

Two Approximate Analytic Eigensolutions of the Hellmann Potential with any Arbitrary Angular Momentum

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The parametric Nikiforov–Uvarov (pNU) and asymptotic iteration method (AIM) are applied to study the approximate analytic bound state eigensolutions (energy levels and wave functions) of the radial Schrödinger equation (SE) for the Hellmann potential which represents the superposition of the attractive Coulomb potential $(-a/r)$ and the Yukawa potential $b \exp(-\delta r)/r$ of arbitrary strength b and screening parameter δ in closed form. The analytical expressions to the energy eigenvalues E_{nl} yield quite accurate results for a wide range of n, l in the limit of very weak screening but the results become gradually worse as the strength b and the screening coefficient δ increase. The calculated bound state energies have been compared with available numerical data. Special cases of our solution like pure Coulomb and Yukawa potentials are also investigated.

Key words: Schrödinger Equation; Hellmann Potential; Nikiforov–Uvarov Method; Asymptotic Iteration Method.

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1. Introduction

A two particle system interacting through a combination of the attractive Coulomb and the Yukawa potential can be expressed as

$$V(r) = -\frac{a}{r} + \frac{b}{r} e^{-\delta r}, \quad (1)$$

where the parameters a and b denote the strength of the Coulomb and Yukawa potentials, respectively, δ denotes the screening parameter, and r is the distance between two particles. The parameters a and δ are positive, and b can be both positive as well as negative. The Hellmann potential with $b > 0$ was firstly proposed by Hellmann [1–3] and thereafter has been used to include both positive and negative b . It has many applications in atomic and condensed matter physics; e. g., the electron-core [4, 5], electron–ion interactions [6, 7], inner-shell ionization problem [8], alkali hydride molecules [9], solid-state physics [10, 11], etc.

Over the past years, the potential model (1) has received much concern from many authors. The radial Schrödinger equation (SE) does not admit exact analytical solutions and one has to resort to the approximate methods such as the variational or perturbative techniques [12]. It is worth to be noted that the bound state spectra of this potential model are the presence of complex states crossings [12] and the absence of accidental degeneracies (characteristics of the pure Coulomb potential). Shortly later, shifted large N expansion technique [13] reported the energy levels with more or less accuracy as those of perturbative technique. This prescription yields reasonably accurate results for very weak screenings and gradually worsens as b and δ increases. Attempts have been made to use the first-order Rayleigh–Schrödinger perturbation theory to obtain approximate formulae for bound eigenstates [14]. Lately, the combined Hellmann–Feynman theorem has also been used to study bound states [15].

In the current treatment of this potential model, many difficulties have been faced that deserve careful

and thorough examinations. In the recent years, many excellent and highly accurate energy states have been achieved for both the Coulomb and Yukawa potentials [16–18] within the framework of some perturbative approaches.

Thus a general reliable formalism or a nonperturbative approach which can offer accurate results for wide range arbitrary values of the interaction parameters for both lower and higher states would have their merits. The generalized pseudospectral (GPS) method that reports accurate eigenvalues for all $n \leq 5$ states have been recently used with the advantage of varying the interaction parameters covering a large range of parameters [19].

The trajectories of the poles of the S-matrix for the Hellmann potential in the complex energy plane have been studied near the critical screening parameter [20]. The calculation has been performed using the J-matrix approach which uses a suitable L^2 basis to tridiagonalize the reference Hellmann matrix. This calculated bound and resonance state energies have been compared with available normalized data.

The method of potential envelopes is used to analyze the bound state spectrum of the Schrödinger equation with Hellmann potential [21]. They established simple formulae yielding upper and lower bounds for all the energy eigenvalues.

In our recent work, we applied a methodology to study the bound states of the Hellmann potential based on the decomposition of the radial SE into two pieces having an exactly solvable part and an additional piece leading to either analytic solution or approximate treatment depending on the nature of the additional perturbed potential [22]. The bound state energy eigenvalues of the generalized Hellmann potential are obtained using the hypervirial $1/N$ expansion method together with the Hellmann–Feynman theorem [23]. Results are analytically given up to the fourth order of screening parameters λ and μ .

An alternative and accurate solution of the radial SE for the Hellmann potential has been found within the framework of the asymptotic iteration method (AIM) [24]. It was shown that the bound state energy eigenvalues can be obtained easily for any n and l values without using any approximations required by other methods.

The priority purpose of the present work is to solve the Schrödinger equation for the Hellmann potential and to calculate the energy eigenvalues and the corre-

sponding wave functions which are expressed in terms of the Jacobi polynomials for any orbital quantum number l . We computed the energy spectrum numerically for weak and strong screening parameter δ and strong coupling b . The parametric Nikiforov–Uvarov (pNU) and AIM methods are used in present calculations

The article is organized as follows: Section 2 gives a brief outline of the pNU method [25–28] used to solve the SE in the presence of the Hellmann potential. Analytical expressions for energy levels and corresponding wave functions are obtained for any n and l quantum numbers in Section 3. We finalize with a few concluding remarks in Section 4.

