Classification and Approximate Functional Separable Solutions to the Generalized Diffusion Equations with Perturbation

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In this paper, the generalized diffusion equation with perturbation $u_t = A(u,u_x)u_{xx} + \varepsilon B(u,u_x)$ is studied in terms of the approximate functional variable separation approach. A complete classification of these perturbed equations which admit approximate functional separable solutions is presented. Some approximate solutions to the resulting perturbed equations are obtained by examples.

Key words: Diffusion Equation; Approximate Functional Separable Solutions; Approximate Functional Variable Separation Approach.
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1. Introduction

The symmetry group method plays an important role in reducing and finding exact solutions of partial differential equations (PDEs). Some of these methods have been employed to seek exact solutions of PDEs for decades, for example, the Lie symmetry method [1 – 5], the generalized conditional symmetry (GCS) method [6, 7], and the potential symmetry method [8], etc. Moreover, some variable separation approaches were presented, and have been effectively used to construct exact solutions of PDEs, such as the classical method [9], the differential geometry method [10], the ansatz-based method [11 – 13], the formal variable separation approach [14], and the multi-linear variable separation approach [15, 16]. As far as the symmetry group and the ansatz of the solution form of PDEs are concerned, we point out two types of the approaches, namely, the functional variable separation approach (FVSA) [17] and the derivative-dependent functional variable separation approach (DDFVSA) [18]. Both methods are used to investigate variable separation of the generalized nonlinear evolution equations.

On the other hand, in recent years, more and more researchers have been engaging in the study of the nonlinear evolution equations with a small parameter that were arising from science, technology, and engineering. In order to solve such perturbed systems, there are some approximate methods that are commonly used, for instance, the numerical and the perturbation methods [19], the approximate conditional symmetry method [20], the approximate potential symmetry method [21], the approximate generalized conditional symmetry method (AGCS) [22], the approximate homotopy direct reduction method [23], and the approximate direct reduction method [24].

Recently, we introduced the concept of the approximate functional separable solutions (AFSSs), and proposed the approximate functional variable separation approach (AFVSA) that based on AGCS, and it was applied to discuss the perturbed evolution equations [25, 26]. In this paper, we consider the approximate functional variable separation of the following generalized diffusion equations with perturbation:

$$u_t = A(u,u_x)u_{xx} + \varepsilon B(u,u_x)\,, \tag{1}$$

where $A(u,u_x)$ and $B(u,u_x)$ are smooth functions of the indicated variables, ε is a small parameter.

The outline of the paper is as follows. In Section 2, a complete classification to the generalized diffusion equations with perturbation which admit AFSSs is obtained. In Section 3, some AFSSs to the resulting perturbed equations are constructed by way of examples. The last section is reserved for conclusion and discussion.

2. Classification of (1) Which Admits the AFSSs

Consider a kth-order differential system $[E]$, which is perturbed up to the first order in the small parameter ε, viz.

$$\begin{aligned} E^{\beta}\left(x,u,u_{(1)},\ldots,u_{(k)};\varepsilon\right) &\equiv E_0^{\beta}\left(x,u,u_{(1)},\ldots,u_{(k)}\right) \\ &+\varepsilon E_1^{\beta}\left(x,u,u_{(1)},\ldots,u_{(k)}\right)=0\,,\ \ \beta=1,\ldots,q\,, \end{aligned} \tag{2}$$

where $x=\left(x^1,x^2,\ldots,x^n\right)$, $u=\left(u^1,u^2,\ldots,u^m\right)$, E_i^{β} are smooth functions in their arguments, ε is a small parameter, $u_{(i)}$ $(i=1,\ldots,k)$ is the collection of ith-order partial derivatives, and

$$D_i=\frac{\partial}{\partial x^i}+u_i^{\alpha}\frac{\partial}{\partial u^{\alpha}}+u_{ij}^{\alpha}\frac{\partial}{\partial u_j^{\alpha}}+\cdots,\ \ i=1,\ldots,n,$$

denotes the operator of total derivative with respect to x^i.

Let

$$V=\eta\frac{\partial}{\partial u}\equiv\eta(x,t,u;\varepsilon)\frac{\partial}{\partial u} \tag{3}$$

be an evolutionary vector field and η its characteristic.

