Exact Solutions for Unsteady Magnetohydrodynamic Oscillatory Flow of a Maxwell Fluid in a Porous Medium

Ilyas Khan, Farhad Ali, Samiulhaq, and Sharidan Shafie

Department of Mathematical Sciences, Faculty of Science, University of Technology Malaysia, 81310 UTM Skudai, Malaysia

Reprint requests to I. K.; E-mail: ilyaskhanqau@yahoo.com

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In this paper, exact solutions of velocity and stresses are obtained for the magnetohydrodynamic (MHD) flow of a Maxwell fluid in a porous half space by the Laplace transform method. The flows are caused by the cosine and sine oscillations of a plate. The derived steady and transient solutions satisfy the involved differential equations and the given conditions. Graphs for steady-state and transient velocities are plotted and discussed. It is found that for a large value of the time t, the transient solutions disappear, and the motion is described by the corresponding steady-state solutions.

Key words: Exact Solutions; MHD Flow; Maxwell Fluid; Porous Medium.

1. Introduction

The problems resulting from the flows of incompressible non-Newtonian fluids have been of great and increasing interest for the last five decades. Such fluids differ from the Newtonian fluids in that the relationship between the shear stress and the velocity gradient is more complicated. Examples of the non-Newtonian fluids are coal water, jellies, toothpaste, ketchup, food products, inks, glues, soaps, blood, and polymer solutions. It is well accepted now that the flow behaviour of the non-Newtonian fluids cannot be described by the Navier–Stokes equations. The constitutive equations of these fluids lead to flow problems in which the order of the differential equations exceeds the number of available conditions. The solutions of resulting problems for these fluids are in general more difficult to obtain and more complex than the Navier–Stokes equations. This is not only true for exact analytic solutions but even for numerical solutions. With all these difficulties, several recent researchers [1 – 14] are still involved in the study of steady and unsteady flows of the non-Newtonian fluids.

Further, the flows of non-Newtonian fluids filling a porous medium are of considerable practical and theoretical interest. The applications are in numerous areas such as ground water flow, irrigation problems, thermal and insulating engineering, ventilation of rooms, grain storage devices, chemical catalytic reactors, and many others. Having such motivation in mind, some contributions [15 – 18] discuss the flows in a porous medium.

In recent years, the rate type fluid models have received special attention. The first viscoelastic rate type model, which is still used widely, is due to Maxwell. Maxwell himself recognized that some liquids have a trend for storing energy and a means for dissipating energy, the storing of energy characterizing the fluid's elastic response and the dissipation of energy characterizing its viscous nature. The Maxwell model is the simplest subclass of rate type fluids which takes into consideration the stress relaxation effect. Having this motivation in mind, Fetecau et al. [19] studied the Stokes second problem in a Maxwell fluid. Vieru and Rauf [20] obtained the exact solutions of Stokes flows for a Maxwell fluid whereas Vieru and Zafar [21] recently investigated some Couette flows of a Maxwell fluid. In both of these papers the slip boundary condition is used and the solutions are obtained using the Laplace transform technique [22, 23].

The aim of the current study is to extend the flow analysis of [19] in two directions, i. e. for magnetohydrodynamic (MHD) effects and porous medium. With this motivation, the paper is organized as follows: the next section follows the problem formulation; Sections three to five present the solutions, graphical results, and conclusions.

2. Definition of the Problem

Let us consider the unsteady unidirectional flow of a MHD incompressible Maxwell fluid filling the semi-infinite porous space bounded by an infinite plate. A uniform magnetic field of strength B_0 is extended normal to the flow direction. At time $t = 0^+$, both the fluid and plate are at rest. After time $t \geq 0$ the plate starts its oscillations in its own plane and induces the motion in the fluid. The governing equations are [10]

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial u(y,t)}{\partial t} = \nu\frac{\partial^2 u(y,t)}{\partial y^2} - \frac{\sigma B_0^2}{\rho}\left(1+\lambda\frac{\partial}{\partial t}\right) \cdot u(y,t) - \frac{\nu\varphi}{k}u(y,t);\; y,t>0, \tag{1}$$

$$\left(1+\lambda\frac{\partial}{\partial t}\right)T(y,t) = \mu\frac{\partial u(y,t)}{\partial y};\; y,t>0, \tag{2}$$

where ρ is the fluid density, μ the dynamic viscosity, σ the electrical conductivity, λ the relaxation time, and T the tangential stress; k (>0) and φ $(0<\varphi<1)$ are respectively the permeability and porosity of the porous space. The subjected initial and boundary conditions are written in the following form:

$$\frac{\partial u(y,0)}{\partial t} = u(y,0) = 0;\; y\geq 0, \tag{3}$$

$$u(0,t) = U_0\cos(\omega_0 t) \text{ or } u(0,t) = U_0\sin(\omega_0 t);\; t>0, \tag{4}$$

$$u(y,t)\to 0,\; T(y,t)\to 0;\; y\to\infty;\; t>0, \tag{5}$$

where U_0 signifies the amplitude and ω_0 the frequency of oscillation of the plate.

3. Solution of the Problem

Inserting the following dimensionless variables

$$\tau = \frac{U_0^2 t}{\nu},\; \xi = \frac{U_0 y}{\nu},\; U = \frac{u}{U_0},\; S = \frac{T}{\rho U_0^2},\; \lambda_1 = \frac{\lambda U_0^2}{\nu},\; \omega = \frac{\omega_0 \nu}{U_0^2}, \tag{6}$$

into (1) and (2), we get

$$\left(1+\lambda_1\frac{\partial}{\partial \tau}\right)\frac{\partial U(\xi,\tau)}{\partial \tau} = \frac{\partial^2 U(\xi,\tau)}{\partial \xi^2} - M\left(1+\lambda_1\frac{\partial}{\partial \tau}\right)U(\xi,\tau) - \frac{1}{K}U(\xi,\tau);\; \xi,\tau>0, \tag{7}$$

$$\left(1+\lambda_1\frac{\partial}{\partial \tau}\right)S(\xi,\tau) = \frac{\partial U}{\partial \xi}(\xi,\tau);\; \xi,\tau>0, \tag{8}$$

where

$$M = \frac{\sigma B_0^2 \nu}{\rho U_0^2},\; \frac{1}{K} = \frac{\nu^2\varphi}{k U_0^2}.$$

The dimensionless initial and boundary conditions are

$$\frac{\partial U(\xi,0)}{\partial \tau} = U(\xi,0) = 0;\; \xi>0, \tag{9}$$

$$U(0,\tau) = \cos(\omega\tau) \text{ or } U(0,\tau) = \sin(\omega\tau);\; \tau>0, \quad U(\xi,\tau), S(\xi,\tau)\to 0;\; \xi\to\infty;\; \tau>0. \tag{10}$$

By taking the Laplace transform of (7), (8), and (10), using (9), one obtains

$$\frac{d^2\bar{U}}{d\xi^2} - \left[\lambda_1 q^2 + (M\lambda_1+1)q + M + \frac{1}{K}\right]\bar{U} = 0, \tag{11}$$

$$(1+\lambda_1 q)\bar{S}(\xi,q) = \frac{d\bar{U}}{d\xi}, \tag{12}$$

$$\bar{U}(0,q) = \frac{q}{q^2+\omega^2} \text{ or } \bar{U}(0,q) = \frac{\omega}{q^2+\omega^2}, \quad \bar{U}(\xi,q), \bar{S}(\xi,q)\to 0 \text{ as } \xi\to\infty, \tag{13}$$

where $\bar{U}(\xi,q)$ and $\bar{S}(\xi,q)$ denote the Laplace transforms of $U(\xi,\tau)$ and $S(\xi,\tau)$, respectively.

The solutions of (11) subjected to the boundary conditions (13) take the form

$$\bar{U}_c(\xi,q) = \frac{q}{q^2+\omega^2} \cdot \exp\left(-\xi\sqrt{\lambda_1 q^2 + (M\lambda_1+1)q + M + \frac{1}{K}}\right), \tag{14}$$

$$\bar{U}_s(\xi,q) = \frac{\omega}{q^2+\omega^2} \cdot \exp\left(-\xi\sqrt{\lambda_1 q^2 + (M\lambda_1+1)q + M + \frac{1}{K}}\right), \tag{15}$$

where the subscripts c and s refer to cosine and sine oscillations of the plate.

