General Solutions for Magnetohydrodynamic Natural Convection Flow with Radiative Heat Transfer and Slip Condition over a Moving Plate

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General solutions for the magnetohydrodynamic (MHD) natural convection flow of an incompressible viscous fluid over a moving plate are established when thermal radiation, porous effects, and slip condition are taken into consideration. These solutions, obtained in closed-form by Laplace transform technique, depend on the slip coefficient and the three essential parameters Gr, $\mathrm{Pr}_{\mathrm{eff}}$, and K_{eff}. They satisfy all imposed initial and boundary conditions and can generate a large class of exact solutions corresponding to different fluid motions with technical relevance. For illustration, two special cases are considered and some interesting results from the literature are recovered as limiting cases. The influence of pertinent parameters on the fluid motion is graphically underlined.

Key words: Natural Convection Flow; Radiative Heat Transfer; Slip Condition; General Solutions.

1. Introduction

Free convection flow over a moving vertical plate is extensively studied in the literature due to its wide applications in different engineering and environmental processes. It is also of great interest in industrial applications, and different investigations have been performed using analytical or numerical methods under different thermal conditions. The first exact solutions for the free convection flow of an incompressible viscous fluid past an impulsively started vertical plate seem to be those of Soundalgekar [1] and Ingham [2]. The free convection flow of such a fluid past an accelerated vertical plate has been later studied by Soundalgekar and Gupta [3], Raptis and Singh [4] and Singh and Kumar [5]. The influence of a magnetic field on the flow was also taken into consideration by Rapthis and Singh. This type of flow has large applications in polymer industry and metallurgy. On the other hand, flows through porous media also have numerous engineering and geophysical applications, and problems of free convection and heat transfer through such media under the influence of a magnetic field have attracted the attention of many researchers. The most recent analytical solutions for such flows seem to be those of Toki [6], Rajesh [7], Narahari and Ishak [8], Seth et al. [9], Samiulhaq et al. [10, 11] and Fetecau et al. [12]. However, in all these papers the possibility of fluid slippage at walls is not taken into consideration.

The phenomenon of slippage on the solid boundary appears in many applications and attracted the attention of many researchers. Khaled and Vafai [13] established exact solutions for the second problem of Stokes under slip condition. Mansour et al. [14] studied the magnetohydrodynamic (MHD) free convection flow of a micropolar fluid through a porous medium with periodic temperature and slip condition. Recently, Hamza et al. [15] brought to light the influence of magnetic field, radiative heat transfer, and slip condition on the unsteady flow of a viscous fluid through a channel filled with a porous medium and with an oscillating temperature on the boundary. The solutions that have been obtained are important since they help us to see if the wall slip has significant effects on the fluid velocity or it can be neglected.

The aim of this work is to provide general exact solutions for the unsteady MHD natural convection flow of an incompressible viscous fluid over a moving infinite plate with radiative heat transfer and slip condition. The viscous dissipation is neglected but porous effects are taken into consideration. The dimensionless governing equations are solved using Laplace transforms and exact solutions for temperature and velocity are established in integral forms in terms of three essential parameters only (effective Prandtl number $\mathrm{Pr}_{\mathrm{eff}}$, Grashof number Gr, and the effective permeability number K_{eff}). In order to illustrate the theoretical and practical value of general solutions, two special cases are considered and some known results from the literature are recovered as limiting cases. Finally, the influence of the slip parameter on the fluid motion, as well as the effects of pertinent parameters on the dimensionless velocity, is graphically underlined. The required time to reach the thermal steady-state in the case of oscillatory heating of the boundary is also determined.

2. Statement of the Problem

Let us consider the flow of an incompressible electrically conducting viscous fluid over an infinite vertical plate embedded in a porous medium. A uniform transverse magnetic field B_0 acts perpendicular to the plate. Initially, at time $t = 0$, both the fluid and plate are at rest at the constant temperature T_∞. At time $t = 0^+$, the plate starts to move in its plane with a variable velocity $U f_0(t)$, and its temperature is raised or lowered to the value $T_\infty + T_w h_0(t)$. The functions $f_0(\cdot)$ and $h_0(\cdot)$ are piecewise continuous and $f_0(0) = h_0(0) = 0$. We also take into consideration the possibility of fluid slippage at the wall. More precisely, the relative velocity between the fluid at the wall and the wall is assumed to be proportional to the shear rate at the wall.

The x-axis of the coordinate system is taken along the plate in the upward direction and the y-axis is normal to the plate. The induced magnetic field produced by the fluid motion is assumed to be negligible in comparison with the applied one. The radiative heat flux along the plate is also negligible in comparison to the y-direction. The plate is electrically nonconducting and all physical quantities, excepting the pressure, are functions of y and t only. Bearing in mind the above assumptions, neglecting the viscous dissipation and using the usual Boussinesq's approximation, the equations governing the laminar natural convection flow of an incompressible viscous fluid are [9]

$$\frac{\partial u(y,t)}{\partial t} = \nu \frac{\partial^2 u(y,t)}{\partial y^2} - \left(\frac{\sigma B_0^2}{\rho} + \frac{\nu}{K}\right) u(y,t) + g\beta\left[T(y,t) - T_\infty\right];\ y,t > 0, \tag{1}$$

$$\rho c_{\mathrm{p}} \frac{\partial T(y,t)}{\partial t} = k\frac{\partial^2 T(y,t)}{\partial y^2} - \frac{\partial q_{\mathrm{r}}(y,t)}{\partial y};\ y,t > 0, \tag{2}$$

where u, T, ν, σ, ρ, K, g, β, c_{p}, k, and q_{r} are, respectively, the velocity of the fluid, temperature, kinematic viscosity, electrical conductivity, fluid density, permeability of the porous medium, gravitational acceleration, volumetric coefficient of thermal expansion, specific heat at constant pressure, thermal conductivity, and the radiative heat flux.