2. The Nikiforov–Uvarov (NU) Method

This powerful mathematical tool could be used to solve second-order differential equations. Considering the following differential equation [29]:

$$\psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)} \psi_n'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)} \psi_n(s) = 0, \quad (2)$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials of second degree at most, and $\tilde{\tau}(s)$ is a first-degree polynomial. To make the application of the NU method [29] simpler and the checking of the validity of solution unnecessary, we write a shortcut of the method. At first, we write the general form of the Schrödinger-like equation (2) in a more general form [25–28]:

$$\begin{aligned} \psi_n''(s) + \left(\frac{c_1 - c_2s}{s(1 - c_3s)} \right) \psi_n'(s) \\ + \left(\frac{-Q_2s^2 + Q_1s - Q_0}{s^2(1 - c_3s)^2} \right) \psi_n(s) = 0, \end{aligned} \quad (3)$$

where the wave functions satisfy

$$\psi_n(s) = \phi(s)y_n(s). \quad (4)$$

Now, comparing (3) with its counterpart (2), we can obtain

$$\begin{aligned} \tilde{\tau}(s) &= c_1 - c_2s, \quad \sigma(s) = s(1 - c_3s), \\ \tilde{\sigma}(s) &= -Q_2s^2 + Q_1s - Q_0. \end{aligned} \quad (5)$$

Further, according to the NU method [8], one can obtain the bound-state energy equation [25]

$$c_2n - (2n + 1)c_5 + (2n + 1)(\sqrt{c_9} + c_3\sqrt{c_8}) + n(n - 1)c_3 + c_7 + 2c_3c_8 + 2\sqrt{c_8c_9} = 0. \tag{6}$$

In addition, we also find that the functions

$$\rho(s) = s^{c_{10}}(1 - c_3s)^{c_{11}}, \quad \phi(s) = s^{c_{12}}(1 - c_3s)^{c_{13}}, \quad c_{12} > 0, \quad c_{13} > 0, \tag{7}$$

$$y_n(s) = P_n^{(c_{10}, c_{11})}(1 - 2c_3s), \quad c_{10} > -1, \quad c_{11} > -1,$$

are necessary in calculating the wave functions

$$\psi_{nl}(s) = N_{nl}s^{c_{12}}(1 - c_3s)^{c_{13}}P_n^{(c_{10}, c_{11})}(1 - 2c_3s), \tag{8}$$

where N_{nl} is a normalization constant and $P_n^{(\mu, \nu)}(x)$, $\mu > -1$, $\nu > -1$, $x \in [-1, 1]$, are Jacobi polynomials with constant parameters [25]

$$\begin{aligned} c_4 &= \frac{1}{2}(1 - c_1), \quad c_5 = \frac{1}{2}(c_2 - 2c_3), \\ c_6 &= c_5^2 + Q_2, \quad c_7 = 2c_4c_5 - Q_1, \\ c_8 &= c_4^2 + Q_0, \quad c_9 = c_3(c_7 + c_3c_8) + c_6, \\ c_{10} &= c_1 + 2c_4 + 2\sqrt{c_8} - 1 > -1, \\ c_{11} &= 1 - c_1 - 2c_4 + \frac{2}{c_3}\sqrt{c_9} > -1, \quad c_3 \neq 0, \\ c_{12} &= c_4 + \sqrt{c_8} > 0, \\ c_{13} &= -c_4 + \frac{1}{c_3}(\sqrt{c_9} - c_5) > 0, \quad c_3 \neq 0, \end{aligned} \tag{9}$$

with $c_{12} > 0$, $c_{13} > 0$, and $s \in [0, 1/c_3]$, $c_3 \neq 0$. Also, the e wave function (8) can be expressed in terms of the hypergeometric function as

$$\psi_{nl}(s) = N_{nl}s^{c_{12}}(1 - c_3s)^{c_{13}} \cdot {}_2F_1(-n, 1 + c_{10} + c_{11} + n; c_{10} + 1; c_3s). \tag{10}$$

In the more special case of $c_3 = 0$, the wave function (8) becomes

$$\begin{aligned} \lim_{c_3 \rightarrow 0} P_n^{(c_{10}, c_{11})}(1 - 2c_3s) &= L_n^{(c_{10})}(c_{11}s), \\ \lim_{c_3 \rightarrow 0} s^{c_{12}}(1 - c_3s)^{c_{13}} &= s^{c_{12}}e^{c_{13}s}, \\ \psi_{nl}(s) &= N_{nl}s^{c_{12}}e^{c_{13}s}L_n^{(c_{10})}(c_{11}s), \end{aligned} \tag{11}$$

where $L_n^{(\alpha)}(x)$ are the associated Laguerre polynomials.