In order to find the dimensionless velocity, we write (14) and (15) in the following forms:

$$\bar{U}_c(\xi,q) = \bar{U}_1(q)\bar{U}_3(\xi,q), \tag{16}$$

$$\bar{U}_s(\xi,q) = \bar{U}_2(q)\bar{U}_3(\xi,q), \tag{17}$$

with

$$\bar{U}_1(q) = \frac{q}{q^2+\omega^2},\; \bar{U}_2(q) = \frac{\omega}{q^2+\omega^2}, \tag{18}$$

$$\bar{U}_3(\xi,q) = \exp\left(-\xi\sqrt{\lambda_1}\sqrt{(q+b_0)^2 - a^2}\right), \tag{19}$$

where

$$b_0=\frac{M\lambda_1+1}{2\lambda_1}\,,\ a^2=b_0^2-\frac{1}{\lambda_1}\left(M+\frac{1}{K}\right).$$

Writing $U_1(\tau)=\mathcal{L}^{-1}\{\bar{U}_1(q)\}$, $U_2(\tau)=\mathcal{L}^{-1}\{\bar{U}_2(q)\}$, and $U_3(\xi,\tau)=\mathcal{L}^{-1}\{\bar{U}_3(\xi,q)\}$, and using the convolution theorem [20], one obtains

$$\begin{aligned}U_{\mathrm{c}}(\xi,\tau)&=(U_1\cdot U_3)(\tau)\\&=\int_0^\tau U_1(\tau-s)U_3(\xi,s)\,\mathrm{d}s\,,\end{aligned}\tag{20}$$

$$\begin{aligned}U_{\mathrm{s}}(\xi,\tau)&=(U_2\cdot U_3)(\tau)\\&=\int_0^\tau U_2(\tau-s)U_3(\xi,s)\,\mathrm{d}s\,,\end{aligned}\tag{21}$$

where $\mathcal{L}^{-1}$ is denoting the inverse Laplace transform.

Laplace inversion of (18) leads to the following expressions:

$$U_1(\tau)=\cos(\omega\tau)\,,\ U_2(\tau)=\sin(\omega\tau)\,.\tag{22}$$

In order to find the Laplace inverse of $\bar{U}_3(\xi,q)$, we use a similar procedure as in [19] and write

$$\begin{aligned}U_3(\xi,\tau)&=\mathrm{e}^{-b_0\xi\sqrt{\lambda_1}}\delta\left(\tau-\xi\sqrt{\lambda_1}\right)\\&+\begin{cases}0\ \text{ for }\ 0<\tau<\xi\sqrt{\lambda_1}\,,\\ \frac{a\xi\sqrt{\lambda_1}\,\mathrm{e}^{-b_0\tau}}{\sqrt{\tau^2-\xi^2\lambda_1}}I_1\left(a\sqrt{\tau^2-\xi^2\lambda_1}\right)\\ \qquad\text{for }\ \tau>\xi\sqrt{\lambda_1}\,,\end{cases}\end{aligned}\tag{23}$$

where $\delta(\cdot)$ is the Dirac delta function, and $I_1(\cdot)$ is the modified Bessel function of the first kind of order one. Now using equations (22) and (23) into equations (20) and (21) and using the filtration property of the Dirac delta function [23], we get

$$U_{\mathrm{c}}(\xi,\tau)=\begin{cases}0\ \text{ for }\ 0<\tau<\xi\sqrt{\lambda_1}\,,\\ \mathrm{e}^{-b_0\xi\sqrt{\lambda_1}}\cos\left(\omega\left(\tau-\xi\sqrt{\lambda_1}\right)\right)\\ +a\xi\sqrt{\lambda_1}\int_{\xi\sqrt{\lambda_1}}^{\tau}\frac{\mathrm{e}^{-b_0s}}{\sqrt{s^2-\xi^2\lambda_1}}\\ \cdot\cos\left(\omega(\tau-s)\right)I_1\left(a\sqrt{s^2-\xi^2\lambda_1}\right)\mathrm{d}s\\ \qquad\text{for }\ \tau>\xi\sqrt{\lambda_1}\,,\end{cases}\tag{24}$$

$$U_{\mathrm{s}}(\xi,\tau)=\begin{cases}0\ \text{ for }\ 0<\tau<\xi\sqrt{\lambda_1}\,,\\ \mathrm{e}^{-b_0\xi\sqrt{\lambda_1}}\sin\left(\omega\left(\tau-\xi\sqrt{\lambda_1}\right)\right)\\ +a\xi\sqrt{\lambda_1}\int_{\xi\sqrt{\lambda_1}}^{\tau}\frac{\mathrm{e}^{-b_0s}}{\sqrt{s^2-\xi^2\lambda_1}}\\ \cdot\sin\left(\omega(\tau-s)\right)I_1\left(a\sqrt{s^2-\xi^2\lambda_1}\right)\mathrm{d}s\\ \qquad\text{for }\ \tau>\xi\sqrt{\lambda_1}\,.\end{cases}\tag{25}$$

The starting solutions $U_{\mathrm{c}}(\xi,\tau)$ and $U_{\mathrm{s}}(\xi,\tau)$ given by (24) and (25) are rather complicated. Hence, we derive approximate expressions for these velocities corresponding to small and large values of time. This time is important, especially for those who need to eliminate transients from their rheological measurements. In order to determine this time, we need first to write the starting solutions as the sum of the steady state and transient solutions. Therefore, we decompose the integrals from (24) and (25) under the form [19]

$$\begin{aligned}\int_{\xi\sqrt{\lambda_1}}^{\tau}f(\xi,\tau,s)\,\mathrm{d}s&=\int_{\xi\sqrt{\lambda_1}}^{\infty}f(\xi,\tau,s)\,\mathrm{d}s\\&-\int_{\tau}^{\infty}f(\xi,\tau,s)\,\mathrm{d}s\end{aligned}\tag{26}$$

and obtain

$$\begin{aligned}U_{\mathrm{c}}(\xi,\tau)&=U_{\mathrm{cs}}(\xi,\tau)+U_{\mathrm{c}\tau}(\xi,\tau)\,,\\U_{\mathrm{s}}(\xi,\tau)&=U_{\mathrm{ss}}(\xi,\tau)+U_{\mathrm{s}\tau}(\xi,\tau)\,,\end{aligned}\tag{27}$$

where the steady state solutions are written as

$$\begin{aligned}U_{\mathrm{cs}}(\xi,\tau)&=\mathrm{e}^{-m\xi}\cos(\omega\tau-n\xi)\,,\\U_{\mathrm{ss}}(\xi,\tau)&=\mathrm{e}^{-m\xi}\sin(\omega\tau-n\xi)\,,\end{aligned}\tag{28}$$

and the transient solutions are written as

$$U_{\mathrm{ct}}(\xi,\tau)=\begin{cases}0\ \text{ for }\ 0<\tau<\xi\sqrt{\lambda_1}\,,\\ -a\xi\sqrt{\lambda_1}\int_\tau^\infty\frac{\mathrm{e}^{-b_0s}\cos(\omega(\tau-s))}{\sqrt{s^2-\xi^2\lambda_1}}I_1\\ \cdot\left(a\sqrt{s^2-\xi^2\lambda_1}\right)\mathrm{d}s\ \text{ for }\ \tau>\xi\sqrt{\lambda_1}\,,\end{cases}\tag{29}$$

$$U_{\mathrm{st}}(\xi,\tau)=\begin{cases}0\ \text{ for }\ 0<\tau<\xi\sqrt{\lambda_1}\,,\\ -a\xi\sqrt{\lambda_1}\int_\tau^\infty\frac{\mathrm{e}^{-b_0s}\sin(\omega(\tau-s))}{\sqrt{s^2-\xi^2\lambda_1}}I_1\\ \cdot\left(a\sqrt{s^2-\xi^2}\right)\mathrm{d}s\ \text{ for }\ \tau>\xi\sqrt{\lambda_1}\,,\end{cases}\tag{30}$$

with

$$\begin{aligned}m^2=\frac{\lambda_1}{2}&\left(\sqrt{\left(b_0^2-\omega^2-a^2\right)^2+4b_0^2\omega^2}\right.\\&\left.+\left(b_0^2-\omega^2-a^2\right)\right),\end{aligned}\tag{31}$$

$$\begin{aligned}n^2=\frac{\lambda_1}{2}&\left(\sqrt{\left(b_0^2-\omega^2-a^2\right)^2+4b_0^2\omega^2}\right.\\&\left.-\left(b_0^2-\omega^2-a^2\right)\right).\end{aligned}\tag{32}$$

It should be pointed out through (27), (29), and (30) that for large time τ the starting solutions $U_c(\xi,\tau)$ and $U_s(\xi,\tau)$ tend to the steady-state solutions $U_{cs}(\xi,\tau)$ and $U_{ss}(\xi,\tau)$ which are periodic in time and independent of the initial conditions. However, these solutions satisfy the governing equations and boundary conditions.

