Sever, C. Tezcan, M. Aktaş, and O. Yeşiltaş, J. Math. Chem. **43**, 845 (2007).
[27] S. M. Ikhdair and R. Sever, J. Mol. Struct.: Theochem **806**, 155 (2007).
[28] S. H. Dong, X. Y. Gu, Z. Q. Ma, and S. Dong, Int. J. Mod. Phys. E **11**, 483 (2002).
[29] D. G. C. McKeon and G. V. Leeuwen, Mod. Phys. Lett. A **17**, 1961 (2002).
[30] S. M. Ikhdair and R. Sever, Cent. Eur. J. Phys. **5**, 516 (2007).
[31] M. Hamzavi, H. Hassanabadi, and A. A. Rajabi, Int. J. Mod. Phys. E **19**, 2189 (2010).
[32] M. Hamzavi, H. Hassanabadi, and A. A. Rajabi, Mod. Phys. Lett. A **25**, 2447 (2010).
[33] M. Hamzavi, A. A. Rajabi, and H. Hassanabadi, Phys. Lett. A **374**, 4303 (2010).
[34] L. H. Zhang, X. P. Li, and C. S. Jia, Phys. Scr. **80**, 035003 (2009).
[35] S. H. Dong and M. Lozada-Cassou, Int. J. Mod. Phys. E **13**, 917 (2004).
[36] S. M. Ikhdair and R. Sever, Int. J. Mod. Phys. C **19**, 1425 (2008).
[37] B. A. Kagali, N. A. Rao, and V. Sivramkrishna, Mod. Phys. Lett. A **17**, 2049 (2002).
[38] S. M. Ikhdair and R. Sever, J. Phys. A: Math. Theor. **44**, 355301 (2011).
[39] S. H. Dong, Appl. Math. Lett. **16**, 199 (2003).
[40] M. C. Zhang, G. Q. Huang-Fu, and B. An, Phys. Scr. **80**, 65018 (2009).
[41] G. F. Wei and S. H. Dong, Phys. Lett. B **686**, 288 (2010).
[42] S. M. Ikhdair, Phys. Scr. **83**, 015010 (2011).
[43] K. J. Oyewumi, F. O. Akinpelu, and A. D. Agboola, Int. J. Theor. Phys. **47**, 1039 (2008).
[44] S. M. Ikhdair and B. J. Falaye, Chem. Phys. **421**, 84 (2013).
[45] S. M. Ikhdair and R. Sever, Appl. Math. Comput. **218**, 10082 (2012).
[46] S. M. Ikhdair, Cent. Eur. J. Phys. **10**, 361 (2012).
[47] S. M. Ikhdair and B. J. Falaye, Phys. Scr. **87**, 35002 (2013).
[48] S. M. Ikhdair and M. Hamzavi, Few-Body Syst. **53**, 487 (2012).
[49] H. Taseli, Int. J. Quant. Chem. **63**, 949 (1997).
[50] M. W. Kermode, M. L. J. Allen, J. P. Mctavish, and A. Kervell, J. Phys. G: Nucl. Part. Phys. **10**, 773 (1984).
[51] W. Greiner, Relativistic Quantum Mechanics: Wave Equations, Springer, Berlin 2000.
[52] A. D. Alhaidari, H. Bahlouli, and A. Al-Hasan, Phys. Lett. A **349**, 87 (2006).
[53] Y. Xu, S. He, and C. S. Jia, Phys. Scr. **81**, 45001 (2010).
[54] S. M. Ikhdair, J. Quantum Infor. Science **1**, 73 (2011).
[55] S. M. Ikhdair and M. Hamzavi, Chin. Phys. B **22**, 040302 (2013).
[56] M. Eshghi, M. Hamzavi, and S. M. Ikhdair, Chin. Phys. B **22**, 030303 (2013).
[57] B. J. Falaye and S. M. Ikhdair, Chin. Phys. B **22**, 060305 (2013).
[58] R. L. Greene and C. Aldrich, Phys. Rev. A **14**, 2363 (1976).
[59] W. C. Qiang and S. H. Dong, Phys. Lett. A **368**, 13 (2007).
[60] S. H. Dong, W. C. Qiang, G. H. Sun, and V. B. Bezerra, J. Phys. A: Math. Theor. **40**, 10535 (2007).
[61] S. M. Ikhdair and R. Sever, J. Math. Chem. **42**, 461 (2007).
[62] A. F. Nikiforov and V. B. Uvarov, Special Functions of Mathematical Physics, Birkhuser, Berlin 1988.
[63] S. M. Ikhdair, Int. J. Mod. Phys. C **20**, 1563 (2009).
[64] S. M. Ikhdair, Eur. Phys. J. A **39**, 307 (2009).
[65] S. M. Ikhdair, Chem. Phys. **361**, 9 (2009).
[66] S. M. Ikhdair and R. Sever, J. Math. Chem. **45**, 1137 (2009).
[67] M. Hamzavi, S. M. Ikhdair, and K. E. Thylwe, Z. Naturforsch. **67a**, 567 (2012).
[68] S. M. Ikhdair, Few Body Syst. (2013), doi:10.1007/s00601-013-0693-2 .
[69] M. Hamzavi, S. M. Ikhdair, and M. Solaimani, Int. J. Mod. Phys. E **21**, 1250016 (2012).
[70] M. Hamzavi, S. M. Ikhdair, and K. E. Thylwe, J. Math. Chem. **51**, 227 (2013).