A New Reliable Approach for Two-Dimensional and Axisymmetric Unsteady Flows Between Parallel Plates

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The main aim of this work is to present a new reliable approach to compute an approximate solution of the system of nonlinear differential equations governing the problem of two-dimensional and axisymmetric unsteady flows due to normally expanding or contracting parallel plates by the homotopy perturbation method, and the Sumudu transform is adopted in the solution procedure. The method finds the solution without any discretization or restrictive assumptions and avoids the round-off errors. The numerical solutions obtained by the proposed technique indicate that the approach is easy to implement and computationally very attractive.

Key words: Nonlinear Equation; Two-Dimensional Viscous Flow; Homotopy Perturbation Method; Sumudu Transform; Incompressible Fluid.

1. Introduction

The problem of unsteady squeezing of a viscous incompressible fluid between two parallel plates in motion normal to their own surfaces independent of each other and arbitrary with respect to time is a fundamental type of unsteady flow which is met frequently in many hydrodynamical machines and apparatuses. Some practical examples of squeezing flow include polymer processing, compression, and injection molding. In addition, the lubrication system can also be modelled by squeezing flows. The poineering work on squeezing flow by using lubrication approximation was conducted by Stefan [1]. Further, Reynolds [2] derived a solution for elliptic plates, and Archibald [3] investigated this problem for rectangular plates. The theoretical and experimental studies of squeezing flows have been conducted by many research workers [4 – 14]. Earlier studies of squeezing flow are based on Reynolds equation. The inadequacy of Reynolds equation in the analysis of porous thrust bearings and squeeze films involving high velocity has been demonstrated by Jackson [13], Ishizawa [14], and others. The general study of the problem with full Navier–Stokes equations involves extensive numerical study requiring more computer time and larger memory. However, many of the important features of this problem can be grasped by prescribing the relative velocity of the plates suitably. If the relative normal velocity is proportional to $(1-\alpha t)^{1/2}$, where t is the time and α a constant of dimension $[T^{-1}]$ which characterizes unsteadiness, then the unsteady Navier–Stokes equations admit a similarity solution.

The homotopy perturbation method (HPM) was first introduced by the Chinese researcher J. H. He in 1998, and was further developed by him [15 – 20]. The HPM is in fact a coupling of the traditional perturbation method and homotopy in topology [21]. This method was applied to axisymmetric flow over a stretching sheet [22], thermal boundary-layer problems in a semi-infinite plate [23], nonlinear Jeffery–Hamel flow [24], coupled nonlinear partial differential equations [25], Abel integral equation [26], peristaltic flow of a magnetohydrodynamic (MHD) Newtonian fluid in an asymmetric channel [27], MHD flow over a nonlinear stretching sheet [28], generalized Burger and Burger–Fisher equations [29], wave and nonlinear diffusion equations [30], time-fractional reaction-diffusion equation of Fisher type [31], motion of a spherical solid particle in plane coquette fluid flow [32], and heat transfer of copper–water nanofluid flow between parallel plates [33]. Natural convection heat transfer of a copper–water nanofluid in a cold outer circular enclosure containing a hot inner sinusoidal circular cylinder in the presence of a horizontal magnetic field was investigated numerically using the control volume based finite element method (CVFEM) [34]. Natural convection in a concentric annulus between a cold outer square and heated inner circular cylinders in presence of a static radial magnetic field was investigated numerically using the lattice Boltzmann method [35]. Mixed convection of a nanofluid consisting of water and SiO_2 in an inclined enclosure cavity was studied numerically [36]. The variational iteration method (VIM) was applied to solve the nonlinear settling particle equation of motion [37]. In recent years, many authors have paid attention to study the solutions of linear and nonlinear partial differential equations by using the HPM with the Laplace transform [38, 39] and the Sumudu transform [40].