By adopting the Rosseland approximation for the radiative heat flux q_{r} [9, Eq. (4)] and assuming small temperature difference between the fluid temperature T and the free stream temperature T_∞, (2) becomes

$$\frac{\partial T(y,t)}{\partial t} = \frac{k}{\rho c_{\mathrm{p}}}\left(1 + \frac{16\sigma^* T_\infty^3}{3kk^*}\right)\frac{\partial^2 T(y,t)}{\partial y^2};\ y,t > 0, \tag{3}$$

where k^* is the mean absorption coefficient and σ^* is the Stefan–Boltzmann constant. The appropriate initial and boundary conditions are

$$u(y,0) = 0,\ T(y,0) = T_\infty;\ y \ge 0, \tag{4}$$

$$u(0,t) - \alpha \left.\frac{\partial u(y,t)}{\partial y}\right|_{y=0} = U f_0(t), \quad T(0,t) = T_\infty + T_w h_0(t);\ t \ge 0, \tag{5}$$

$$u(y,t) \to 0,\ T(y,t) \to T_\infty \ \text{as}\ y \to \infty, \tag{6}$$

where $\alpha \ge 0$ is the slip coefficient. The negative values of α, as it results from [13], do not correspond to physical cases.

Introducing the following non-dimensional quantities

$$\begin{gathered} y^* = \frac{U}{\nu}y,\ t^* = \frac{U^2}{\nu}t,\ u^* = \frac{u}{U},\ T^* = \frac{T - T_\infty}{T_w}, \\ M^* = \frac{\nu\sigma B_0^2}{\rho U^2},\ K^* = \frac{U^2}{\nu^2}K,\ \alpha^* = \frac{U}{\nu}\alpha, \\ \mathrm{Gr} = \frac{\nu g\beta T_w}{U^3},\ \mathrm{Pr} = \frac{\mu c_p}{k},\ \mathrm{Nr} = \frac{16\sigma^* T_\infty^3}{3kk^*}, \\ f(t^*) = f_0\left(\frac{\nu t^*}{U^2}\right),\ h(t^*) = h_0\left(\frac{\nu t^*}{U^2}\right) \end{gathered} \tag{7}$$

and dropping out the star notation, we obtain the next non-dimensional initial-boundary value problem:

$$\frac{\partial^2 u(y,t)}{\partial y^2} - \frac{\partial u(y,t)}{\partial t} - K_{\text{eff}} u(y,t) + \text{Gr}T(y,t) = 0; \; y,t > 0, \tag{8}$$

$$\frac{\partial^2 T(y,t)}{\partial y^2} - \text{Pr}_{\text{eff}} \frac{\partial T(y,t)}{\partial t}; \; y,t > 0, \tag{9}$$

$$u(y,0) = 0, \; T(y,0) = 0; \; y \geq 0, \tag{10}$$

$$u(0,t) - \alpha \left.\frac{\partial u(y,t)}{\partial y}\right|_{y=0} = f(t), \quad T(0,t) = h(t); \; t \geq 0, \tag{11}$$

$$u(y,t) \to 0, \; T(y,t) \to 0 \text{ as } y \to \infty, \tag{12}$$

where Gr and Pr are Grashof and Prandtl numbers, Nr is the radiation-conduction parameter, $K_{\text{eff}} = M + 1/K$, and $\text{Pr}_{\text{eff}} = \text{Pr}/(1+\text{Nr})$ is the effective Prandtl number [16, Eq. (10)].

It is worth pointing out that the fluid velocity $u(y,t)$ does not depend on the magnetic and permeability parameters M and K, independently, but only by a combination of them K_{eff} that can be called the effective permeability. Consequently, the study of a fluid motion through a porous medium with or without magnetic effects is practically the same problem and a 'two parameter approach' is superfluous. The velocity of the fluid is the same for an infinite set of values of parameters M and K which correspond to the same effective permeability K_{eff}.

3. Solution of the Problem

In the following, the solutions of partial differential equations (8) and (9), with the initial and boundary conditions (10) – (12), will be determined by means of Laplace transforms. The energy equation (9) is not coupled to the momentum equation (8). Therefore, we shall firstly establish the exact solution for the temperature.

3.1. Temperature Distribution

Applying the Laplace transform to (9) and $(11)_2$ and using the initial condition $(10)_2$, we find that

$$\frac{\partial^2 \bar{T}(y,q)}{\partial y^2} = \text{Pr}_{\text{eff}} q \bar{T}(y,q), \; \bar{T}(0,q) = \bar{h}(q); \quad \bar{T}(y,q) \to 0 \text{ as } y \to \infty, \tag{13}$$

where $\bar{T}(y,q)$ and $\bar{h}(q)$ are the Laplace transforms of $T(y,t)$ and $h(t)$, respectively, and q is the transform parameter. The solution of the problem (13) is

$$\bar{T}(y,q) = \bar{h}(q) \exp\left(-y\sqrt{\text{Pr}_{\text{eff}} q}\right). \tag{14}$$

Taking the inverse Laplace transform of (14) and using (A1) from the Appendix and the convolution theorem, we find for the temperature $T(y,t)$ the integral expression

$$T(y,t) = \frac{y\sqrt{\text{Pr}_{\text{eff}}}}{2\sqrt{\pi}} \int_0^t \frac{h(t-s)}{s\sqrt{s}} \exp\left(-\frac{\text{Pr}_{\text{eff}} y^2}{4s}\right) \mathrm{d}s. \tag{15}$$

Equation (15) can be written in the equivalent form

$$T(y,t) = \frac{2}{\sqrt{\pi}} \int_{\frac{y\sqrt{\text{Pr}_{\text{eff}}}}{2\sqrt{t}}}^{\infty} h\left(t - \frac{\text{Pr}_{\text{eff}} y^2}{4s^2}\right) \mathrm{e}^{-s^2} \mathrm{d}s, \tag{16}$$

from which the boundary condition $(11)_2$ is clearly satisfied. In order to determine the Nusselt number, which is a measure of the surface heat transfer rate, we use the equality

$$\frac{\partial T(y,t)}{\partial y} = \frac{\sqrt{\text{Pr}_{\text{eff}}}}{2\sqrt{\pi}} \int_0^t \frac{h(t-s)}{s\sqrt{s}} \exp\left(-\frac{\text{Pr}_{\text{eff}} y^2}{4s}\right) \mathrm{d}s - \frac{\sqrt{\text{Pr}_{\text{eff}}}}{\sqrt{\pi}} \int_0^t \frac{h(t-s)}{\sqrt{s}} \frac{\text{Pr}_{\text{eff}} y^2}{4s^2} \exp\left(-\frac{\text{Pr}_{\text{eff}} y^2}{4s}\right) \mathrm{d}s, \tag{17}$$