Definition 1. The approximate solution $u=u(x,t;\varepsilon)$ of (2) is said to be an approximate functional separable solution (AFSS) if it satisfies (2) and the ansatz

$$\begin{aligned} f(u)+\varepsilon g(u)=\psi(x)+\phi(t)& \\ +\varepsilon\big(\omega(x)+\theta(t)\big)+O\big(\varepsilon^2\big)\,,& \end{aligned} \tag{4}$$

where $f(u,u_x)$ and $g(u,u_x)$ are smooth functions of u and u_x, and $\psi(x)$, $\phi(t)$, $\omega(x)$, and $\theta(t)$ are some smooth functions of x and t.

In particular, with respect to any perturbed (1+1)-dimensional nonlinear evolution equations, we have definition as follows:

Definition 2. The evolutionary vector field (3) is said to be an AGCS of the perturbed nonlinear evolution equation

$$u_t=K(x,t,u;\varepsilon) \tag{5}$$

if and only if

$$V^{(k)}\big(u_t-K(x,t,u;\varepsilon)\big)\big|_{[W]\cap[E]}=O\big(\varepsilon^2\big)\,, \tag{6}$$

whenever $u_t=K(x,t,u;\varepsilon)$, where $V^{(k)}$ denotes the kth-order prolongation to (3), K and η are differentiable functions of t,x and $u,u_x,u_{xx},\ldots,[W]$ indicates the set of all differential consequences of $\eta=O\big(\varepsilon^2\big)$ with respect to x, that is, $D_x^j\eta=O\big(\varepsilon^2\big)$, $j=0,1,2,\ldots,$ and $[E]$ denotes the solution manifold of (5).

Considering to the results in [26], we have the following theorem.

Theorem 1. *Equation (5) possesses AFSS (4) if and only if it admits the AGCS*

$$V=\eta\frac{\partial}{\partial u}\equiv\left[u_{xt}+\Big(p(u)+\varepsilon q(u)\Big)u_xu_t\right]\frac{\partial}{\partial u}\,, \tag{7}$$

where

$$p(u)=\big(\ln(f')\big)'\,,\ \ q(u)=\left(\frac{g'}{f'}\right)'\,, \tag{8}$$

where $f\equiv f(u)$, $g\equiv g(u)$, and the prime denotes the corresponding order derivative with respect to u.

Next we classify to (1) that admits AGCS (3) by means of AFVSA. By Definition 2 and Theorem 1, we know that classification of (1) which possesses the AFSSs is equivalent to obtaining the AGCS of (3) satisfying

$$\begin{aligned} &V^{(2)}\Big(u_t-A(u,u_x)u_{xx}-\varepsilon B(u,u_x)\Big)\Big|_{[W]\cap[E]} \\ &=\Big[D_t\eta-(A_u\eta+A_{u_x}D_x\eta)u_{xx} \\ &\quad-\varepsilon(B_u\eta+B_{u_x}D_x\eta)-AD_x^2\eta\Big]\Big|_{[W]\cap[E]}\,. \end{aligned} \tag{9}$$

Using expressions $D_x^i\eta=0$, $i=0,1,2,\ldots,$ and (1), and excluding the higher-order derivative of u in (9), we obtain an expression of independent derivative of u, which is

$$\begin{aligned} D_t\eta\big|_{[W]\cap[E]}=\varepsilon&\Big(\Xi_1u_{xx}^3+\Xi_2u_{xx}^2+\Xi_3u_{xx}+\Xi_4\Big) \\ &+\Xi_5u_{xx}^3+\Xi_6u_{xx}^2+\Xi_7u_{xx}=O\big(\varepsilon^2\big)\,, \end{aligned} \tag{10}$$

where

$$\begin{aligned} &\Xi_i\equiv\Xi_i(u,u_x)=O(\varepsilon)\,,\ \ i=1,2,3,4\,, \\ &\Xi_j\equiv\Xi_j(u,u_x)=O\big(\varepsilon^2\big)\,,\ \ j=5,6,7\,. \end{aligned} \tag{11}$$

Since the expressions for Ξ_j $(j=1,2,\ldots,7)$ are lengthy, we omit them here.