3. Bound State Solutions of the Schrödinger Equation for the Hellmann Potential

3.1. Solutions via pNU Method

The three-dimensional Schrödinger equation for two interacting particles via the Hellmann potential field given in (1) takes the form [30, 31]

$$\begin{aligned} \left(-\frac{\hbar^2}{2\mu}\nabla^2 - \frac{a}{r} + \frac{b}{r}e^{-\delta r} \right) \psi(\mathbf{r}) &= E\psi(\mathbf{r}), \\ \psi(\mathbf{r}) &= \frac{1}{r}u_{nl}(r)Y_{lm}(\theta, \varphi). \end{aligned} \tag{12}$$

Using the separation of variables, we can obtain the following radial SE:

$$\begin{aligned} \left[\frac{d^2}{dr^2} + \frac{2\mu}{\hbar^2} \left(E + \frac{a}{r} - b\frac{e^{-\delta r}}{r} \right) \right. \\ \left. - \frac{l(l+1)}{r^2} \right] u_{nl}(r) = 0. \end{aligned} \tag{13}$$

Since the above radial SE with the Hellmann potential has no exact solution, we resort to an approximation scheme to deal with the rotational centrifugal term as

$$\frac{1}{r^2} \approx \frac{\delta^2}{(1 - e^{-\delta r})^2}, \quad \frac{1}{r} \approx \frac{\delta}{1 - e^{-\delta r}}, \tag{14}$$

which is valid only for $\delta r \ll 1$ [31]. Therefore, the Hellmann potential in (1) can be expressed in the form [32–34]

$$V(r) \simeq -\delta \frac{(a - be^{-\delta r})}{1 - e^{-\delta r}}. \tag{15}$$

To show the accuracy of the present approximation, we have sketched the Hellmann potential (1) and its approximation (15) with parameter values $a = 2$, $b = -4$, and $\delta = 0.01$ in Figure 1.

Now substituting (14) into (13) gives

$$\begin{aligned} \left[\frac{1}{\delta^2} \frac{d^2}{dr^2} - \varepsilon + \frac{\alpha}{1 - e^{-\delta r}} - \frac{\beta e^{-\delta r}}{1 - e^{-\delta r}} \right. \\ \left. - \frac{\Lambda}{(1 - e^{-\delta r})^2} \right] u_{nl}(r) = 0, \quad u_{nl}(0) = 0, \end{aligned} \tag{16}$$

with the following identifications:

$$\begin{aligned} \varepsilon = -\frac{2\mu E}{\hbar^2\delta^2}, \quad \alpha = \frac{2\mu a}{\hbar^2\delta}, \quad \beta = \frac{2\mu b}{\hbar^2\delta}, \\ \Lambda = l(l + 1). \end{aligned} \tag{17}$$

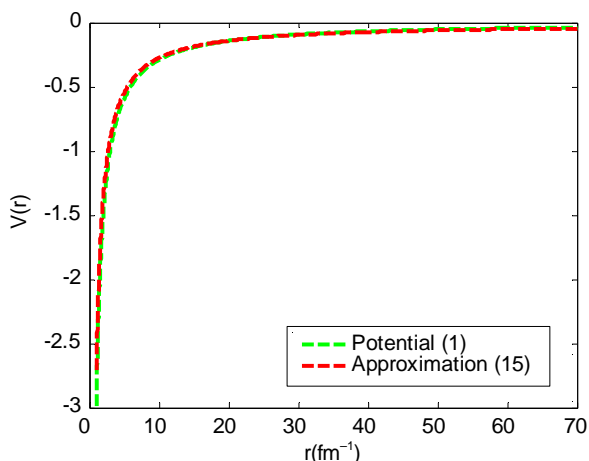


Fig. 1 (colour online). Variation of the Hellmann potential and its approximation with r .

To solve (16) by means of the pNU method, we make an appropriate change of variables $s = e^{-\delta r}$, $s \in (0, 1)$, and hence it recasts as

$$\frac{d^2 u_{nl}(s)}{ds^2} + \frac{1-s}{s(1-s)} \frac{du_{nl}(s)}{ds} + \frac{1}{s^2(1-s)^2} \left[-(\varepsilon - \beta)s^2 + (2\varepsilon - \alpha - \beta)s - (\varepsilon + \Lambda - \alpha) \right] u_{nl}(s) = 0 \quad (18)$$

with $u_{nl}(s=0) = 0$ and $u_{nl}(s=1) = 0$. Comparing (18) with (3), we can easily obtain the coefficients c_k ($k = 1, 2, 3$) together with the analytical expressions Q_i ($i = 1, 2, 3$) as follows:

$$\begin{aligned} c_1 &= 1, & Q_2 &= \varepsilon - \beta, \\ c_2 &= 1, & Q_1 &= 2\varepsilon - \alpha - \beta, \\ c_3 &= 1, & Q_0 &= \varepsilon + \Lambda - \alpha. \end{aligned} \quad (19)$$

By using (9), we find the remaining values of other constants c_k ($k = 4, 5, \dots, 13$) for the Hellmann potential model as displayed in Table 1. Further, using (6), we can obtain energy formula for the Hellmann potential as

$$E_{nl} = \frac{\hbar^2 l(l+1)\delta^2}{2\mu} - a\delta - \frac{\mu}{2\hbar^2} \left[\frac{a-b - \hbar^2 l(l+1)\delta/(2\mu)}{(n+l+1)} - \frac{\hbar^2(n+l+1)\delta}{2\mu} \right]^2, \quad (20)$$

where $n = 0, 1, 2, \dots, n_{\max}$.