resulting from (15), and integrate by parts the second integral. Direct computations show that

$$\frac{\partial T(y,t)}{\partial y} = -h(0)\frac{\sqrt{\text{Pr}_{\text{eff}}}}{\sqrt{\pi t}} \exp\left(-\frac{\text{Pr}_{\text{eff}} y^2}{4t}\right) - \frac{\sqrt{\text{Pr}_{\text{eff}}}}{\sqrt{\pi}} \int_0^t \frac{h'(t-s)}{\sqrt{s}} \exp\left(-\frac{\text{Pr}_{\text{eff}} y^2}{4s}\right) \mathrm{d}s. \tag{18}$$

Because $h(0) = 0$, it clearly results that

$$\text{Nu} = -\left.\frac{\partial T(y,t)}{\partial y}\right|_{y=0} = \frac{\sqrt{\text{Pr}_{\text{eff}}}}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{s}} h'(t-s) \,\mathrm{d}s. \tag{19}$$

3.2. Calculation of the Velocity Field

Applying the Laplace transform to (8) and using the corresponding initial and boundary conditions, we obtain the next problem for $\bar{u}(y,q)$:

$$\frac{\partial^2 \bar{u}(y,q)}{\partial y^2} - (q + K_{\text{eff}})\bar{u}(y,q) + \text{Gr}\bar{T}(y,q) = 0; \; y > 0, \tag{20}$$

$$\bar{u}(0,q) - \alpha \left.\frac{\partial \bar{u}(y,q)}{\partial y}\right|_{y=0} = \bar{f}(q); \quad \bar{u}(y,q) \to 0 \text{ as } y \to \infty, \tag{21}$$

where $\bar{T}(y,q)$ is given by (14). The solution of this problem is

$$\bar{u}(y,q) = \left[\bar{f}(q) - \mathrm{Gr}\bar{h}(q)\bar{a}(q)\right]\bar{b}(y,q) + \mathrm{Gr}\bar{h}(q)\bar{c}(y,q)\,, \tag{22}$$

where

$$\bar{a}(q) = \frac{1+\alpha\sqrt{\mathrm{Pr_{eff}}q}}{q(1-\mathrm{Pr_{eff}})+K_{\mathrm{eff}}}\,,$$
$$\bar{b}(y,q) = \frac{\exp(-y\sqrt{q+K_{\mathrm{eff}}})}{1+\alpha\sqrt{q+K_{\mathrm{eff}}}}\,, \quad \text{and}$$
$$\bar{c}(y,q) = \frac{\exp(-y\sqrt{\mathrm{Pr_{eff}}q})}{q(1-\mathrm{Pr_{eff}})+K_{\mathrm{eff}}}\,.$$

In order to obtain the (y,t)-domain solution, we need the inverse Laplace transforms of the functions $\bar{a}(q)$, $\bar{b}(y,q)$, and $\bar{c}(y,q)$. Writing $\bar{a}(q)$ in the suitable form

$$\bar{a}(q) = \frac{1}{1-\mathrm{Pr_{eff}}}\frac{1}{q+d} + \frac{\alpha\sqrt{\mathrm{Pr_{eff}}}}{1-\mathrm{Pr_{eff}}}\frac{q^{1/2}}{q+d}\,,$$
$$d = \frac{K_{\mathrm{eff}}}{1-\mathrm{Pr_{eff}}} \quad \text{for } \mathrm{Pr_{eff}} \neq 1$$

and using (A2) (see also [17]), we find that

$$a(t) = L^{-1}\{\bar{a}(q)\} = \frac{1}{1-\mathrm{Pr_{eff}}}\mathrm{e}^{-\mathrm{d}t} + \alpha\frac{\sqrt{\mathrm{Pr_{eff}}}}{1-\mathrm{Pr_{eff}}}\frac{1}{\sqrt{t}}E_{1,1/2}(-\mathrm{d}t)\,. \tag{23}$$

Here $E_{m,n}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(km+n)}$, with $m>0$ and $n>0$, is the Mittag–Leffler function [18].

The inverse Laplace transform of $\bar{b}(y,q)$, namely

$$b(y,t) = \frac{1}{\alpha\sqrt{\pi t}}\exp\left(-\frac{y^2}{4t} - K_{\mathrm{eff}}t\right) - \frac{1}{\alpha^2}\exp\left(\frac{y}{\alpha} + \frac{t}{\alpha^2} - K_{\mathrm{eff}}t\right)\mathrm{erfc}\left(\frac{y}{2\sqrt{t}} + \frac{\sqrt{t}}{\alpha}\right), \tag{24}$$

is obtained using (A3) and the properties (A4) from Appendix.

Finally, in view of (A5), it results that

$$c(y,t) = \frac{\mathrm{e}^{-\mathrm{d}t}}{2(1-\mathrm{Pr_{eff}})}\left[\mathrm{e}^{y\sqrt{-d\mathrm{Pr_{eff}}}}\mathrm{erfc}\left(\frac{y\sqrt{\mathrm{Pr_{eff}}}}{2\sqrt{t}} + \sqrt{-dt}\right) + \mathrm{e}^{-y\sqrt{-\mathrm{d}\mathrm{Pr_{eff}}}}\mathrm{erfc}\left(\frac{y\sqrt{\mathrm{Pr_{eff}}}}{2\sqrt{t}} - \sqrt{-dt}\right)\right], \tag{25}$$

while the velocity field $u(y,t)$ can be presented in the simple form

$$u(y,t) = \int_0^t f(t-s)b(y,s)\,\mathrm{d}s - \mathrm{Gr}\int_0^t\int_0^s h(t-s) \cdot a(s-\tau)b(y,\tau)\,\mathrm{d}\tau\,\mathrm{d}s + \mathrm{Gr}\int_0^t h(t-s)c(y,s)\,\mathrm{d}s\,, \tag{26}$$