By solving over-determined system of differential equations (11), we obtain the classification theorem as follows:

Theorem 2. *Suppose* $A(u,u_x) \neq 0$ *and* $B(u,u_x) \neq$ constant, *the perturbed equation*

$$u_t = A(u,u_x)u_{xx} + \varepsilon B(u,u_x)$$

admits AFSS of the (4) if and only if it is equivalent to one of the following equations, up to first order in ε*:*

(1)
$$u_t = c_1^2 u^\alpha u_x^\alpha u_{xx} + \varepsilon\left[\left(F_1\left(\frac{u_x}{u}\right)+c_3\right)u^{2\alpha+1}+c_2 u\right], \tag{12}$$
$$\eta = u_{xt} - \frac{1}{u}u_x u_t; \tag{13}$$

(2)
$$u_t = c_1\,\mathrm{e}^{\gamma u}u_x^\alpha u_{xx} + \varepsilon\left[\left(\frac{c_1c_2(2\alpha+1)}{\alpha+2} + c_1c_3\,\mathrm{e}^{\gamma(\alpha-1)u/\alpha}\right)u_x^{\alpha+2} + \frac{F_2(u_x)\,\mathrm{e}^{\gamma u}}{\gamma} + c_4\right],$$
$$\eta = u_{xt} + \varepsilon\left(c_2\,\mathrm{e}^{-\gamma u} + c_3\,\mathrm{e}^{-\gamma u/\alpha}\right)u_x u_t,$$
$$\alpha\big(|c_2|+|c_3|\big)\neq 0;$$

(3)
$$u_t = \frac{c_1\,\mathrm{e}^{\alpha u}}{u_x^2}u_{xx} + \varepsilon\left[\frac{F_2(u_x)\,\mathrm{e}^{\alpha u}}{\alpha} - 3c_1c_2\ln(u_x) + c_1c_3\,\mathrm{e}^{3\alpha u/2} + c_4\right],$$
$$\eta = u_{xt} + \varepsilon\left[c_2\,\mathrm{e}^{\alpha u/2} + c_3\,\mathrm{e}^{-\alpha u}\right]u_x u_t,$$
$$\big(|c_2|+|c_3|\big)\neq 0;$$

(4)
$$u_t = c_1 u_x^\alpha u_{xx} + \varepsilon\left[\left(\frac{2c_1c_2(\alpha+1)u_x^{\alpha+2}}{\alpha+2} + c_4\right)u + F_2(u_x)\right], \tag{14}$$
$$\eta = u_{xt} + \varepsilon(c_2 u + c_3)u_x u_t,\quad \alpha\big(|c_2|+|c_3|\big)\neq 0; \tag{15}$$

(5)
$$u_t = \frac{c_1}{u_x^2}u_{xx} + \varepsilon\Big[\big(c_4 - 2c_1c_2\ln(u_x)\big)u + F_2(u_x)\Big],$$
$$\eta = u_{xt} + \varepsilon(c_2u+c_3)uu_xu_t,\quad \big(|c_2|+|c_3|\big)\neq 0;$$

(6)
$$u_t = c_1\,\mathrm{e}^{\alpha u}u_{xx} + \varepsilon\left[\frac{1}{2}c_1c_2u_x^2 + F_2(u_x)\,\mathrm{e}^{\alpha u} + c_3\right],$$
$$\eta = u_{xt} + \varepsilon c_2\,\mathrm{e}^{-\alpha u}u_xu_t,\quad c_2\alpha\neq 0;$$

(7)
$$u_t = c_1u_{xx} + \varepsilon\left[\left(uF_3(u) - \int^u \xi F_3(\xi)\,\mathrm{d}\xi\right)u_x^2 + F_2(u_x) + c_2u\right],$$
$$\eta = u_{xt} + \varepsilon\left(\frac{uF_3(u) - \int^u \xi F_3(\xi)\,\mathrm{d}\xi}{c_1}\right)u_xu_t;$$

(8)
$$u_t = c_1u^\alpha u_{xx} + \varepsilon\left[\left(F_1\left(\frac{u_x}{u}\right)+c_4\right)u^{\alpha+1} + \left(\frac{c_1c_2(\alpha-6)u_x^2}{2(\alpha-2)}+c_3\right)u\right], \tag{16}$$
$$\eta = u_{xt} + \left(-\frac{1}{u} + \varepsilon c_2u^{1-\alpha}\right)u_xu_t; \tag{17}$$

(9)
$$u_t = c_1u^2u_{xx} + \varepsilon\left[\left(F_1\left(\frac{u_x}{u}\right)+c_3\right)u^3 + \left(2c_1c_2\ln(u)u_x^2 + c_4\right)u\right],$$
$$\eta = u_{xt} + \left(-\frac{1}{u} + \varepsilon\frac{c_2}{u}\right)u_xu_t;$$