Table 1. Parametric constants for the Hellmann potential.

constant	Value
c_4	0
c_5	$-1/2$
c_6	$\varepsilon - \beta + 1/4$
c_7	$\alpha + \beta - 2\varepsilon$
c_8	$\varepsilon + \Lambda - \alpha$
c_9	$(l+1/2)^2$
c_{10}	$2\sqrt{\varepsilon + \Lambda - \alpha}$
c_{11}	$2l+1$
c_{12}	$\sqrt{\varepsilon + \Lambda - \alpha}$
c_{13}	$l+1$

3.2. Solutions via the Asymptotic Iteration Method

By using the AIM [35–51], we can also solve differential equation (18). The details about this method can be found in [50, 51]. For this purpose, we use a transformation of the form

$$\begin{aligned} u_{nl}(s) &= s^p(1-s)^q R(s), & p &= \sqrt{\varepsilon + \Lambda - \alpha}, \\ q &= l+1, \end{aligned} \quad (21)$$

by which (18) is easily transformed into the more convenient second-order homogeneous linear differential equation

$$\begin{aligned} R''(s) + \left[\frac{2p+1-s(2p+2q+1)}{s(1-s)} \right] R'(s) \\ - \left[\frac{(p+q)^2 - \varepsilon + \beta}{s(1-s)} \right] R(s) = 0 \end{aligned} \quad (22)$$

with a solution being found by using AIM [50, 51].

The systematic procedure of the AIM begins by rewriting (22) in the form [50, 51]

$$R''(s) - \lambda_k(s)R'(s) - S_k(s)R(s) = 0. \quad (23)$$

For sufficiently large k [50, 51], the following recurrence relation can be used to determine the λ_k and S_k ($k = 1, 2, 3, \dots$) values:

$$\begin{aligned} \lambda_k(s) &= \lambda'_{k-1}(s) + S_{k-1}(s) + \lambda_0(s)\lambda_{k-1}(s), \\ S_k(s) &= S'_{k-1}(s) + S_0(s)\lambda_{k-1}(s), \quad k = 1, 2, 3, \dots \end{aligned} \quad (24)$$

In accordance with AIM [50, 51], the energy eigenvalue equations are obtained from the roots of the equation

$$\delta_k = \begin{vmatrix} \lambda_k(s) & S_k(s) \\ \lambda_{k-1}(s) & S_{k-1}(s) \end{vmatrix} = 0, \quad k = 1, 2, 3, \dots \quad (25)$$

Table 2. Energy eigenvalues of the special case $a = 0, b \rightarrow -V_0$ in units $\hbar = \mu = 1$. For comparison with other methods, we set $V_0 = \sqrt{2}$ and $\delta = gb$.

States	g	Present (AIM & pNU)	SUSY [52]	AIM [53]	Numerical [54]
1s	0.002	-0.998001000	-0.99601	-0.996006	-0.99600
1s	0.005	-0.995006250	-0.99004	-0.990037	-0.99000
1s	0.010	-0.990025000	-0.98015	-0.980149	-0.98010
1s	0.020	-0.980100000	-0.96059	-0.960592	-0.96060
1s	0.025	-0.975156250	-0.95092	-0.950922	-0.95090
1s	0.050	-0.950625000	-0.90363	-0.903632	-0.90360
2s	0.002	-0.248004000	-0.24602	-0.246023	-0.24600
2s	0.005	-0.245025000	-0.24015	-0.240148	-0.24010
2s	0.010	-0.240100000	-0.23059	-0.230586	-0.23060
2s	0.020	-0.230400000	-0.21230	-0.212296	-0.21230
2s	0.025	-0.225625000	-0.20355	-0.203551	-0.20360
2s	0.050	-0.202500000	-0.16351	-0.163542	-0.16350
2p	0.002	-0.247001000	-0.24602	-0.246019	-0.24600
2p	0.005	-0.242506250	-0.24012	-0.240123	-0.24010
2p	0.010	-0.235025000	-0.23049	-0.230490	-0.23050
2p	0.020	-0.220100000	-0.21192	-0.211926	-0.21190
2p	0.025	-0.212656250	-0.20299	-0.202984	-0.20300
2p	0.050	-0.175625000	-0.16144	-0.161480	-0.16150
3p	0.002	-0.108672111	-0.10716	-0.107160	-0.10720
3p	0.005	-0.105034028	-0.10142	-0.101416	-0.10140
3p	0.010	-0.099025000	-0.09231	-0.092306	-0.09231
3p	0.020	-0.087211111	-0.07570	-0.075704	-0.07570
3p	0.025	-0.081406250	-0.06814	-0.068157	-0.06816
3p	0.050	-0.053402778	-0.03739	-0.037115	-0.03712
3d	0.002	-0.107778778	-0.10715	-0.107152	-0.10720
3d	0.005	-0.102784028	-0.10114	-0.101368	-0.10140
3d	0.010	-0.094469444	-0.09212	-0.092122	-0.09212
3d	0.020	-0.077877778	-0.07502	-0.075030	-0.07503
3d	0.025	-0.069600694	-0.06713	-0.067146	-0.06715
3d	0.050	-0.028402778	-0.03388	-0.033831	-0.03383