where $a(t)$, $b(y,t)$, and $c(y,t)$ are given by (23)–(25). Neglecting the thermal effects, the velocity field reduces to

$$u_m(y,t) = \int_0^t f(t-s)b(y,s)\,\mathrm{d}s\,. \tag{27}$$

3.3. Solution in the Case of no-slip Condition (the Case $\alpha = 0$)

Equations (15) or (16) and (26) provide solutions in integral form for the fluid temperature and velocity corresponding to the flow with slip boundary condition. In order to underline the effects of the slippage on the fluid flow, we need the velocity field corresponding to $\alpha = 0$. It can be obtained starting again from (22) with $\bar{a}_0(q) = \frac{1}{q(1-\mathrm{Pr_{eff}})+K_{\mathrm{eff}}}$ and $\bar{b}_0(y,q) = \mathrm{e}^{-y\sqrt{q+K_{\mathrm{eff}}}}$ instead of $\bar{a}(q)$ and $\bar{b}(y,q)$.

The inverse Laplace transform of $\bar{a}_0(q)$ can be directly obtained making $\alpha = 0$ into (23), while the inverse Laplace transform $b_0(y,t)$ of $\bar{b}_0(y,q)$, namely

$$b_0(y,t) = \frac{y}{2t\sqrt{\pi t}}\exp\left(-\frac{y^2}{4t} - K_{\mathrm{eff}}t\right), \tag{28}$$

can be obtained using (A1) and the first shift property of the Laplace transform (see the property $(A4)_1$). The corresponding velocity field is

$$u_0(y,t) = \int_0^t f(t-s)b_0(y,s)\,\mathrm{d}s - \mathrm{Gr}\int_0^t\int_0^s h(t-s) \cdot a_0(s-\tau)b_0(y,\tau)\,\mathrm{d}\tau\,\mathrm{d}s + \mathrm{Gr}\int_0^t h(t-s)c_0(y,s)\,\mathrm{d}s\,, \tag{29}$$

where $c_0(y,t) = c(y,t)$ and $a_0(t) = \mathrm{e}^{-\mathrm{d}t}/(1-\mathrm{Pr_{eff}})$. The second term of this solution can be further simplified using a result obtained in [19] and given here by (A6). Indeed, in view of (A6), we have

$$L^{-1}\{\bar{a}_0(q)\mathbf{b}_0(y,q)\} = (a_0 * b_0)(t) = \frac{\mathrm{e}^{-\mathrm{d}t}}{2(1-\mathrm{Pr_{eff}})} \cdot \left[\mathrm{e}^{y\sqrt{-\mathrm{Pr_{eff}}\mathrm{d}}}\mathrm{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{-\mathrm{Pr_{eff}}\,\mathrm{d}t}\right) + \mathrm{e}^{-y\sqrt{-\mathrm{Pr_{eff}}\mathrm{d}}} \cdot \mathrm{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{-\mathrm{Pr_{eff}}\,\mathrm{d}t}\right)\right], \tag{30}$$

where the star notation denotes the convolution product.

In the absence of thermal effects, our solution (29) becomes

$$u_{0m}(y,t) = \frac{y}{2\sqrt{\pi}} \int_0^t \frac{f(t-s)}{s\sqrt{s}} \cdot \exp\left(-\frac{y^2}{4s} - K_{\text{eff}} s\right) \mathrm{d}s\,. \tag{31}$$

In view of an important remark resulting from [20, Eq. (35) with $\alpha = 0$] (namely, for such motions of Newtonian fluids the shear stress satisfies the same partial differential equation as does the velocity), it is clear that our last result is in accordance with a known result from [12]. Indeed, the non-dimensional shear stress

$$\tau_m(y,t) = \frac{\partial u_m(y,t)}{\partial y} = \frac{y}{2\sqrt{\pi}} \int_0^t \frac{f(t-s)}{s\sqrt{s}} \exp\left(-\frac{y^2}{4s} - K_p s\right) \mathrm{d}s\,, \tag{32}$$

as it results from [12, Eq. (19)], is identical as form to our solution (31) for the non-dimensional velocity $u_{0m}(y,t)$.

4. Applications

The general solutions (15), (26), and (29) can be used to give the temperature and velocity distributions for any motion problem with physical relevance. For illustration, two special cases are considered and some known results from the literature are recovered.

4.1. Flow over an Isothermal Suddenly Moved Plate

Let us now consider the flow over an infinite plate which is initially at rest and is suddenly moved in its own plane with the velocity $U(1-\gamma e^{\delta t})$ with $\gamma \geq 0$ and $\delta > 0$. The temperature of the plate is T_w for $t > 0$, and the functions $f(\cdot)$ and $h(\cdot)$ become identically to $H(t)(1-\gamma e^{\delta t})$ and $H(t)$, where $H(\cdot)$ is the Heaviside unit step function. Temperature distribution and the Nusselt number, as it results from (16), (19), and (A7), are identically to those obtained in [12, Eq. (12)] (see also [9, Eqs. (19) and (23)]), namely

$$T(y,t) = \operatorname{erfc}\left(\frac{y}{2}\sqrt{\frac{\text{Pr}_{\text{eff}}}{t}}\right), \quad \text{Nu} = \sqrt{\frac{\text{Pr}_{\text{eff}}}{\pi t}}\,. \tag{33}$$