Following a similar method for solutions as in the velocity case, the corresponding shear stresses $S_c(\xi,\tau)$ and $S_s(\xi,\tau)$ can also be presented as convolution products

$$S_c(\xi,\tau) = (S_1 \cdot S_3)(\tau) = \int_0^\tau S_1(\tau-s)S_3(\xi,s)\,\mathrm{d}s, \quad (33)$$

$$S_s(\xi,\tau) = (S_2 \cdot S_3)(\tau) = \int_0^\tau S_2(\tau-s)S_3(\xi,s)\,\mathrm{d}s, \quad (34)$$

where

$$S_1(\tau) = \mathcal{L}^{-1}\left\{\bar{S}_1(q)\right\}, \quad S_2(\tau) = \mathcal{L}^{-1}\left\{\bar{S}_2(q)\right\},$$
$$S_3(\xi,\tau) = \mathcal{L}^{-1}\left\{\bar{S}_3(\xi,q)\right\}.$$

Now using (14) and (15) into (12), we obtain

$$\bar{S}_c(\xi,q) = -\bar{S}_1(q)\bar{S}_3(\xi,q), \quad (35)$$
$$\bar{S}_s(\xi,q) = -\bar{S}_2(q)\bar{S}_3(\xi,q), \quad (36)$$

where

$$\bar{S}_1(q) = \frac{1}{\sqrt{\lambda_1}}\left[\frac{\left(\lambda_1 q^2+(M\lambda_1+1)q+M+\frac{1}{K}\right)q}{(1+\lambda_1 q)(q^2+\omega^2)}\right], \quad (37)$$

$$\bar{S}_2(q) = \frac{1}{\sqrt{\lambda_1}}\left[\frac{\left(\lambda_1 q^2+(M\lambda_1+1)q+M+\frac{1}{K}\right)\omega}{(1+\lambda_1 q)(q^2+\omega^2)}\right], \quad (38)$$

$$\bar{S}_3(\xi,q) = \frac{\exp\left(-\xi\sqrt{\lambda_1}\sqrt{(q+b_0)^2-a^2}\right)}{\sqrt{(q+b_0)^2-a^2}}. \quad (39)$$

The Laplace inverses of (37) and (38) yield to

$$S_1(\tau) = \frac{\delta(\tau)}{\sqrt{\lambda_1}} - a_1\,\mathrm{e}^{-\frac{\tau}{\lambda_1}} + a_2\cos(\omega\tau) - a_3\sin(\omega\tau), \quad (40)$$

$$S_2(\tau) = a_4\,\mathrm{e}^{-\frac{\tau}{\lambda_1}} + a_3\cos(\omega\tau) + a_2\sin(\omega\tau), \quad (41)$$

where

$$a_1 = \frac{1}{K\sqrt{\lambda_1}\left(1+\omega^2\lambda_1^2\right)}, \quad a_2 = \frac{1+KM\left(1+\omega^2\lambda_1^2\right)}{K\sqrt{\lambda_1}\left(1+\omega^2\lambda_1^2\right)},$$

$$a_3 = \frac{\omega\left(K\left(1+\omega^2\lambda_1^2\right)-\lambda_1\right)}{K\sqrt{\lambda_1}\left(1+\omega^2\lambda_1^2\right)}, \quad a_4 = a_1\omega\sqrt{\lambda_1}.$$

The Laplace inverse of (39) using formulae (A2) and (A7) from Appendix A in [19], is given as

$$S_3(\xi,\tau) = \begin{cases} 0 \ \text{ for } \ 0<\tau<\xi\sqrt{\lambda_1}, \\ \mathrm{e}^{-b_0\tau} I_0\left(a\sqrt{\tau^2-\xi^2\lambda_1}\right) \\ \qquad \text{for } \ \tau>\xi\sqrt{\lambda_1}, \end{cases} \quad (42)$$

where $I_0(\cdot)$ denotes the modified Bessel function of order zero and type one. Substituting (40) – (42) into (33) and (34), we obtain the following expressions for the shear stress:

$$S_c(\xi,\tau) = -\begin{cases} 0 \ \text{ for } \ 0<\tau<\xi\sqrt{\lambda_1}, \\ \frac{1}{\sqrt{\lambda_1}}\mathrm{e}^{-b_0\tau} I_0\left(a\sqrt{\tau^2-\xi^2\lambda_1}\right) - a_1\,\mathrm{e}^{-\frac{\tau}{\lambda_1}} \\ \cdot\int_{\xi\sqrt{\lambda_1}}^{\tau} I_0\left(a\sqrt{s^2-\xi^2\lambda_1}\right) \\ \cdot\exp\left(\left(\frac{1}{\lambda_1}-b_0\right)s\right)\mathrm{d}s \\ +a_2\int_{\xi\sqrt{\lambda_1}}^{\tau}\mathrm{e}^{-b_0 s} \\ \cdot\cos(\omega(\tau-s))I_0\left(a\sqrt{s^2-\xi^2\lambda_1}\right)\mathrm{d}s \\ -a_3\int_{\xi\sqrt{\lambda_1}}^{\tau}\mathrm{e}^{-b_0 s}\sin(\omega(\tau-s))I_0 \\ \left(a\sqrt{s^2-\xi^2\lambda_1}\right)\mathrm{d}s \\ \qquad \text{for } \ \tau>\xi\sqrt{\lambda_1}. \end{cases} \quad (43)$$

$$S_s(\xi,\tau) = -\begin{cases} 0 \ \text{ for } \ 0<\tau<\xi\sqrt{\lambda_1}, \\ a_4\,\mathrm{e}^{-\frac{\tau}{\lambda_1}}\int_{\xi\sqrt{\lambda_1}}^{\tau}\exp\left(\left(\frac{1}{\lambda_1}-b_0\right)s\right)I_0 \\ \cdot\left(a\sqrt{s^2-\xi^2\lambda_1}\right)\mathrm{d}s + a_3 \\ \cdot\int_{\xi\sqrt{\lambda_1}}^{\tau}\exp(-b_0 s)\cos(\omega(\tau-s))I_0 \\ \cdot\left(a\sqrt{s^2-\xi^2\lambda_1}\right)\mathrm{d}s + a_2 \\ \cdot\int_{\xi\sqrt{\lambda_1}}^{\tau}\exp\left(-\left(\frac{\tau-s}{\lambda_1}+b_0 s\right)\right)I_0 \\ \cdot\left(a\sqrt{s^2-\xi^2\lambda_1}\right)\mathrm{d}s \\ \qquad \text{for } \ \tau>\xi\sqrt{\lambda_1}. \end{cases} \quad (44)$$