The objective of this paper is to present a simple recursive algorithm based on the homotopy perturbation method, Sumudu transform method, and He's polynomials, and is mainly due to Ghorbani and Saberi-Nadjafi [41] and Ghorbani [42] which produce the series solution of the two-dimensional and axisymmetric unsteady flows due to normally expanding or contracting parallel plates. The advantage of this technique is its capability of combining two powerful methods for obtaining exact and approximate analytical solutions for nonlinear equations. It is worth mentioning that the proposed method is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the numerical result; the size reduction amounts to an improvement of the performance of the approach.

2. Sumudu Transform

The Sumudu transform [43] is defined over the set of functions

$$A = \Big\{ f(t) | \exists M, \tau_1, \tau_2 > 0, |f(t)| < M \mathrm{e}^{|t|/\tau_j}, \text{ if } t \in (-1)^j \times [0, \infty) \Big\}$$

by the following formula:

$$\bar{f}(u) = S[f(t)] = \int_0^\infty f(ut)\,\mathrm{e}^{-t}\,\mathrm{d}t\,, \quad u \in (-\tau_1, \tau_2)\,. \quad (1)$$

For further detail and properties of this transform, see [44–49].

3. Basic Idea of Homotopy Perturbation Method Using Sumudu Transform

To illustrate the basic idea of this method, we consider a general nonlinear non-homogenous partial differential equation of the form

$$LU + RU + NU = g(x)\,, \quad (2)$$

where L is the highest-order linear differential operator, R is the linear differential operator of less order than L, N represents the general nonlinear differential operator, and $g(x)$ is the source term. By applying the Sumudu transform on both sides of (2), we get

$$S[U] = u^n \sum_{k=0}^{n-1} \frac{U^{(k)}(0)}{u^{(n-k)}} + u^n S[g(x)] - u^n S[RU + NU] = 0\,. \quad (3)$$

Now applying the inverse Sumudu transform on both sides of (3), we get

$$U = G(x) - S^{-1}\left[u^n S[RU + NU]\right]\,, \quad (4)$$

where $G(x)$ represents the term arising from the source term and the prescribed initial conditions. Now, we apply the HPM

$$U = \sum_{m=0}^{\infty} p^m U_m \quad (5)$$

and the nonlinear term can be decomposed as

$$NU = \sum_{m=0}^{\infty} p^m H_m\,, \quad (6)$$

for some He's polynomials that are given by

$$H_m(U_0, U_1, \ldots, U_m) = \frac{1}{m!}\frac{\partial^m}{\partial p^m}\left[N\left(\sum_{i=0}^{\infty} p^i U_i\right)\right]_{p=0}\,, \quad m = 0, 1, 2, 3, \ldots\,. \quad (7)$$

Substituting (5) and (6) in (4), we get

$$\sum_{m=0}^{\infty} p^m U_m = G(x) - p\left(S^{-1}\left[u^n S\left[R\sum_{m=0}^{\infty} p^m U_m + \sum_{m=0}^{\infty} p^m H_m\right]\right]\right), \tag{8}$$

which is the coupling of the Sumudu transform and the HPM using He's polynomials. Comparing the coefficient of like powers of p, the following approximations are obtained:

$$p^0 : U_0(x) = G(x),$$
$$p^m : U_m(x) = -S^{-1}\left[u^n S\left[RU_{m-1}(x) + H_{m-1}(U)\right]\right],$$
$$m = 1,2,3,\ldots. \tag{9}$$

Finally, we approximate the analytical solution U by truncated series

$$U = \lim_{N\to\infty} \sum_{m=0}^{N} U_m. \tag{10}$$

The above series solutions generally converge very rapidly. A classical approach of convergence of this type of series is already presented by Abbaoui and Cherruault [50].

4. Mathematical Formulation

Let the position of the two plates be at $z = \pm\ell(1-\alpha t)^{1/2}$, where ℓ is the position at time $t = 0$ as depicted in Figure 1. We consider that the length 1 (in the two-dimensional case) or the diameter D (in the axisymmetric case) is much larger than the gap width $2z$ at any time such that the end effects can be neglected.