On the other hand, direct computations show that in this case

$$(h*a)(t) = \frac{1-e^{-\mathrm{d}t}}{K_{\text{eff}}} + \alpha \frac{\sqrt{\text{Pr}_{\text{eff}} t}}{1-\text{Pr}_{\text{eff}}} E_{1,3/2}(-\mathrm{d}t)\,, \tag{34}$$

$$(h*c)(y,t) = \frac{1}{K_{\text{eff}}} \operatorname{erfc}\left(\frac{y\sqrt{\text{Pr}_{\text{eff}}}}{2\sqrt{t}}\right) - \frac{e^{-\mathrm{d}t}}{2K_{\text{eff}}} \cdot \left[e^{y\sqrt{-\mathrm{d}\text{Pr}_{\text{eff}}}} \operatorname{erfc}\left(\frac{y\sqrt{\text{Pr}_{\text{eff}}}}{2\sqrt{t}} + \sqrt{-\mathrm{d}t}\right) + e^{-y\sqrt{-\mathrm{d}\text{Pr}_{\text{eff}}}} \operatorname{erfc}\left(\frac{y\sqrt{\text{Pr}_{\text{eff}}}}{2\sqrt{t}} - \sqrt{-\mathrm{d}t}\right)\right] \tag{35}$$

and the corresponding velocity

$$u(y,t) = \int_0^t (1-\gamma e^{\delta(t-s)}) b(y,s)\,\mathrm{d}s - \text{Gr} \int_0^t (h*a)(s) \cdot b(y,t-s)\,\mathrm{d}s + \text{Gr}(h*c)(t) \tag{36}$$

is obtained introducing (34) and (35) into (26) with $f(t) = H(t)(1-\gamma e^{\delta t})$.

Lengthy but straightforward computations show that the solution corresponding to the no-slip condition on the boundary, namely

$$\begin{aligned} u_0(y,t) &= \frac{1}{2}\left(1 - \frac{\text{Gr}}{K_{\text{eff}}}\right)\left[e^{y\sqrt{K_{\text{eff}}}} \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{K_{\text{eff}} t}\right) + e^{-y\sqrt{K_{\text{eff}}}} \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{K_{\text{eff}} t}\right)\right] - \frac{\gamma}{2} e^{-\delta t}\left[e^{y\sqrt{K_{\text{eff}}-\delta}}\right. \\ &\cdot \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{(K_{\text{eff}}-\delta)t}\right) + e^{-y\sqrt{K_{\text{eff}}-\delta}} \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} \left. - \sqrt{(K_{\text{eff}}-\delta)t}\right)\right] + \frac{\text{Gr}}{2} \frac{e^{-\mathrm{d}t}}{K_{\text{eff}}}\left[e^{y\sqrt{-\mathrm{d}\text{Pr}_{\text{eff}}}} \operatorname{erfc}\left(\frac{y}{2\sqrt{t}}\right.\right. \\ &\left. + \sqrt{-\text{Pr}_{\text{eff}}\mathrm{d}t}\right) + e^{-y\sqrt{-\mathrm{d}\text{Pr}_{\text{eff}}}} \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{-\text{Pr}_{\text{eff}}\mathrm{d}t}\right)\Big] \\ &+ \frac{\text{Gr}}{K_{\text{eff}}} \operatorname{erfc}\left(\frac{y\sqrt{\text{Pr}_{\text{eff}}}}{2\sqrt{t}}\right) - \frac{\text{Gr}\,e^{-\mathrm{d}t}}{2K_{\text{eff}}}\left[e^{y\sqrt{-\mathrm{d}\text{Pr}_{\text{eff}}}}\right. \\ &\cdot \operatorname{erfc}\left(\frac{y\sqrt{\text{Pr}_{\text{eff}}}}{2\sqrt{t}} + \sqrt{-\mathrm{d}t}\right) + e^{-y\sqrt{-\mathrm{d}\text{Pr}_{\text{eff}}}} \\ &\left. \cdot \operatorname{erfc}\left(\frac{y\sqrt{\text{Pr}_{\text{eff}}}}{2\sqrt{t}} - \sqrt{-\mathrm{d}t}\right)\right], \end{aligned} \tag{37}$$

reduces to that obtained by Seth et al. [9, Eq. (20)] if $\gamma = 0$. In the absence of thermal effects and if $\gamma = 0$, (37) takes the simple form

$$u_m(y,t) = \frac{1}{2}\left[e^{y\sqrt{K_{\text{eff}}}} \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{K_{\text{eff}} t}\right) + e^{-y\sqrt{K_{\text{eff}}}} \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{-K_{\text{eff}} t}\right)\right]. \tag{38}$$

By neglecting porous and magnetic effects, the classical solution is recovered.

4.2. Flow over an Oscillating Plate with Oscillatory Heating

Let us suppose that after time $t = 0$ the infinite plate is oscillating in its plane and its temperature is also oscillatory. In this case, the functions $f(\cdot)$ and $h(\cdot)$ can be taken to be $\sin(\omega t)$, $H(t)\cos(\omega t)$ or a combination of them if the frequencies of thermal and mechanical oscillations are the same. By making $h(t) = \sin(\omega t)$ into (15), where ω is the frequency of oscillations, it results the staring solution for temperature

$$T(y,t) = \frac{y\sqrt{\mathrm{Pr}_{\mathrm{eff}}}}{2\sqrt{\pi}} \int_0^t \frac{\sin[\omega(t-s)]}{s\sqrt{s}} \cdot \exp\left(-\frac{\mathrm{Pr}_{\mathrm{eff}} y^2}{4s}\right) \mathrm{d}s\,. \tag{39}$$

This solution can be written as a sum of the steady-state

$$T_{\mathrm{s}}(y,t) = \frac{y\sqrt{\mathrm{Pr}_{\mathrm{eff}}}}{2\sqrt{\pi}} \int_0^\infty \frac{\sin[\omega(t-s)]}{s\sqrt{s}} \cdot \exp\left(-\frac{\mathrm{Pr}_{\mathrm{eff}} y^2}{4s}\right) \mathrm{d}s\,, \tag{40}$$

and transient

$$T_{\mathrm{t}}(y,t) = -\frac{y\sqrt{\mathrm{Pr}_{\mathrm{eff}}}}{2\sqrt{\pi}} \int_t^\infty \frac{\sin[\omega(t-s)]}{s\sqrt{s}} \cdot \exp\left(-\frac{\mathrm{Pr}_{\mathrm{eff}} y^2}{4s}\right) \mathrm{d}s \tag{41}$$

solutions. The steady-state solution $T_{\mathrm{s}}(y,t)$ can be further processed to give the simple form (see (A8) or [21, Eq. (3.928)] after a suitable change of variable)

$$T_{\mathrm{s}}(y,t) = \exp\left(-y\sqrt{\frac{\omega \mathrm{Pr}_{\mathrm{eff}}}{2}}\right) \cdot \sin\left(\omega t - y\sqrt{\frac{\omega \mathrm{Pr}_{\mathrm{eff}}}{2}}\right). \tag{42}$$