Thus, by using the above quantization condition (25) and recurrence relations (24), we can establish the following relations:

$$\begin{aligned}
 S_0\lambda_1 - S_1\lambda_0 &= 0 \Rightarrow p + q = 0 + \sqrt{\varepsilon - \beta}, \quad (26) \\
 S_1\lambda_2 - S_2\lambda_1 &= 0 \Rightarrow p + q = -1 + \sqrt{\varepsilon - \beta}, \\
 S_2\lambda_3 - S_3\lambda_2 &= 0 \Rightarrow p + q = -2 + \sqrt{\varepsilon - \beta}, \\
 &\vdots \\
 S_n\lambda_{n+1} - S_{n+1}\lambda_n &= 0 \Rightarrow p + q = -n + \sqrt{\varepsilon - \beta}.
 \end{aligned}$$

The energy eigenvalues can so be easily found by using the n th term of the series, i. e.

$$\begin{aligned}
 p + q &= -n + \sqrt{\varepsilon - \beta} \\
 \Leftrightarrow p^2 &= \left[-(n + q) + \sqrt{\varepsilon - \beta} \right]^2 \quad (27)
 \end{aligned}$$

or more explicitly as

$$E_{nl} = -b\delta - \frac{\mu}{2\hbar^2} \left[\frac{\hbar^2(n+l+1)\delta}{2\mu} - \frac{b - \alpha + \hbar^2 l(l+1)\delta/(2\mu)}{(n+l+1)} \right]^2. \quad (28)$$

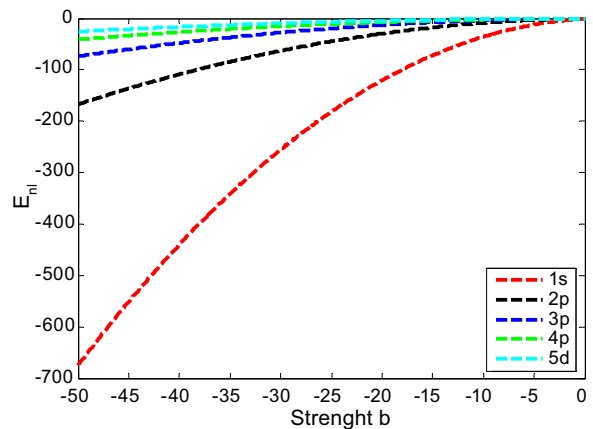


Fig. 2 (colour online). Variation of energy eigenvalues as function of strength b for various quantum states.

Table 3. Comparison of the calculated energy eigenvalues with the literature for $\delta = 0.01$ as a functions of the strength b .

b\state	Present (AIM & NU) (2p)	[35] (3p)	Present (AIM & NU) (3p)	[35] (3p)	Present (AIM & NU) (3d)	[35] (3d)
1	-0.06875625	-0.0720203	-0.034756250	-0.0366436	-0.033617361	-0.0368131
0.5	-0.14500625	-0.1454630	-0.067950694	-0.0671090	-0.066256250	-0.0671683
0.2	-0.20575625	-0.2044460	-0.094534028	-0.0918682	-0.092506250	-0.0918884
0	-0.25250625	-0.2500000	-0.115034028	-0.1110000	-0.112784028	-0.1110000
-0.2	-0.30425625	-0.3005450	-0.137756250	-0.1325530	-0.135284028	-0.1325370
-0.5	-0.39125625	-0.3857230	-0.176006250	-0.1688520	-0.173200694	-0.1688150
-1	-0.56125625	-0.5526640	-0.250867361	-0.2404040	-0.247506250	-0.2403410
-2	-0.99500625	-0.9802480	-0.442256250	-0.4250550	-0.437784028	-0.4249590
-5.0	-3.04625625	-3.0128600	-1.349756250	-1.3119900	-1.341950694	-1.3118500
-10	-8.96500625	-8.9004200	-3.973367361	-3.9010300	-3.960006250	-3.9008700
-20	-30.1775063	-30.050500	-13.38725625	-13.245600	-13.36278403	-13.245400

Table 4. Comparison of the calculated energy eigenvalues with the literature for $b = -10$ as a functions of the strength δ .