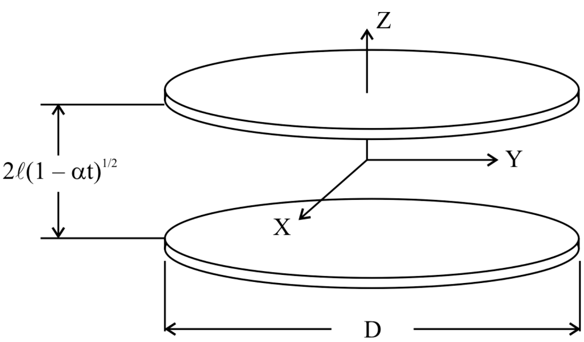

Fig. 1. Schematic diagram of the problem.

When α is positive, the two plates are squeezed until they touch at $t = 1/a$. When α is negative, the two plates are separated. Let U, V, and W be the velocity components along $x-$, $y-$, and z-axis, respectively. For a two-dimensional flow, Wang introduced the following transform [51]:

$$U = \frac{\alpha x}{[2(1-\alpha t)]} f'(\eta), \quad W = -\frac{\alpha\ell}{[2(1-\alpha t)^{1/2}]} f(\eta), \tag{11}$$

where

$$\eta = \frac{z}{[\ell(1-\alpha t)^{1/2}]}. \tag{12}$$

Substituting (11) into the unsteady two-dimensional Navier–Stokes equations transform nonlinear differential equation in the form

$$f'''' + s\{-\eta f''' - 3f'' - f'f'' + ff'''\} = 0, \tag{13}$$

where $s = \alpha\ell^2/2\nu$ (squeeze number) is the non-dimensional parameter. The flow is characterized by this parameter. The boundary conditions are such that on the plates, the lateral velocities are zero and the normal velocity is equal to the velocity of the plate, that is,

$$f(0) = 0, \;\; f''(0) = 0, \;\; f(1) = 1, \;\; f'(1) = 0. \tag{14}$$

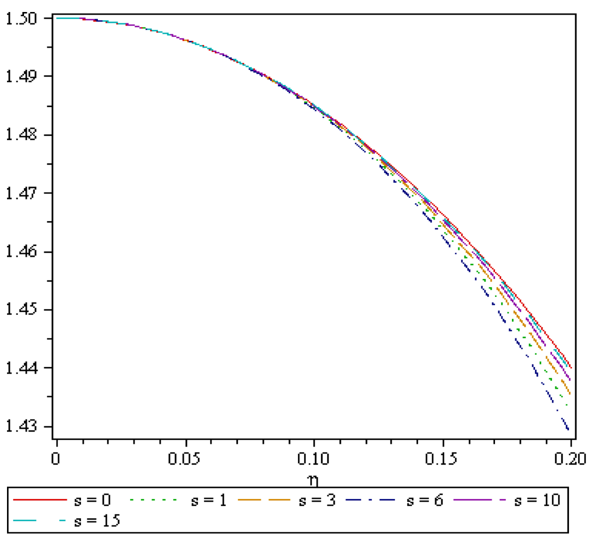

Fig. 2 (colour online). Influence of positive s on $f'(\eta)$ for the two-dimensional case for $\beta = 1$.

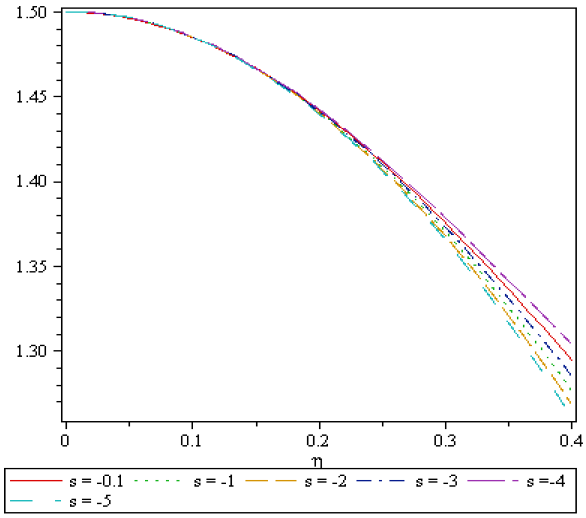

Fig. 3 (colour online). Influence of negative s on $f'(\eta)$ for the axisymmetric case for $\beta = 0$.