The Nusselt number, as it results from (19), is

$$\mathrm{Nu} = \sqrt{2\omega \mathrm{Pr}_{\mathrm{eff}}}\left[C(\omega t)\cos(\omega t) + S(\omega t)\sin(\omega t)\right], \tag{43}$$

where C(t) and S(t) are the Fresnel cosine and sine integrals [21]. It can be also written as a sum of steady-state and transient components. Its steady-state component Nu_{s} can be written in the simple form (see for instance (A9) with $p = 2$)

$$\mathrm{Nu}_{\mathrm{s}} = \sqrt{\omega \mathrm{Pr}_{\mathrm{eff}}} \sin\left(\omega t + \frac{\pi}{4}\right). \tag{44}$$

The corresponding velocity field, resulting from (26) for $f(t) = h(t) = \sin(\omega t)$, can be also processed but the final result is not simpler.

However, it is worth pointing out that in the absence of thermal effects our velocity field

$$\begin{aligned} u(y,t) = {} & \frac{1}{\alpha\sqrt{\pi}} \int_0^t \frac{\sin(t-s)}{\sqrt{s}} \exp\left(-\frac{y^2}{4s} - K_{\mathrm{eff}} s\right) \mathrm{d}s \\ & - \frac{1}{\alpha^2} \mathrm{e}^{y/\alpha} \int_0^t \sin(t-s) \exp\left(\frac{s}{\alpha^2} - K_{\mathrm{eff}} s\right) \\ & \cdot \mathrm{erfc}\left(\frac{y}{2\sqrt{s}} + \frac{\sqrt{s}}{\alpha}\right) \mathrm{d}s \end{aligned} \tag{45}$$

is identical to that obtained by Hayat et al. [22, Eq. (12)].

The general solution corresponding to the no-slip condition can be obtained in the same way from (29). It also can be written as a sum between steady-state and transient solutions. In absence of thermal effects, the steady-state component $u_{0ms}(y,t)$ of

$$u_{0m}(y,t) = \frac{y}{2\sqrt{\pi}} \int_0^t \frac{\sin[\omega(t-s)]}{s\sqrt{s}} \exp\left(-\frac{y^2}{4s} - K_{\mathrm{eff}} s\right) \mathrm{d}s\,, \tag{46}$$

for instance, can be written in the simple form

$$u_{0ms}(y,t) = \mathrm{e}^{-my}\sin(\omega t - ny)\,; \quad m,n = \sqrt{\frac{\sqrt{\omega^2 + K_{\mathrm{eff}}^2} \pm K_{\mathrm{eff}}}{2}}\,. \tag{47}$$

By neglecting the porous and magnetic effects, this is taking $K_{\mathrm{eff}} = 0$, our last relation reduces to the well-known equality (17) from [23] (see also the corresponding solution of Rajagopal [24, page 371 with $\alpha_1 = 0$]).

5. Graphical Results, Discussion, and Conclusions

A theoretical study of the MHD natural convection flow of an incompressible viscous fluid over an infinite moving plate is developed when radiative, porous, and slippage effects are taken into consideration. General exact solutions are established for the dimensionless temperature, velocity, and surface heat transfer rate when the plate is sliding in its plane with an arbitrary velocity $Uf_0(t)$, and its temperature $T_w h_0(t)$ is also arbitrary. They satisfy all imposed initial and boundary conditions and can be used to generate exact solutions for various fluid motions with technical relevance. For illustration, as well as for a check of general results, two interesting cases are considered, and some known results from the literature are recovered as limiting cases. In the case of oscillating plate with oscillatory heating, the starting solutions can be presented as a sum of steady-state and transient solutions.

The starting solutions describe the motion of the fluid some time after its initiation. After that time, when the transients disappear, they tend to the steady-state solutions that are periodic in time and independent of the initial conditions. However, the steady-state solutions satisfy the governing equations and boundary conditions. Such solutions are important for those who want to eliminate the transients from their experiments.

Now, in order to bring to light some relevant physical aspects of results that have been obtained, the velocity and temperature profiles are presented for different situations with typical values of parameters. The influence of $\mathrm{Pr_{eff}}$ on the temperature, in the case of an isothermal suddenly moved plate, was shown in [12, Fig. 6]. Our interest here is to determine the required time to reach the steady-state in the case of an oscillatory heating on the boundary. This time, as it results from (40) and (43), depends on the effective Prandtl number and the frequency of oscillations ω. Figures 1 and 2 clearly show that the required time to reach the thermal steady-state increases with respect to $\mathrm{Pr_{eff}}$ and decreases with regard to ω. After this time, that seems to be small enough, the dimensionless temperature distribution in fluid varies according to the steady-state solution (43).

In the first case, of the motion over an isothermal suddenly moved plate, we are equally interested on the influence of slip parameter α and of pertinent parameters $\mathrm{Pr_{eff}}$, K_{eff}, and Gr on the fluid motion. From Figure 3, that presents profiles of the velocities $u(y,t)$ and $u_0(y,t)$ against y, it is clearly seen that the slip parameter α has a significant influence on the fluid motion. Furthermore, as expected, the fluid velocity is a decreasing function with respect to α, and the velocity profiles corresponding to $u(y,t)$ tend to superpose over that of $u_0(y,t)$ as α approaches to zero. All velocity profiles smoothly decrease from maximum values at the boundary to a minimum value for large values of y. The influence of $\mathrm{Pr_{eff}}$ and K_{eff} on the fluid motion is underlined by Figures 4 and 5. The dimensionless velocity of the fluid is a decreasing function with respect to both numbers. Velocity profiles monotonically decay for all values of $\mathrm{Pr_{eff}}$ from maximum values at the wall to zero in the free stream. However, Figure 5 shows that for smaller values of K_{eff} (i. e. 3 and 4) there are velocity over-shoots close to the moving plate. Then, the velocity profiles smoothly descend to their lowest values for large values of y. Velocity profiles against y are also depicted in Figure 6 for different positive and negative values of Gr. Positive or negative values of Gr correspond to the cooling, respectively heating of the plate by natural convection. It is clearly seen that the velocity is an increasing function with regard to Gr in the case of cooling and a reverse effect is observed in the case of heating of the plate. For positive values of Gr, for instance, the values of the velocity at any distance y are always higher for $\mathrm{Gr}=3$ than that for $\mathrm{Gr}=1$ or 2. Furthermore, the boundary layer thickness increases with respect to Gr and decreases if $\mathrm{Pr_{eff}}$ or K_{eff} increases.