The corresponding steady-state and transient solutions are (see Appendix for details)

$$S_c(\xi,\tau) = S_{cs}(\xi,\tau) + S_{ct}(\xi,\tau), \quad (45)$$
$$S_s(\xi,\tau) = S_{ss}(\xi,\tau) + S_{st}(\xi,\tau), \quad (46)$$

where

$$S_{cs}(\xi,\tau)=\frac{\sqrt{\lambda_1}\,\mathrm{e}^{-m\xi}}{m^2+n^2}\Big[-a_2\big(m\cos(\omega\tau-n\xi)+n\sin(\omega\tau-n\xi)\big)+a_3\big(m\sin(\omega\tau-n\xi)-n\cos(\omega\tau-n\xi)\big)\Big], \tag{47}$$

$$S_{ss}(\xi,\tau)=-\frac{\mathrm{e}^{-m\xi}}{m^2+n^2}\Big[a_2\big(m\sin(\omega\tau-n\xi)-n\cos(\omega\tau-n\xi)\big)+a_3\big(m\cos(\omega\tau-n\xi)+n\sin(\omega\tau-n\xi)\big)\Big] \tag{48}$$

are the shear stresses corresponding to the steady state, whereas

$$S_{ct}(\xi,\tau)=\begin{cases}0 \ \text{ for } \ 0<\tau<\xi\sqrt{\lambda_1}, \\ -\frac{1}{\sqrt{\lambda_1}}\mathrm{e}^{-b_0\tau}I_0\Big(a\sqrt{\tau^2-\xi^2\lambda_1}\Big) \\ +\frac{a_1\lambda_1\mathrm{e}^{-\frac{\tau}{\lambda_1}}\exp\Big(-\frac{\xi}{\sqrt{\lambda_1}}\sqrt{\big(\lambda_1 b_0-1\big)^2-a^2\lambda_1}\Big)}{\sqrt{\big(\lambda_1 b_0-1\big)^2-a^2\lambda_1}} \\ -a_1\mathrm{e}^{-\frac{\tau}{\lambda_1}}\int_\tau^\infty \exp\Big(\Big(\frac{1}{\lambda_1}-b_0\Big)s\Big)I_0 \\ \cdot\Big(a\sqrt{s^2-\xi^2\lambda_1}\Big)\,\mathrm{d}s+a_2\int_\tau^\infty \mathrm{e}^{-b_0 s} \\ \cdot\cos(\omega(\tau-s))I_0\Big(a\sqrt{s^2-\xi^2\lambda_1}\Big)\,\mathrm{d}s \\ -a_3\int_\tau^\infty \mathrm{e}^{-b_0 s}\sin(\omega(\tau-s))I_0 \\ \cdot\Big(a\sqrt{s^2-\xi^2\lambda_1}\Big)\,\mathrm{d}s \\ \qquad \text{for } \ \tau>\xi\sqrt{\lambda_1},\end{cases} \tag{49}$$

$$S_{st}(\xi,\tau)=\begin{cases}0 \ \text{ for } \ 0<\tau<\xi\sqrt{\lambda_1}, \\ -\frac{a_4\lambda_1\exp\Big(-\Big(\frac{\tau}{\lambda_1}+\frac{\xi}{\sqrt{\lambda_1}}\sqrt{(\lambda_1 b_0-1)^2-a^2\lambda_1}\Big)\Big)}{\sqrt{(\lambda_1 b_0-1)^2-a^2\lambda_1}} \\ +a_4\mathrm{e}^{-\frac{\tau}{\lambda_1}}\int_\tau^\infty \exp\Big(\Big(\frac{1}{\lambda_1}+b_0\Big)s\Big)I_0 \\ \cdot\Big(a\sqrt{s^2-\xi^2\lambda_1}\Big)\,\mathrm{d}s+a_3\int_\tau^\infty \mathrm{e}^{-b_0 s} \\ \cdot\cos(\omega(\tau-s))I_0\Big(a\sqrt{s^2-\xi^2\lambda_1}\Big)\,\mathrm{d}s \\ +a_2\int_\tau^\infty \mathrm{e}^{-b_0 s}\sin(\omega(\tau-s))I_0 \\ \cdot\Big(a\sqrt{s^2-\xi^2\lambda_1}\Big)\,\mathrm{d}s; \\ \qquad \text{for } \ \tau>\xi\sqrt{\lambda_1}\end{cases} \tag{50}$$

are the adequate transient parts. Furthermore, the present solutions are more general and all solutions in [19] appear as the limiting cases. Hence, this provides a useful mathematical check to our calculi.

4. Graphical Results and Discussion

This section is devoted to various results obtained from the flow analyzed in this paper. The graphical interpretations for different values of the involved parameters on the velocity profiles are given. Special attention has been focused on the variations of the magnetic parameter M, the permeability parameter K, the non-Newtonian fluid parameter λ_1, and the non-dimensional time τ on the profiles of steady-state and transient velocities for both the cosine and sine oscillations of the plate. Therefore, Figures 1 – 10 are displayed. In these, Figures 1 – 4 are plotted for the steady-state velocities U_{cs} for the cosine oscillations and U_{ss} for the sine oscillations of the plate. Similarly,

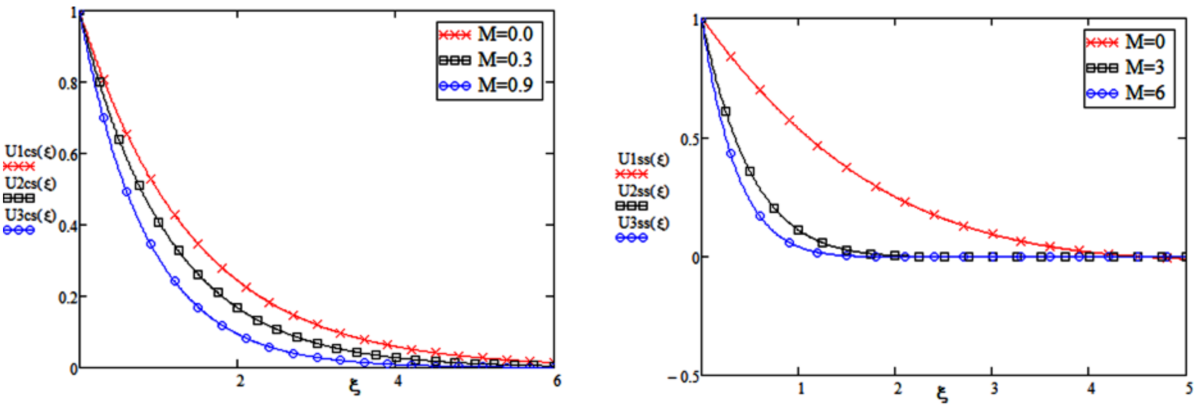

Fig. 1 (colour online). Steady-state velocity given by (28) for different values of M when $\lambda_1=0.2$ and $K=1$.

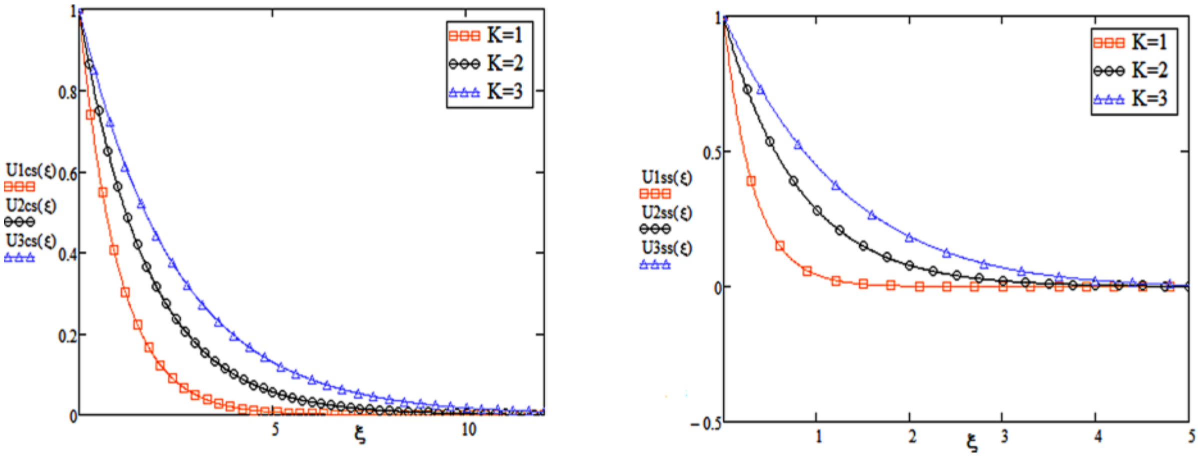

Fig. 2 (colour online). Steady-state velocity given by (28) for different values of K when $\lambda_1 = 0.2$ and $M = 1$.