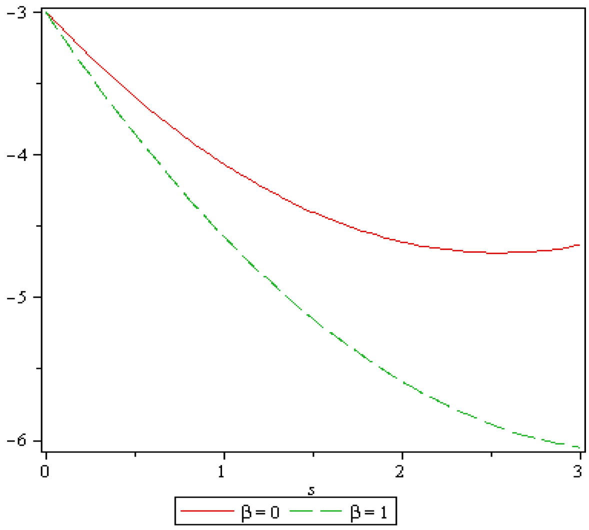

Fig. 4 (colour online). Skin fraction $f''(1)$ for the axisymmetric and two-dimensional cases.

Similarly, Wang's transforms [51] for axisymmetric flow are

$$U = \frac{\alpha x}{[4(1-\alpha t)]} f'(\eta), \quad V = \frac{\alpha y}{[4(1-\alpha t)]} f'(\eta), \quad W = -\frac{\alpha \ell}{[2(1-\alpha t)^{1/2}]} f(\eta). \tag{15}$$

Using transforms (15), the unsteady axisymmetric Navier–Stokes equations reduce to

$$f'''' + s\Big\{-\eta f''' - 3f'' + f f'''\Big\} = 0, \tag{16}$$

subjected to the boundary conditions (14).

Consequently, we should solve the nonlinear ordinary differential equation

$$f'''' + s\Big\{-\eta f''' - 3f'' - \beta f' f'' + f f'''\Big\} = 0, \tag{17}$$

where

$$\beta = \begin{cases} 0, & \text{Axisymmetric,} \\ 1, & \text{Two-dimensional,} \end{cases} \tag{18}$$

and subject to the boundary conditions (14). The two-dimensional and axisymmetric unsteady flows due to normally expanding or contracting parallel plates have also been studied by Dinarvand and Moradi [52].

5. Solution of the Problem

In this section, we apply the homotopy perturbation method using the Sumudu transform to obtain an approximate solution of (17). By applying the Sumudu transform on both sides of (17), we have

$$S[F(\eta)] = au + bu^3 - u^4 S\Big[s\Big\{-\eta f''' - 3f'' - \beta f' f'' + f f'''\Big\}\Big], \tag{19}$$

The inverse Sumudu transform gives

$$F(\eta) = a\eta + \frac{1}{6}b\eta^3 - S^{-1}\Big[u^4 S\Big[s\Big\{-\eta f''' - 3f'' - \beta f' f'' + f f'''\Big\}\Big]\Big]. \tag{20}$$

Now applying the HPM, we get

$$\sum_{m=0}^{\infty} p^m F_m(\eta) = a\eta + \frac{1}{6}b\eta^3 - S^{-1}\Bigg[u^4 S\Bigg[s\Bigg(-\eta \sum_{m=0}^{\infty} p^m f_m'''(\eta) - 3\sum_{m=0}^{\infty} p^m f_m''(\eta) - \beta \sum_{m=0}^{\infty} p^m H_m(\eta) + \sum_{m=0}^{\infty} p^m H_m'(\eta)\Bigg)\Bigg]\Bigg], \tag{21}$$

where $H_m(\eta)$ and $H_m'(\eta)$ are He's polynomials that represents the nonlinear terms. So, He's polynomials are given by

$$\sum_{m=0}^{\infty} p^m H_m(\eta) = f' f''. \tag{22}$$

The first few components of He's polynomials, are given by

$$H_0(\eta) = f_0'(\eta) f_0''(\eta), \quad H_1(\eta) = f_0'(\eta) f_1''(\eta) + f_1'(\eta) f_0''(\eta), \cdots, \tag{23}$$

and for $H_m'(\eta)$, we find that

$$\sum_{m=0}^{\infty} p^m H_m'(\eta) = f(\eta) f'''(\eta), \tag{24}$$

$$H_0'(\eta) = f_0(\eta) f_0'''(\eta), \quad H_1'(\eta) = f_0(\eta) f_1'''(\eta) + f_1(\eta) f_0'''(\eta), \cdots. \tag{25}$$