Finally, for comparison, profiles of the velocity $u(y,t)$ given by (36) and of its thermal component $u_t(y,t)$ are presented in Figure 7 against y for different values of time and the same values of common parameters. It is clearly seen that the thermal effects are significant and the difference $u(y,t)-u_t(y,t)$ monoton-

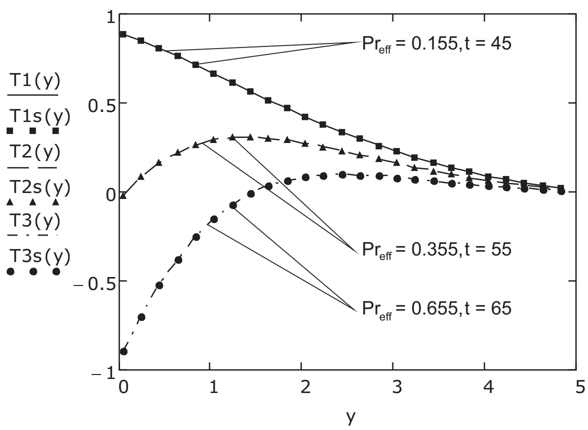

Fig. 1. Required time to reach the thermal steady-state in the case of oscillatory heating of the plate, with an error of 10^{-4}, for $\omega=2$ and different values of $\mathrm{Pr_{eff}}$.

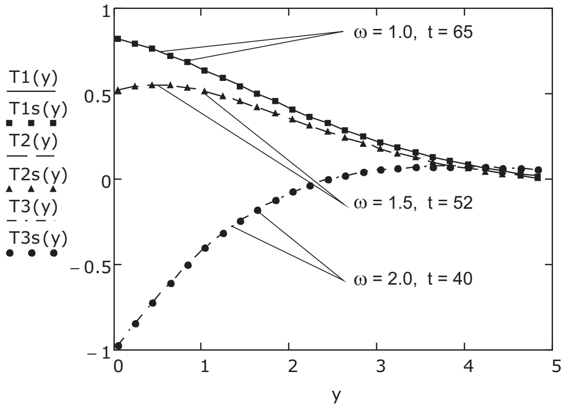

Fig. 2. Required time to reach the thermal steady-state in the case of oscillatory heating of the plate, with an error of 10^{-4}, for $\mathrm{Pr_{eff}} = 0.355$ and different values of ω.

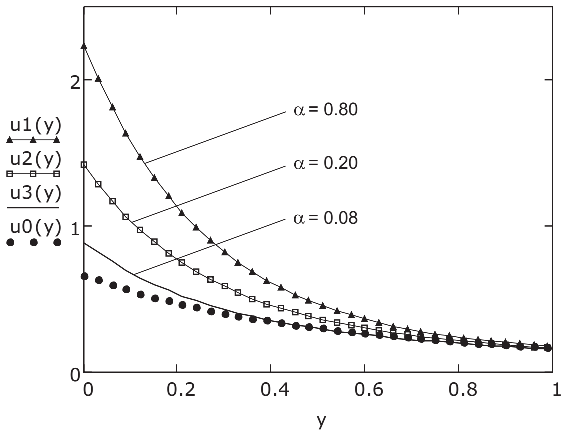

Fig. 3. Profiles of velocities $u(y,t)$ and $u_0(y,t)$ given by (36) and (37) for $\mathrm{K} = 7$, $\mathrm{Pr} = 0.355$, $\mathrm{Gr} = 2$, $\gamma = 0.5$, $\delta = 0.8$, $t = 0.5$, and different values of α.

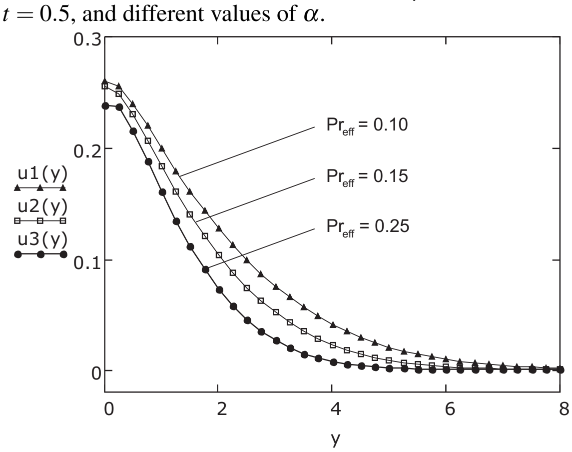

Fig. 4. Profiles of velocity $u(y,t)$ given by (36) for $\mathrm{K_{eff}} = 7$, $\mathrm{Gr} = 2$, $\alpha = 0.4$, $\gamma = 0.5$, $\delta = 0.8$, $t = 0.5$, and different values of $\mathrm{Pr_{eff}}$.

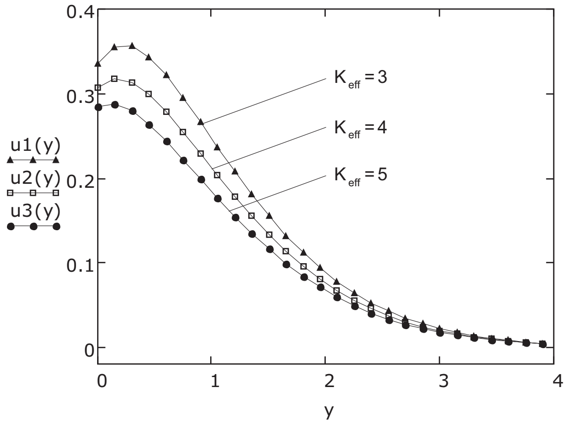

Fig. 5. Profiles of velocity $u(y,t)$ given by (36) for $\mathrm{Pr_{eff}} = 0.355$, $\mathrm{Gr} = 2$, $\alpha = 0.4$, $\gamma = 0.5$, $\delta = 0.8$, $t = 0.5$, and different values of $\mathrm{K_{eff}}$.