states	Present (AIM & NU) $\delta = 0.001$	[35] $\delta = 0.001$	Present (AIM & NU) $\delta = 0.01$	[35] $\delta = 0.01$	Present (AIM & NU) $\delta = 0.1$	[35] $\delta = 0.1$
1s	-35.99800006	-35.9900	-35.98000625	-35.9001	-35.80062500	-35.0124
2s	-8.998000250	-8.99000	-8.980025000	-8.90050	-8.802500000	-8.04820
2p	-8.998000250	-8.99000	-8.965006250	-8.90042	-8.650625000	-8.04037
3s	-3.998000562	-3.99001	-3.980056250	-3.90112	-3.805625000	-3.10435
3p	-3.997333674	-3.99001	-3.973367361	-3.90103	-3.736736111	-3.09701
3d	-3.996000062	-3.99001	-3.960006250	-3.90087	-3.600625000	-3.08240
4s	-2.248001000	-2.24002	-2.230100000	-2.15197	-2.060000000	-1.42667
4p	-2.247625766	-2.24002	-2.226326562	-2.15189	-2.020156250	-1.41993
4d	-2.246875391	-2.24002	-2.218789062	-2.15173	-1.941406250	-1.40652
4f	-2.245750062	-2.24001	-2.207506250	-2.15148	-1.825625000	-1.38656

In Figures 2 and 3, we show the variation of the energy eigenvalues as function of strength parameter and screening parameter, respectively, for various quantum states. Some numerical results are given in Tables 2–4. In Table 2, we take the parameter values $\hbar = \mu = 1$, $a = \sqrt{2}$, $b = 1$, and $\delta = 0.002, 0.005, 0.010, 0.020,$

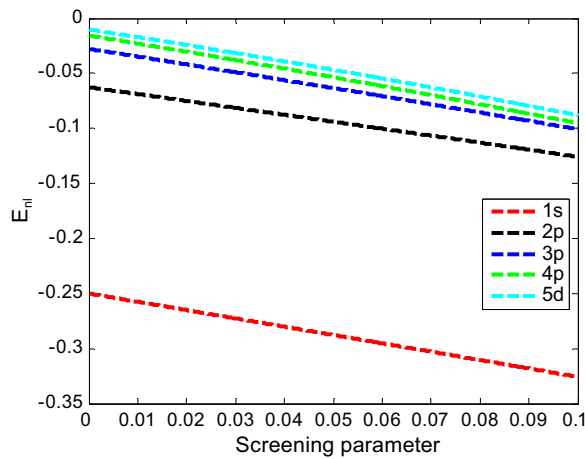


Fig. 3 (colour online). Variation of energy eigenvalues as function of screening parameter for various quantum states.

0.025, 0.050 to obtain the energy eigenvalues of the Hellmann potential for various states and compare them with those ones obtained by other methods. In Table 3, we show the numerical results with parameter $\delta = 0.01$ for various b , and also we keep δ constant at -10 and vary b in Table 4. Our numerical results are further compared with those obtained by the findings of other screening parameter methods.

Table 5. Comparison of the calculated negative eigenvalues (in a. u.) with the literature for the 2s states as functions of b and δ .

b	δ	Present (AIM & NU)	[19]
0.5	0.001	-0.0320005	-0.03174701400990
0.5	0.005	-0.035012500	-0.03367675354994
0.5	2	-3.5312500	-0.11290716132278
0.5	10	-57.53125000	-0.12339007950313
-0.5	0.001	-0.2815005	-0.28075099844730
-0.5	0.005	-0.282512500	-0.2787748073142
-0.5	2	-2.7812500	-0.14061295116700
-0.5	10	-52.78125000	-0.1268366598878
-2	0.001	-1.1245005	-1.12300199844620
-2	0.005	-1.122512500	-1.1150498066913
-2	2	-2.1250000	-0.20100449384560
-2	10	-46.12500000	-0.1342619146710

In Table 5, we computed the energy eigenvalues for the $2s$ state using three values of the parameter b , viz., 0.5 , -0.5 , and -2 , as a function of the screening parameter and then compare the results with those obtained previously. It is worth to be noted here that our results are only valid for low screening parameter. For instance, when $b = 0.5$ and $\delta = 0.001$, the local truncation error (LTE) is 0.00025349 . When increasing the screening parameter, say $\delta = 0.005$ (i. e. 5 times the formal), the LTE increases to 0.0013 (i. e. approximately 5 times the formal LTE). This shows how sensitive our results are to even a very small increment in the screening parameter. Moreover, while increasing the screening parameter to 2 and 10, it can be readily seen that our results are not in agreement with the ones obtained analytically. Hence, our approximation (14) is valid only for a very low screening parameter.

In the case when the screening parameter $\delta \rightarrow 0$, $b = 0$, and $a = Ze^2$, the potential (1) reduces to an attractive Coulomb potential field. Thus, in this limit the energy formula (20) turns to become the energy levels of the pure Coulomb interaction between electron and nucleus, i. e.,

$$E_{nl} = -\frac{\mu e^4 Z^2}{2\hbar^2 n'^2}, \tag{29}$$

where $n' = n + l + 1$ [30, 31].