Comparing the coefficients of like powers of p, we have

$$p^0 : f_0 = a\eta + \frac{1}{6} b\eta^3, \tag{26}$$

$$p^1 : f_1 = -\frac{bs(a - a\beta - 4)}{120}\eta^5 + \frac{b^2 s(3\beta - 1)}{5040}\eta^7, \tag{27}$$

$$p^2 : f_2 = -\frac{s^2(11a - 3a\beta - 4 - 2a^2 + 2a^2\beta)b}{5040}\eta^7 - \frac{s^2(5a - 5a\beta - 20 - 9b\beta - 53b - 2ab\beta + 10ab)b}{362880}\eta^9 + \frac{s^2(28 - 42\beta + 18b\beta - 6b)b^2}{39916800}\eta^{11}, \tag{28}$$

$$\vdots$$

where $a = f''(0)$ and $b = f'''(0)$ are to be determined from the boundary conditions. The solutions of (17), when $p \to 1$, will be as follows:

$$f(\eta) = f_0(\eta) + f_1(\eta) + f_2(\eta) + \cdots. \tag{29}$$

6. Results and Discussion

In this paper, we have applied the homotopy perturbation method using the Sumudu transform for solving two-dimensional and axisymmetric unsteady flows due to normally expanding or contracting parallel plates. Our main purpose is to find the various values of $f(\eta)$ and $f'(\eta)$. These values describe the flow behaviour. Figure 2 depicts the variation of $f'(\eta)$ with the change in the positive values of s for the two-dimensional case. Figure 3 presents the effect of negative s on $f'(\eta)$ for the axisymmetric flow. It is important that for the large values of s, the results of the similarity analysis are not consistent. $f''(1)$ gives skin friction, $f'''(1)$ represents the pressure gradient and are shown as a function of s in Figures 4 and 5, respectively.

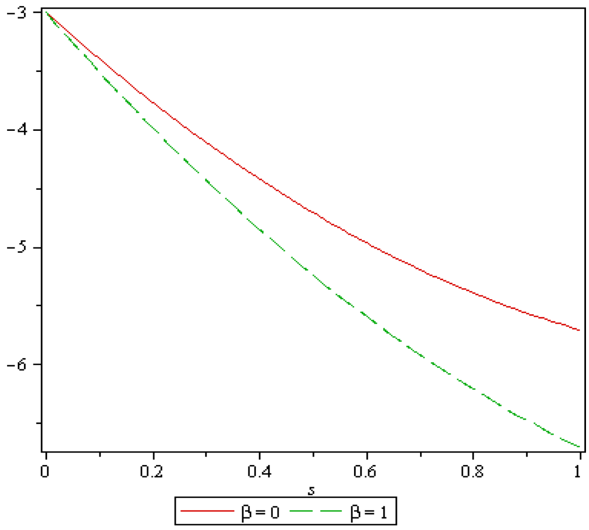

Fig. 5 (colour online). Pressure gradient $f'''(1)$ for the axisymmetric and two-dimensional cases.

7. Conclusions

In this paper, the homotopy perturbation method using the Sumudu transform has been successfully applied for solving two-dimensional and axisymmetric unsteady flows due to normally expanding or contracting parallel plates. Graphical results are presented to investigate the effect of squeeze number on the velocity, skin friction, and pressure gradient. The proposed method requires less computational work as compared to other analytical methods. In conclusion, the homotopy perturbation method using the Sumudu transform may be considered as a nice refinement in existing numerical techniques and might find wide applications.

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