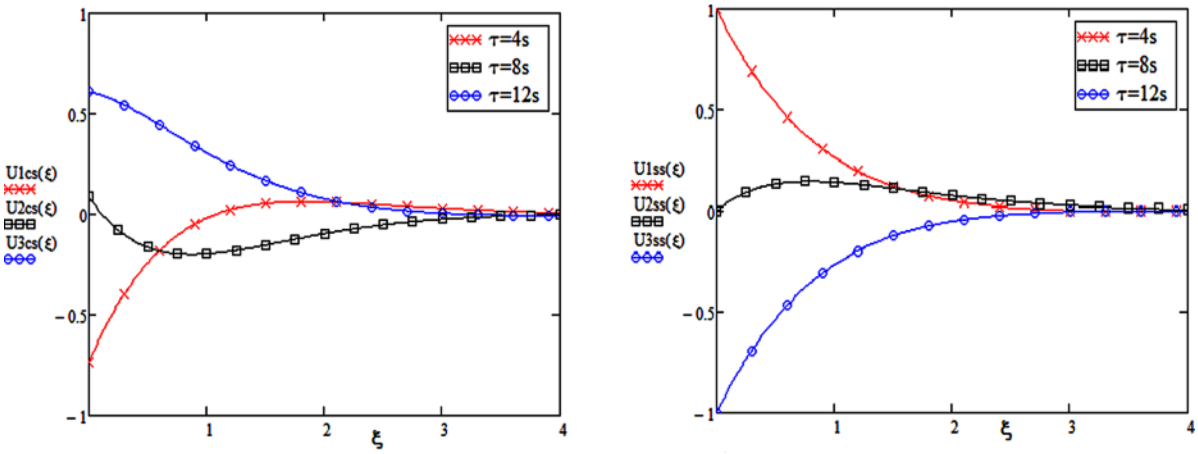

Fig. 3 (colour online). Steady-state velocity given by (28) for different values of τ when $\lambda_1 = 0.2$ and $K = 1$.

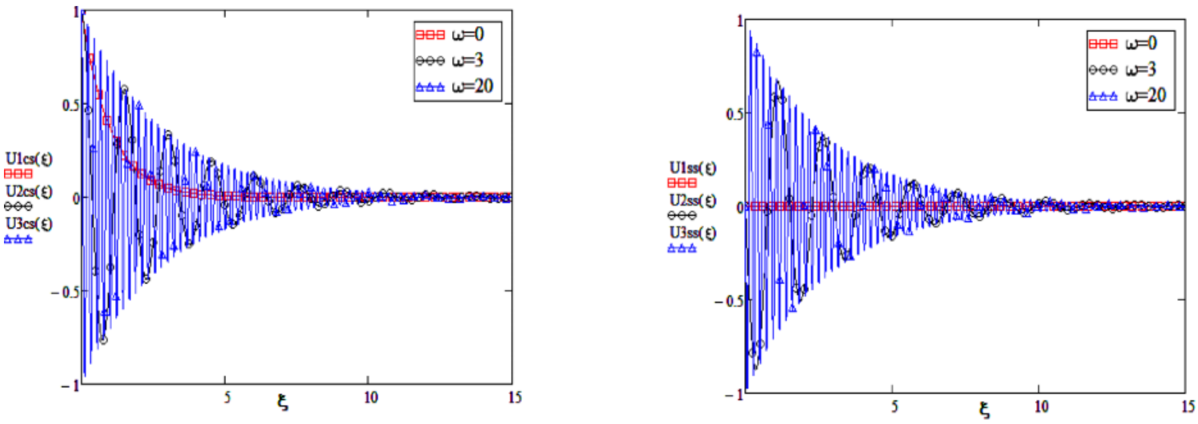

Fig. 4 (colour online). Steady-state velocity given by (28) for different values of ω when $\lambda_1 = 0.2$ and $K = 1$.

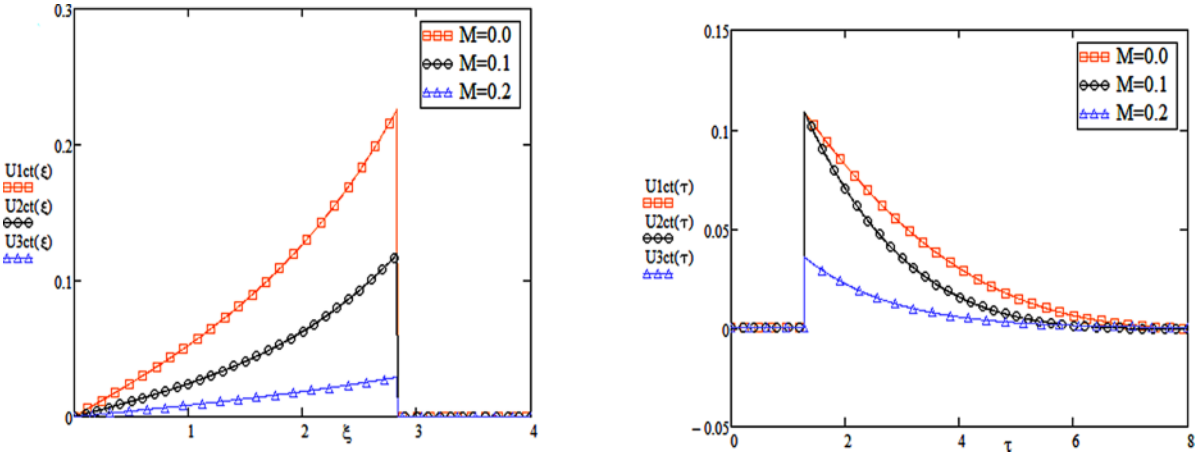

Fig. 5 (colour online). Transient velocity given by (29) for different values of M when $\lambda_1 = 0.2$ and $K = 0.7$.

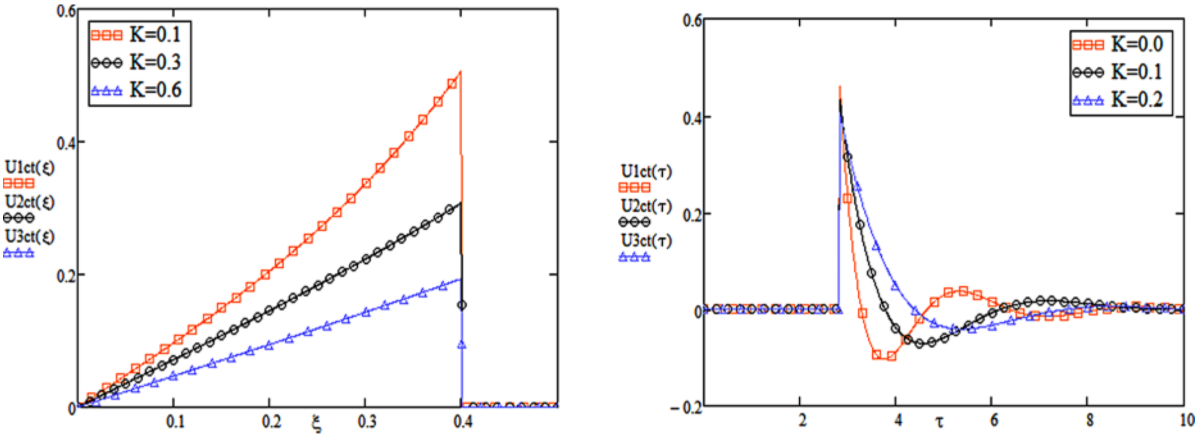

Fig. 6 (colour online). Transient velocity given by (29) for different values of K when $\lambda_1 = 0.2$ and $M = 0.7$.