(10)
$$u_t = \frac{u^\alpha}{u_x^2}u_{xx} + \varepsilon\left[\left(F_1\left(\frac{u_x}{u}\right)+c_5\right)u^{\alpha-1} - 3c_1c_3u\ln(u_x) + \frac{c_1c_2\alpha u^{\frac{3\alpha}{2}-2} - c_4u}{\alpha-2}\right],$$
$$\eta = u_{xt} + \left[-\frac{1}{u} + \varepsilon\left(c_2u^{\frac{1}{2}\alpha-2} + c_3u^{1-\alpha}\right)\right]u_xu_t,$$
$$\big(|c_2|+|c_3|\big)\neq 0;$$

(11)
$$u_t = c_1(u-c)^\alpha u_x^\beta u_{xx} + \varepsilon B(u,u_x),$$
$$c_1\beta(\beta+2)\neq 0,$$
$$\eta = u_{xt} + \left[-\frac{1}{u-c} + \varepsilon\left(c_2(u-c)^{1-\alpha} + c_3(u-c)^{-\frac{\alpha+2\beta}{\beta}}\right)\right]u_xu_t,$$

where $B = B(u,u_x)$ *satisfies*

$$B_u + c_1\left[\frac{c_3\alpha(u-c)^{\frac{\alpha\beta-\alpha-3\beta}{\beta}}}{\beta} + \frac{c_2\big[\alpha(2\beta+1)-6(\beta+1)\big]}{\beta+2}\right]u_x^{\beta+2} + \frac{B_{u_x}u_x - (1+\alpha+\beta)B - (\alpha+\beta)H + K}{u-c} + L = 0,$$

where $H = H(u)$, $K = K(u)$, *and* $L = L(u)$ *satisfy*

$$(u-c)K' - K - (\alpha+\beta)\big[(u-c)H' - H\big] - (u-c)^2L' = 0.$$

(12)
$$u_t = A(u,u_x)u_{xx} + \varepsilon B(u,u_x),$$
$$\eta = u_{xt} + \big(p(u) + \varepsilon q(u)\big)u_xu_t,$$

where $A = A(u,u_x)$, $B = B(u,u_x)$, $p = p(u)$, *and* $q = q(u)$ *satisfy*

$$A(p'-p^2)+c_1u_x^{c_2}=0\,,\;\; c_1\neq 0\,,$$

$$\Big[\Big(2Ap^2+pA_u-2Ap'\Big)A_u-ApA_{uu_x}\Big]u_x$$
$$-2A^2\big(p'-p^2\big)-A(pA_u-A_{uu})-A_u^2=0\,,$$

$$A\Big[\big(qA_u+4Apq-2Aq'\big)A_{u_x}-qAA_{uu_x}\Big]u_{u_x}^2$$
$$+\Big[(4pq-2q')-A^2(pB_{u_xu_x}+qA_u)+A\big(pA_{u_x}B_{u_x}$$
$$-pBAu_xu_x)+pBA_{u_x}^2\Big]u_x-BA_{u_x}(pA+A_uA_{u_x})$$
$$+A\big(AB_{uu_x}-A_uB_{u_x}+BA_{uu_x}\big)=0\,,$$

$$A^2\Big[(2pq-q')A_u-A\big(q''-2pq'-2p'q\big)\Big]u_x^3$$
$$+\Big[\Big(B\Big(pA_u-2p'A+3p^2A\Big)+pAB_u\Big)A_{u_x}$$
$$-A\Big(A(B_{u_x}p'+B_{uu_x}p)+pBA_{uu_x}\Big)\Big]u_x^2$$
$$+\Big[A^2B_{uu}+A(pA-A_u)B_u+\Big(A^2\big(2p^2-p'\big)$$
$$+A(A_{uu}-2pA_u)-A_u^2\Big)B\Big]u_x=0;$$

where α, β, γ, c, and c_i ($i\in\mathbb{Z}$) are arbitrary constants in their sets of definition when not specified.

3. Construction of AFSSs for the Resulting Equations

Using the AFVSA, we present some AFSSs of the resulting equations by the following some examples.