In the case when $a = 0$ and $b = -V_0$, the potential (1) reduces to an attractive Yukawa potential field [35, 52–54]. Thus, in this limit the energy formula (20) turns to become the energy levels of the pure Yukawa interaction, i. e.,

$$E_{nl} = \frac{\hbar^2 l(l+1)\delta^2}{2\mu} - \frac{\hbar^2 \delta^2}{8\mu} \cdot \left[\frac{l(l+1) - 2\mu V_0 / (\delta \hbar^2)}{(n+l+1)} + (n+l+1) \right]^2. \tag{30}$$

To find corresponding wave functions, by using the parametric constants Table 1 and (8), we find the radial wave functions as

$$R_{nl}(s) = s^{\sqrt{\varepsilon+\Lambda-\alpha}}(1-s)^{l+1} \cdot P_n^{(2\sqrt{\varepsilon+\Lambda-\alpha}, 2l+1)}(1-2s) \tag{31}$$

or more explicit by substituting $s = e^{-\delta r}$:

$$u_{nl}(r) = N_{nl} \left(e^{-\delta r} \right)^{\sqrt{\varepsilon+\Lambda-\alpha}} (1 - e^{-\delta r})^{l+1} \cdot P_n^{(2\sqrt{\varepsilon+\Lambda-\alpha}, 2l+1)}(1 - 2e^{-\delta r}), \tag{32}$$

where the normalization constant N_{nl} is obtained as

$$N_{nl} = \left[\left(\delta n! \sqrt{\varepsilon + \Lambda - \alpha} \left[2n + 2l + 2\sqrt{\varepsilon + \Lambda - \alpha} \right] \Gamma \left(n + 2l + 2 + 2\sqrt{\varepsilon + \Lambda - \alpha} \right) \right) \left((n + l + 1) \Gamma(n + 2l + 2) \Gamma(n + 2\sqrt{\varepsilon + \Lambda - \alpha} + 1) \right)^{-1} \right]^{\frac{1}{2}}. \tag{33}$$

4. Conclusion

In this work, we have obtained the bound state solutions of the Schrödinger equation with the Hellmann potential within the framework of the pNU and AIM methods. The energy eigenvalues and the corresponding wave functions are obtained. Some numerical results are given in Tables 2, 3, 4, and 5. The comparison of numerical results with the findings of other method proves the success of the formalism. It is worth to be noted that we found exactly the same results by using both pNU and AIM methods. This has been shown numerically by Tables 2, 3, and 4. We also found that when the screening parameter $\delta \rightarrow 0$, the energy levels approach to the familiar pure Coulomb potential energy levels. In the low screening region where the screening parameter δ is small (i. e., $\delta \ll 1$), the potential reduces to the Killingbeck potential, i. e., $V(r) = ar^2 + br - c/r$ [32–34, 55], where a , b , and c are potential constants that can be obtained after making an expansion to the Hellmann potential. It can also be reduced into the Cornell potential [56–59], i. e., $V(r) = br - c/r$. These two potentials are usually used in the study of mesons and baryons. Further, when the screening parameter approaches to zero, the Hellmann potential turns to become the Coulomb potential.

Finally, the approximation we used in the present work is only valid for a very low screening parameter values.