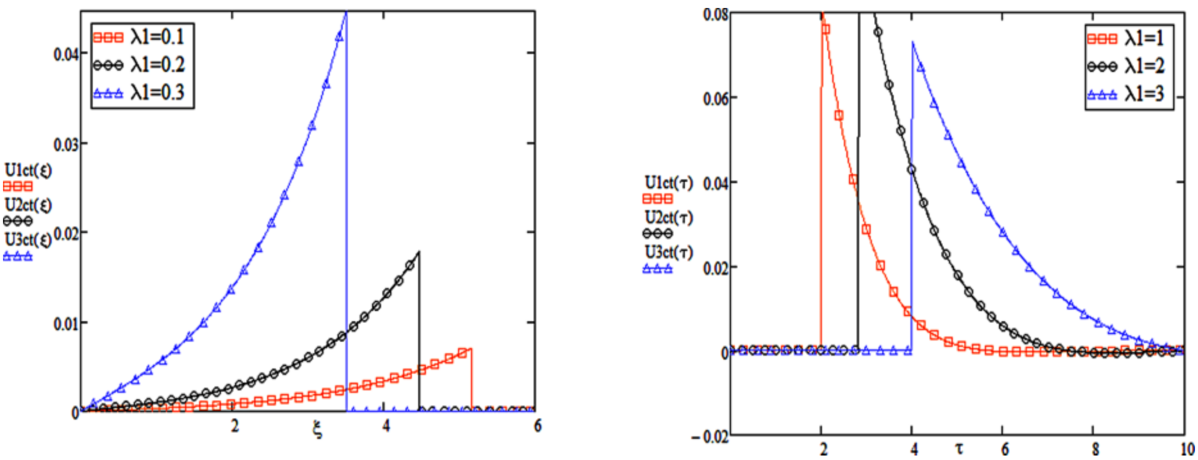

Fig. 7 (colour online). Transient velocity given by (29) for different values of λ_1 when $K = 0.2$ and $M = 0.7$.

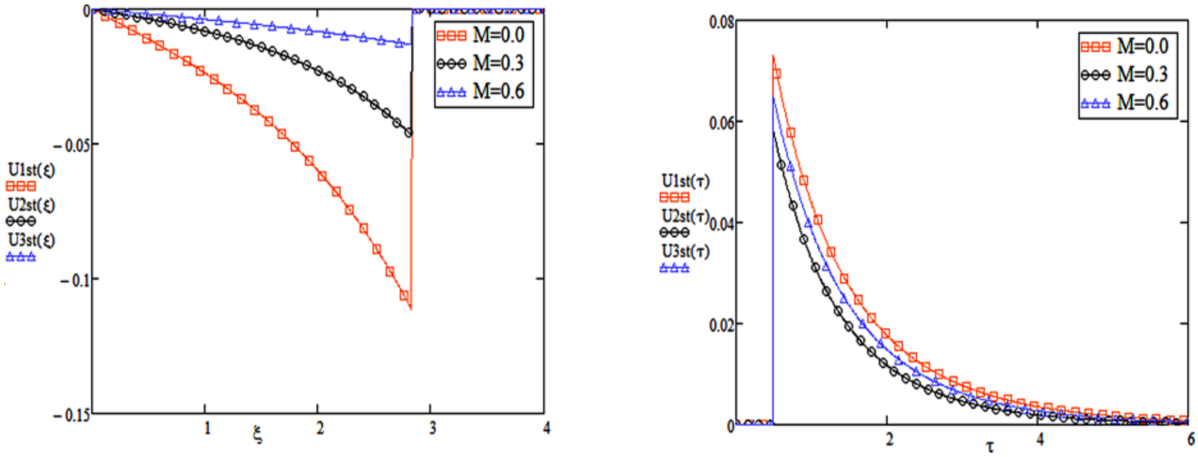

Fig. 8 (colour online). Transient velocity given by (30) for different values of M when $\lambda_1 = 0.2$ and $K = 0.7$.

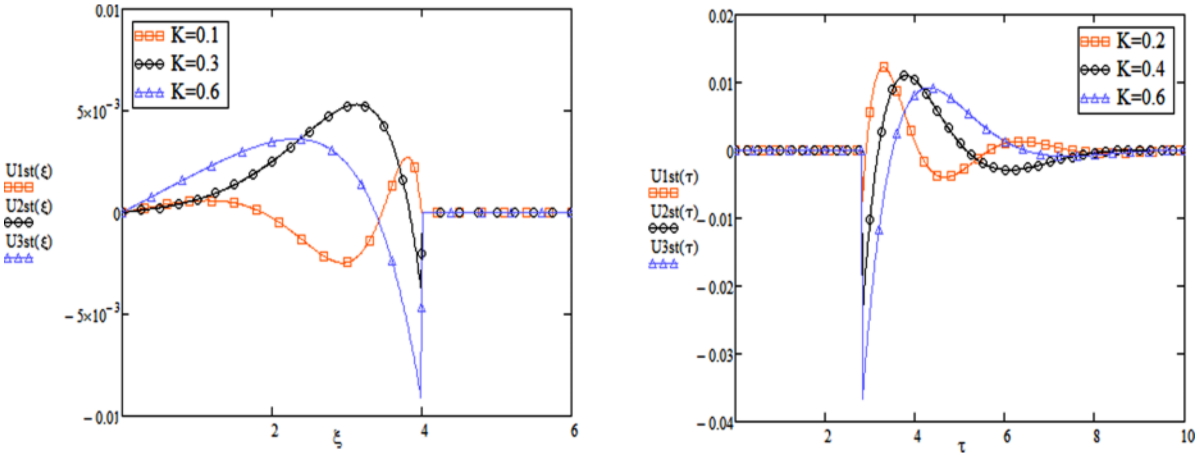

Fig. 9 (colour online). Transient velocity given by (30) for different values of K when $\lambda_1 = 0.2$ and $M = 0.7$.

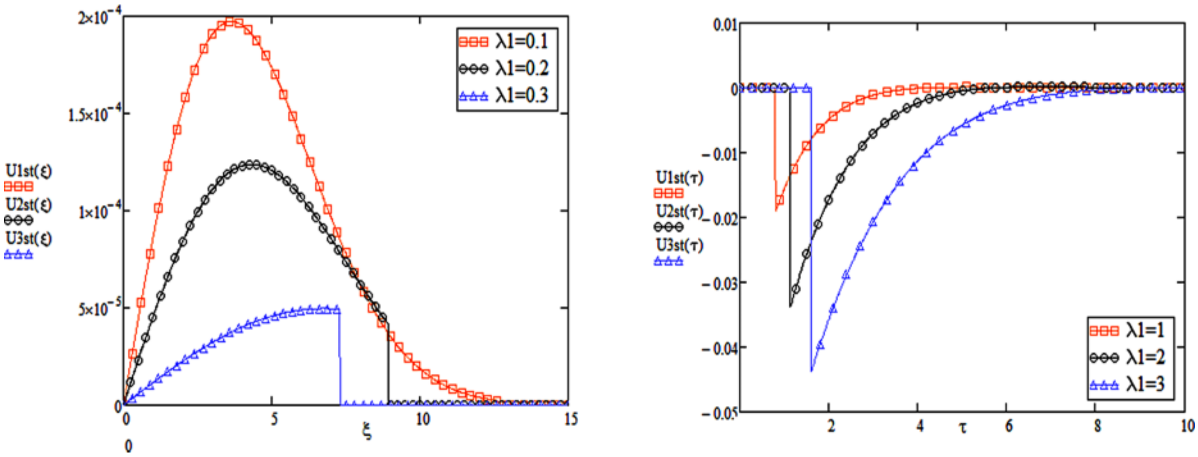

Fig. 10 (colour online). Transient velocity given by (30) for different values of λ_1 when $M = 0.2$ and $K = 0.7$.

in Figures 5 – 7 and 8 – 10 the transient velocities U_{ct} and U_{st} corresponding to cosine and sine oscillations of the plate are plotted.

The influence of the parameter M on the steady-state velocity is shown in Figure 1. It is observed that the amplitude of the velocity as well as the boundary layer thickness decreases when M is increased for both type of oscillations. Physically, it may also be expected due to the fact that the application of a transverse magnetic field results in a resistive type force (called Lorentz force) similar to the drag force, and upon increasing the values of M, the drag force increases which leads to the deceleration of the flow. In Figure 2, the profiles of steady-state velocity versus ξ have been plotted for various values of permeability parameter K by keeping other parameters fixed. It is observed that for large values of K, velocity and boundary layer thickness are increased for both cosine and sine oscillations. This explains the physical situation that as K increases, the resistance of the porous medium is lowered which increases the momentum development of the flow regime, ultimately enhances the velocity field. It is further observed that in case of sine oscillations, the velocity goes to zero before than that for cosine oscillations. Figure 3 shows the variations in steady-state velocities for different values of non-dimensional time τ. It is found that for both types of oscillations, velocities admit an oscillating nature. The steady-state velocities for different values of oscillating frequency ω in Figure 4 show that with increasing values of ω the oscillations in velocities increase.