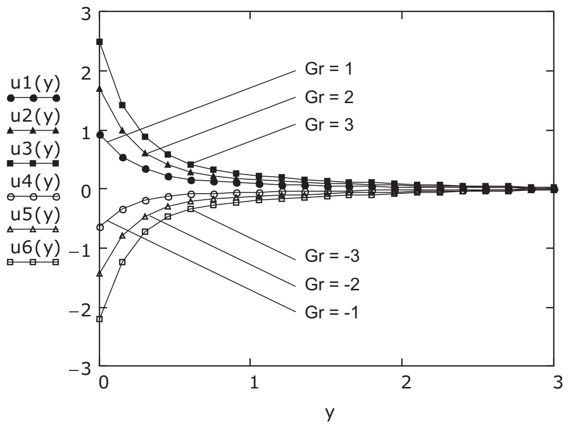

Fig. 6. Profiles of velocity $u(y,t)$ given by (36) for $\mathrm{K_{eff}} = 7$, $\mathrm{Pr_{eff}} = 0.355$, $\alpha = 0.4$, $\gamma = 0.5$, $\delta = 0.8$, $t = 0.5$, and different values of Gr.

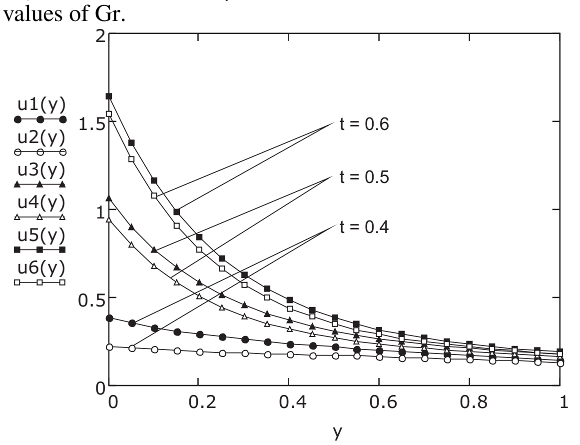

Fig. 7. Comparison between the velocity $u(y,t)$ given by (36) and thermal component $u_t(y,t)$ for $\mathrm{K_{eff}} = 7$, $\mathrm{Pr_{eff}} = 0.25$, $\mathrm{Gr} = 2$, $\alpha = 0.4$, $\gamma = 0.5$, $\delta = 0.8$, and different values of t.

ically decreases both with respect to the temporal variable t and with respect to y. Consequently, the thermal effects as well as the slippage on the solid wall are notable, and they have to be taken into consideration. All graphical representations have been performed using the program Mathcad 14.0.

Appendix

$$L^{-1}\left\{e^{-y\sqrt{aq}}\right\} = \frac{y\sqrt{a}}{2t\sqrt{\pi t}}\exp\left(-\frac{ay^2}{4t}\right), \tag{A1}$$

$$L^{-1}\left\{\frac{q^{\alpha-\beta}}{q^{\alpha}-a}\right\} = t^{\beta-1}E_{\alpha,\beta}(at^{\alpha});\ \alpha,\beta>0, \tag{A2}$$

$$L^{-1}\left\{\frac{e^{-a\sqrt{q}}}{b+\sqrt{q}}\right\} = \frac{1}{\sqrt{\pi t}}\exp\left(-\frac{a^2}{4t}\right) - b\,e^{ab+b^2t}\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}+bt\right), \tag{A3}$$

$$L^{-1}\{F(q+a)\} = f(t)e^{-at},\ L^{-1}\left\{F\left(\frac{q}{a}\right)\right\} = af(at)\ \text{if}\ f(t)=L^{-1}\{F(q)\}, \tag{A4}$$

$$L^{-1}\left\{\frac{e^{-y\sqrt{aq}}}{q+b}\right\} = \frac{e^{-bt}}{2}\left[e^{y\sqrt{-ab}}\operatorname{erfc}\left(\frac{y\sqrt{a}}{2\sqrt{t}}+\sqrt{-bt}\right) + e^{-y\sqrt{-ab}}\operatorname{erfc}\left(\frac{y\sqrt{a}}{2\sqrt{t}}-\sqrt{-bt}\right)\right], \tag{A5}$$

$$L^{-1}\left\{\frac{e^{-y\sqrt{q+a}}}{q-b}\right\} = \frac{e^{bt}}{2}\left[e^{y\sqrt{a+b}}\operatorname{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{(a+b)t}\right) + e^{-y\sqrt{a+b}}\operatorname{erfc}\left(\frac{y\sqrt{a}}{2\sqrt{t}}-\sqrt{(a+b)t}\right)\right], \tag{A6}$$

$$\int_0^t f(s)\delta(t-s)\,ds = f(t) \quad \text{if}\ \delta(\cdot)\ \text{is the Dirac delta function}, \tag{A7}$$

$$\int_0^\infty e^{-a^2s^2}\sin\left(\frac{b^2}{s^2}\right)ds = \frac{\sqrt{\pi}}{2a}e^{-ab\sqrt{2}}\sin(ab\sqrt{2}),\quad \int_0^\infty e^{-a^2s^2}\cos\left(\frac{b^2}{s^2}\right)ds = \frac{\sqrt{\pi}}{2a}e^{-ab\sqrt{2}}\cos(ab\sqrt{2}), \tag{A8}$$

$$\int_0^\infty \sin(as^p)\,ds = \frac{\Gamma\left(\frac{1}{p}\right)\sin\left(\frac{\pi}{2p}\right)}{pa^{1/p}},\quad \int_0^\infty \cos(as^p)\,ds = \frac{\Gamma\left(\frac{1}{p}\right)\cos\left(\frac{\pi}{2p}\right)}{pa^{1/p}};\ a>0,\ p>1. \tag{A9}$$

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