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- [1] H. Hellmann, *Acta Physicochim*, URSS **1**, 913 (1935); *ibid.* **4**, 225 (1936); *ibid.* **4**, 324 (1936).
- [2] H. Hellmann, *J. Chem. Phys.* **3**, 61 (1935).
- [3] H. Hellmann and W. Kassatotchkin, *J. Chem. Phys.* **4**, 324 (1936).
- [4] J. Callaway and P. S. Laghos, *Phys. Rev.* **187**, 192 (1969).
- [5] G. McGinn, *J. Chem. Phys.* **53**, 3635 (1970).
- [6] V. K. Gryaznov, M. V. Zhernokletov, V. N. Zubarev, I. L. Losilevskii, and V. E. Tortov, *Zh. Eksp. Teor. Fiz.* **78**, 573 (1980) [*Sov. Phys. JETP* **51**, 288 (1980)].
- [7] V. A. Alekseev, V. E. Fortov, and I. T. Yakubov, *Usp. Fiz. Nauk.* **139**, 193 (1983) [*Sov. Phys. Usp.* **26**, 99 (1983)].
- [8] J. N. Das and S. Chakravarty, *Phys. Rev. A* **32**, 176 (1985).
- [9] Y. P. Varshni and R. C. Shukla, *Rev. Mod. Phys.* **35**, 130 (1963).
- [10] J. C. Philips and L. Kleinmann, *Phys. Rev. A* **116**, 287 (1959); *ibid.* **118**, 1153 (1960).
- [11] A. J. Hughes and J. Callaway, *Phys. Rev. A* **136**, 1390 (1964).
- [12] J. Adamowski, *Phys. Rev. A* **31**, 43 (1985).
- [13] R. Dutt, U. Mukherji, and Y. P. Varshni, *Phys. Rev. A* **34**, 777 (1986).
- [14] M. Bag, R. Dutt, and Y. P. Varshni, *J. Phys. B* **20**, 5267 (1987).
- [15] M. G. Kwato Njock, M. Nsangou, Z. Bona, S. G. Nana Engo, and B. Oumarou, *Phys. Rev. A* **61**, 42105 (2000).
- [16] E. R. Vrscaj, *Phys. Rev. A* **33**, 1433 (1986).
- [17] S. Stubbins, *Phys. Rev. A* **48**, 220 (1993).
- [18] M. A. Nunez, *Phys. Rev. A* **47**, 3620 (1993); *ibid.* **1591** (2005).
- [19] A. K. Roy, A. F. Jalbout, and E. I. Proynov, *J. Math. Chem.* **44**, 260 (2008).
- [20] I. Nasser and M. S. Abdelmonem, *Phys. Scr.* **83**, 55004 (2011).
- [21] R. L. Hall and Q. D. Katatbeh, *Phys. Lett. A* **287**, 183 (2001).
- [22] S. M. Ikhdair and R. Sever, *J. Mol. Struct. Theochem* **809**, 103 (2007).
- [23] R. Sever and C. Tezcan, *Phys. Scr.* **51**, 13 (1995).
- [24] G. Kocak, O. Bayrak, and I. Boztosun, *J. Theor. Comput. Chem.* **6**, 893 (2007).
- [25] S. M. Ikhdair and R. Sever, *Int. J. Mod. Phys. C* **19**, 1425 (2008).
- [26] S. M. Ikhdair, *J. Math. Phys.* **52**, 52303 (2011).
- [27] S. M. Ikhdair and R. Sever, *J. Math. Phys.* **52**, 122108 (2011).
- [28] S. M. Ikhdair, *J. Math. Phys.* **51**, 23525 (2010).
- [29] A. F. Nikiforov and V. B. Uvarov, *Special Functions of Mathematical Physics*, Birkhäuser, Berlin 1988.
- [30] L. I. Schiff, *Quantum Mechanics*, 3rd Edition, McGraw-Hill Book Co., New York 1955.
- [31] R. L. Liboff, *Introductory Quantum Mechanics*, 4th Edition, Addison Wesley, San Francisco 2003.
- [32] J. Killingbeck, *Phys. Lett. A* **65**, 87 (1987).
- [33] M. Hamzavi, S. M. Ikhdair, and K.-E. Thylwe, *Z. Naturforsch.* **67a**, 567 (2012).
- [34] M. Hamzavi and A. A. Rajabi, *Commun. Theor. Phys.* **55**, 35 (2011).
- [35] S. M. Ikhdair and B. J. Falaye, *Phys. Scr.* **87**, 35002 (2013).
- [36] O. Bayrak, I. Boztosun, and H. Ciftci, *Int. J. Quant. Chem.* **107**, 540 (2007).
- [37] E. Ateser, H. Ciftci, and M. Ugurlu, *Chin J. Phys.* **45**, 346 (2007).
- [38] O. Bayrak and I. Boztosun, *Phys. Scr.* **76**, 92 (2007).
- [39] F. Yasuk and A. Durmus, *Phys. Scr.* **77**, 15005 (2007).
- [40] A. Durmuş and F. Yasuk, *J. Chem. Phys.* **126**, 74108 (2007).
- [41] B. J. Falaye, *Cent. Eur. J. Phys.* **10**, 960 (2012).
- [42] B. J. Falaye, *Few Body Syst.* **53**, 557 (2012); *ibid.* **53**, 563 (2012).
- [43] B. J. Falaye, *J. Math. Phys.* **53**, 82107 (2012).
- [44] B. J. Falaye, K. J. Oyewumi, T. T. Ibrahim, M. A. Punyasena, and C. A. Onate, *Can. J. Phys.* **91**, 98 (2013).
- [45] B. J. Falaye, *Can. J. Phys.* **90**, 1259 (2012).
- [46] B. J. Falaye, *Can. J. Phys.* **91**, 365 (2013).
- [47] H. Ciftci, *Mod. Phys Lett A* **23**, 261 (2008).
- [48] O. Aydoğdu, *Pseudospin Symmetry and its Applications*, Ph. D. Thesis, Middle East Technical University, 2009.
- [49] B. J. Falaye, *Non-Perturbative Techniques and Their Applications*, M.Sc. Thesis, University of Ilorin, Ilorin 2013.
- [50] H. Ciftci, R. L. Hall, and N. Saad, *J. Phys. A: Math Gen.* **36**, 11807 (2003).
- [51] H. Ciftci, R. L. Hall, and N. Saad, *Phys. Lett. A* **340**, 388 (2005).
- [52] B. Gonul, K. Koksall, and E. Bakir, *Phys. Scripta* **73**, 279 (2006).
- [53] M. Karakoc and I. Boztosun, *Int. J. Mod. Phys. E* **15**, 1253 (2006).
- [54] F. J. Rogers, H. C. Jr. Graboske, and D. J. Harwood, *Phys. Rev. A* **1**, 1577 (1970).
- [55] S. M. Ikhdair, *Few Body Syst.* (2013), doi:10.1007/s00601-013-0693-2.
- [56] S. M. Ikhdair and R. Sever, *Int. J. Mod. Phys. A* **18**, 4215 (2003).
- [57] S. M. Ikhdair and R. Sever, *Int. J. Mod. Phys. A* **19**, 1771 (2004).
- [58] S. M. Ikhdair and R. Sever, *Int. J. Mod. Phys. A* **20**, 4035 (2005).
- [59] G. Plante and A. F. Antippa, *J. Math. Phys.* **46**, 62108 (2005).