Furthermore, the profiles of transient velocity for the cosine oscillations of the plate along ξ and τ for different values of magnetic parameter M are show in Figure 5. As it is clear from (29) that for positive values of τ such that $\tau < \xi\sqrt{\lambda_1}$, the fluid is static, whereas it is dynamic for $\tau > \xi\sqrt{\lambda_1}$. A similar behaviour of the transient velocity is observed in Figure 5. Moreover, with increasing M, we found that the velocity decreases due to an increasing resistive force. For large time, the transient velocity disappears and the motion is described by the corresponding steady-state solutions. This time is important for those who need to eliminate transient velocity from their rheological measurements. The transient velocity along ξ and τ corresponding to the cosine oscillations of the plate for different values of permeability parameter K are shown in Figure 6. It is observed that due to less resistance with increasing permeability, the fluid velocity increases.

The influence of the fluid parameter λ_1 on the transient velocity along ξ and τ is studied in Figure 7. It is found that the velocity decreases with increasing values of λ_1. However, after a certain value of $\xi \simeq 2.8$, the velocity becomes zero. Physically, it is true due to the fact that the non-Newtonian fluid parameter has reducing effects on the flow and hence the velocity of the fluid decreases with increasing values of λ_1. Finally, the graphs of transient velocity for the sine oscillations are displayed in Figures 8 – 10. It is investigated from these graphs that the overall behaviour of the velocity is identical to that studied for the cosine oscillations of the plate. However, it is interesting to note that the time required to reach the steady-state for sine oscillations of the plate is smaller than that required for the cosine oscillations of the plate. Of course, the required time to reach the steady-state also depends on the material constant λ_1 together with physical parameters M and K.

5. Conclusions

Exact solutions for the unsteady MHD flow of a Maxwell fluid saturating the porous space are successfully obtained. The motion in the fluid was induced due to the cosine and sine oscillations of the plate. The solutions for velocity distributions and shear stresses are established and then analyzed for small as well as large times. It is noted that for large times, when the transient solutions disappear, the starting solutions reduce to the steady-state solutions which are periodic in time and independent of initial conditions. The transient solutions of the velocity are reduced when time is increased. The graphical results are displayed to see the effects of various indispensable parameters on the velocity for cosine as well as for sine oscillation of the boundary. The solutions in [19] appeared as a special case when $M = \frac{1}{K} = 0$.

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Appendix

In order to derive (47) and (49), we use (26) into (43) to get

$$S_c(\xi,\tau) = -\begin{cases} 0 \quad \text{for } 0<\tau<\xi\sqrt{\lambda_1}\,, \\ \frac{1}{\sqrt{\lambda_1}}\,\mathrm{e}^{-b_0\tau} I_0\left(a\sqrt{\tau^2-\xi^2\lambda_1}\right) \\ -a_1\,\mathrm{e}^{-\frac{\tau}{\lambda_1}}\int_{\xi\sqrt{\lambda_1}}^{\tau}\exp\left(\left(\frac{1}{\lambda_1}-b_0\right)s\right) I_0 \\ \cdot\left(a\sqrt{s^2-\xi^2\lambda_1}\right)\mathrm{d}s + a_1\,\mathrm{e}^{-\frac{\tau}{\lambda_1}}\int_{\tau}^{\infty} \\ \exp\left(\left(\frac{1}{\lambda_1}-b_0\right)s\right) I_0\left(a\sqrt{s^2-\xi^2\lambda_1}\right)\mathrm{d}s \\ +a_2\int_{\xi\sqrt{\lambda_1}}^{\tau}\mathrm{e}^{-b_0 s}\cos(\omega(\tau-s)) I_0 \\ \cdot\left(a\sqrt{s^2-\xi^2\lambda_1}\right)\mathrm{d}s - a_2\int_{\tau}^{\infty}\mathrm{e}^{-b_0 s} \\ \cdot\cos(\omega(\tau-s)) I_0\left(a\sqrt{s^2-\xi^2\lambda_1}\right)\mathrm{d}s \\ -a_3\int_{\xi\sqrt{\lambda_1}}^{\tau}\mathrm{e}^{-b_0 s}\sin(\omega(\tau-s)) I_0 \\ \cdot\left(a\sqrt{s^2-\xi^2\lambda_1}\right)\mathrm{d}s + a_3\int_{\tau}^{\infty}\mathrm{e}^{-b_0 s} \\ \cdot\sin(\omega(\tau-s)) I_0\left(a\sqrt{s^2-\xi^2\lambda_1}\right)\mathrm{d}s \\ \qquad \text{for } \tau>\xi\sqrt{\lambda_1}\,. \end{cases} \tag{A1}$$

We suppose here

$$A = \int_{\xi\sqrt{\lambda_1}}^{\infty}\frac{\mathrm{e}^{-b_0 s}\cos(\omega(\tau-s))}{\sqrt{s^2-\xi^2\lambda_1}} \cdot I_1\left(a\sqrt{s^2-\xi^2\lambda_1}\right)\mathrm{d}s \tag{A2}$$

and

$$B = \int_{\xi\sqrt{\lambda_1}}^{\infty}\frac{\mathrm{e}^{-b_0 s}\sin(\omega(\tau-s))}{\sqrt{s^2-\xi^2\lambda_1}} \cdot I_1\left(a\sqrt{s^2-\xi^2\lambda_1}\right)\mathrm{d}s\,. \tag{A3}$$

Adding (A2) and (A3), we get

$$A+\mathrm{i}B = e^{\mathrm{i}\omega\tau}\int_{\xi\sqrt{\lambda_1}}^{\infty}\frac{\exp\left(-(b_0+\mathrm{i}\omega)s\right)}{\sqrt{s^2-\xi^2\lambda_1}} \cdot I_1\left(a\sqrt{s^2-\xi^2\lambda_1}\right)\mathrm{d}s\,. \tag{A4}$$

Now, by making the substitution

$$\sqrt{s^2-\xi^2\lambda_1} = z\,,\ \ s\in\left[\xi\sqrt{\lambda_1},\infty\right),\ \ z\in\left[0,\infty\right), \\ \sqrt{s^2-\xi^2\lambda_1} = z\,,\ \ \mathrm{d}s = \frac{z\,\mathrm{d}z}{\sqrt{z^2+\xi^2\lambda_1}} \tag{A5}$$

into (A4), we arrive at the following equation:

$$A+\mathrm{i}B = \mathrm{e}^{\mathrm{i}\omega\tau}\int_0^{\infty}\frac{\exp\left(-(b_0+\mathrm{i}\omega)\sqrt{z^2+\xi^2\lambda_1}\right)}{\sqrt{z^2+\xi^2\lambda_1}} \cdot I_1(az)\,\mathrm{d}z\,. \tag{A6}$$

Using the relation (see (A5), Appendix A in [19])

$$\int_0^{\infty}\frac{\exp\left(-a\sqrt{X^2+Y^2}\right)}{\sqrt{X^2+Y^2}}X I_0(bX)\,\mathrm{d}X = \frac{\exp\left(-Y\sqrt{a^2-b^2}\right)}{\sqrt{a^2-b^2}}\,,\ \ \mathrm{Re}(a^2-b^2)>0\,, \tag{A7}$$

in (A6), one obtains the form

$$A+\mathrm{i}B = \left(\sqrt{\lambda_1}\,\mathrm{e}^{\mathrm{i}\omega\tau}\exp\left(-\xi\sqrt{\lambda_1}\sqrt{(b_0+\mathrm{i}\omega)^2-a^2}\right)\right) \cdot\left(\sqrt{\lambda_1\left((b_0+\mathrm{i}\omega)^2-a^2\right)}\right)^{-1}. \tag{A8}$$

By making the substitution

$$\sqrt{\lambda_1\left((b_0+\mathrm{i}\omega)^2-a^2\right)} = m+\mathrm{i}n\,, \tag{A9}$$

(A8) reduces to the following form:

$$A+\mathrm{i}B = \frac{\sqrt{\lambda_1}\exp(-m\xi)}{m^2+n^2}\Big[m\cos(\omega\tau-\xi n) + n\sin(\omega\tau-\xi n) + \mathrm{i}\big(m\sin(\omega\tau-\xi n) - n\cos(\omega\tau-\xi n)\big)\Big]\,. \tag{A10}$$

Separating real and imaginary parts, we get

$$A = \frac{\sqrt{\lambda_1}\exp(-m\xi)}{m^2+n^2}\Big[m\cos(\omega\tau-\xi n) + n\sin(\omega\tau-\xi n)\Big]\,, \tag{A11}$$

$$B = \frac{\sqrt{\lambda_1}\exp(-m\xi)}{m^2+n^2}\Big[m\sin(\omega\tau-\xi n) - n\cos(\omega\tau-\xi n)\Big]\,. \tag{A12}$$

Now using (A5) in the first integral of (A1), we obtain

$$\int_{\xi\sqrt{\lambda_1}}^{\tau}\exp\left(\left(\frac{1}{\lambda_1}-b_0\right)s\right) I_0\left(a\sqrt{s^2-\xi^2\lambda_1}\right)\mathrm{d}s = \frac{\lambda_1\exp\left(-\frac{\xi}{\sqrt{\lambda_1}}\sqrt{(\lambda_1 b_0-1)^2-a^2\lambda_1}\right)}{\sqrt{\sqrt{(\lambda_1 b_0-1)^2-a^2\lambda_1}}}\,. \tag{A13}$$

Substituting (A11) – (A13) into (A1), and then separating steady state and transients parts, we get

$$S_{cs}(\xi,\tau) = \frac{\sqrt{\lambda_1}\exp(-m\xi)}{m^2+n^2}\Big[-a_2\big(m\cos(\omega\tau-\xi n) + n\sin(\omega\tau-\xi n)\big) + a_3\big(m\sin(\omega\tau-n\xi)-n\cos(\omega\tau-n\xi)\big)\Big], \tag{A14}$$

the shear stress corresponding to the steady state, whereas

$$S_{ct}(\xi,\tau) = \begin{cases} 0 \quad \text{for } 0<\tau<\xi\sqrt{\lambda_1}, \\ -\frac{1}{\sqrt{\lambda_1}}\,\mathrm{e}^{-b_0\tau} I_0\Big(a\sqrt{\tau^2-\xi^2\lambda_1}\Big) + a_1\lambda_1 \\ \cdot\,\mathrm{e}^{-\frac{\tau}{\lambda_1}}\frac{\exp\left(-\frac{\xi}{\sqrt{\lambda_1}}\sqrt{(\lambda_1 b_0-1)^2-a^2\lambda_1}\right)}{\sqrt{(\lambda_1 b_0-1)^2-a^2\lambda_1}} \\ -a_1\,\mathrm{e}^{-\frac{\tau}{\lambda_1}}\int_\tau^\infty \exp\left(\left(\frac{1}{\lambda_1}-b_0\right)s\right) I_0 \\ \cdot\Big(a\sqrt{s^2-\xi^2\lambda_1}\Big)\,\mathrm{d}s + a_2\int_\tau^\infty \mathrm{e}^{-b_0 s} \\ \cdot\cos\big(\omega(\tau-s)\big) I_0\Big(a\sqrt{s^2-\xi^2\lambda_1}\Big)\,\mathrm{d}s \\ -a_3\int_\tau^\infty \mathrm{e}^{-b_0 s}\sin\big(\omega(\tau-s)\big) I_0 \\ \cdot\Big(a\sqrt{s^2-\xi^2\lambda_1}\Big)\,\mathrm{d}s \\ \qquad \text{for } \tau>\xi\sqrt{\lambda_1} \end{cases} \tag{A15}$$

is the corresponding transient shear stress. Similarly for the sine case, we write the steady-state and transient parts as

$$S_{ss}(\xi,\tau) = -\frac{\exp(-m\xi)}{m^2+n^2}\Big[a_2\big(m\sin(\omega\tau-n\xi) - n\cos(\omega\tau-n\xi)\big)a_3 \cdot\big(m\cos(\omega\tau-n\xi)+n\sin(\omega\tau-n\xi)\big)\Big], \tag{A16}$$

$$S_{st}(\xi,\tau) = \begin{cases} 0 \quad \text{for } 0<\tau<\xi\sqrt{\lambda_1}, \\ -\frac{a_4\lambda_1\exp\left(-\left(\frac{\tau}{\lambda_1}+\frac{\xi}{\sqrt{\lambda_1}}\sqrt{(\lambda_1 b_0-1)^2-a^2\lambda_1}\right)\right)}{\sqrt{(\lambda_1 b_0-1)^2-a^2\lambda_1}} \\ +a_4\,\mathrm{e}^{-\frac{\tau}{\lambda_1}}\int_\tau^\infty \exp\left(\left(\frac{1}{\lambda_1}+b_0\right)s\right) I_0 \\ \cdot\Big(a\sqrt{s^2-\xi^2\lambda_1}\Big)\,\mathrm{d}s + a_3\int_\tau^\infty \mathrm{e}^{-b_0 s} \\ \cdot\cos\big(\omega(\tau-s)\big) I_0\Big(a\sqrt{s^2-\xi^2\lambda_1}\Big)\,\mathrm{d}s \\ +a_2\int_\tau^\infty \mathrm{e}^{-b_0 s}\sin\big(\omega(\tau-s)\big) I_0 \\ \cdot\Big(a\sqrt{s^2-\xi^2\lambda_1}\Big)\,\mathrm{d}s \\ \qquad \text{for } \tau>\xi\sqrt{\lambda_1}. \end{cases} \tag{A17}$$

[1] M. E. Erdogan and C. E. Imrak, Int. J. Nonlin. Mech. **44**, 27 (2009).
[2] Corina Fetecau, D. Vieru, and C. Fetecau, Int. J. Nonlin. Mech. **43**, 451 (2008).
[3] M. E. Erdogan and C. E. Imrak, Appl. Math. Model. **31**, 170 (2007).
[4] P. M. Jordan, A. Puri, and G. Boros, Int. J. Nonlin. Mech. **39**, 1371 (2004).
[5] W. C. Tan and T. Masuoka, Int. J. Nonlin. Mech. **40**, 515 (2005).
[6] T. Hayat, Y. Wang, and K. Hutter, Int. J. Nonlin. Mech. **39**, 1027 (2004).
[7] M. Khan, S. Hyder Ali, and C. Fetecau, Appl. Math. Comput. **203**, 881 (2008).
[8] T. Hayat, C. Fetecau, and S. Asghar, Int. J. Eng. Sci. **44**, 1423 (2006).
[9] C. Fetecau and C. Fetecau, Proc. Royal Acad. A **5**, 13 (2004).
[10] T. Hayat, C. Fetecau, and M. Sajid, Phys. Lett. A **372**, 1639 (2008).
[11] C. Fetecau, W. Akhtar, M. A, Imran, and D. Vieru, Comput. Math. Appl. **59**, 2836 (2010).
[12] D. Vieru, W. Akhtar, C. Fetecau, and C. Fetecau, Mecca. **42**, 573 (2007).
[13] M. A. Imran, A. Sohail, and N. Shahid, Chem. Eng. Commun. **199**, 1085 (2012).
[14] C. Fetecau, D. Vieru, T. Hayat, and C. Fetecau, Nonlin. Anal.: Real World Appl. **10**, 2183 (2009).
[15] T. Hayat, I. Khan, R. Ellahi, and C. Fetecau, J. Porous Media **11**, 389 (2008).
[16] W. C. Tan and T. Masuoka, Phys. Fluid **17**, 23101 (2005).
[17] M. Hussain, T. Hayat, S. Asghar, and C. Fetecau, Nonlin. Anal.: Real World Appl. **11**, 2403 (2010).
[18] M. Khan, M. Saleem, C. Fetecau, and T. Hayat, Int. J. Nonlin. Mech. **44**, 27 (2009).
[19] C. Fetecau, M. Jamil, C. Fetecau, and I. Siddique, Int. J. Nonlin. Mech. **44**, 1085 (2009).
[20] D. Vieru and A. Rauf, Can. J. Phys. **89**, 1 (2011).
[21] D. Vieru and A. A. Zafar, Appl. Math. Inf. Sci. **7**, 209 (2013).
[22] Samiulhaq, C. Fetecau, I. Khan, F. Ali, and S. Shafie, Z. Naturforsch. **67a**, 572 (2012).
[23] G. E. Roberts and H. Kaufman, Table of Laplace Transforms, W. B. Saunders Company, Philadelphia, London 1968.