Example 1. Equation (12) enjoys the following AFSSs,

$$u=\left(1+\frac{\varepsilon\big(\omega(x)+\theta(t)\big)}{a_2+a_3}\right)\cdot\exp\left(\frac{\psi(x)+\phi(t)-a_1-a_4}{a_2+a_3}\right),\;\; a_2\neq 0\,,$$

where $\psi(x)$, $\phi(t)$, $\omega(x)$, and $\theta(t)$ satisfy

(1) $\alpha=0$.

$$\psi(x)=-\frac{\sqrt{\lambda}}{c_1}x+\frac{a_2}{2c_1}\cdot\ln\left[\frac{c_1^2}{4\lambda}\left(b_1\exp\left(\frac{2\sqrt{\lambda}x}{c_1a_2}\right)-b_2\right)^2\right],$$

$$\phi(t)=\frac{\lambda t}{a_2}+b_3\,,$$

$$a_2^2c_1^2\Big(a_2^2\omega''(x)+2a_2\psi'(x)\omega'(x)-a_3(\psi(x))^2\Big)$$
$$+a_2^5\left[F_1\left(\frac{\psi'(x)}{a_2}\right)+c_3\right]+\lambda_1=0\,,$$

$$\theta(t)=\frac{(c_2a_2^5-\lambda_1)t}{a_2^4}+b_4\,,\;\;\big(|b_1|+|b_2|\neq 0\big);$$

(2) $\alpha=0$, $\lambda<0$.

$$\psi(x)=\frac{a_2}{2}\ln\left[-\frac{c_1^2}{\lambda}\left(b_1\sin\left(\frac{\sqrt{-\lambda}x}{c_1a_2}\right)-b_2\cos\left(\frac{\sqrt{-\lambda}x}{c_1a_2}\right)\right)^2\right],$$

$$\phi(t)=\frac{\lambda t}{a_2}+b_3\,,$$

$$a_2^2c_1^2\Big(a_2^2\omega''(x)+2a_2\psi'(x)\omega'(x)-a_3(\psi(x))^2\Big)$$
$$+a_2^5\left[F_1\left(\frac{\psi'(x)}{a_2}\right)+c_3\right]+\lambda_1=0\,,$$

$$\theta(t)=\frac{(c_2a_2^5-\lambda_1)t}{a_2^4}+b_4\,,\;\;\big(|b_1|+|b_2|\neq 0\big);$$

(3) $\alpha=1$, $\lambda\neq 0$.

$$\int^{\psi(x)}\frac{c_1a_2\,\mathrm{e}^{\xi/a_2}}{\sqrt{a_2\Big[2\lambda\,\mathrm{e}^{2a_1/a_2}\ln(\xi)+h_1a_2c_1^2\Big]}}\,\mathrm{d}\xi=\pm x+h_2\,,$$

$$\phi(t)=\frac{1}{2}a_2\ln\left[-\frac{a_2^3}{2\lambda(t+h_3)}\right],$$

$$a_2^2\Big[c_1^2\big(\psi(x)\big)\Big(a_2^2\omega''(x)+2a_2\psi'(x)\omega'(x)$$
$$-a_3(\psi'(x))^2\Big)+a_2^4\left(F_1\left(\frac{\psi'(x)}{a_2}\right)+c_3\right)\Big]$$
$$\cdot\,\mathrm{e}^{2(\psi(x)-a_1)/a_2}+a_2^3\lambda\,(\psi'(x))^{-1}\omega'(x)$$
$$+a_2\lambda\Big[2a_2\big(\omega(x)-a_4\big)-2a_3\big(\psi(x)-a_1\big)$$
$$-a_2a_3\Big]+\lambda_1=0\,,$$

$$\theta(t)=\frac{1}{2}a_3\left[\ln\left(-\frac{a_2^3}{\lambda(t+h_3)}\right)+1\right]+\frac{a_2c_2t^2}{2(t+h_3)}+\frac{\Big(2a_2^2c_2h_3\lambda-a_2^2a_3\lambda\ln(2)+\lambda_1\Big)t+2h_4a_2^2\lambda}{2a_2^2\lambda(t+h_3)}\,.$$

(4) $\alpha\neq 0,1$, $\lambda\neq 0$.

$$\int^{\psi(x)}\left\{\left[c_1(\alpha-1)\,\mathrm{e}^{\xi/a_2}\,\mathrm{d}\xi\right]\left[a_5c_1^2(\alpha-1)^2-\frac{\lambda}{\xi^\alpha}\left(\frac{2(\alpha-1)\xi}{a_2}\right)^{\alpha/2}\exp\left(\frac{2a_1\alpha-(\alpha-1)\xi}{a_2}\right)\right.\right.$$

$$\cdot H(\xi)\Big]^{-\frac{1}{2}}\Big\} = \pm x + a_6,$$

$$\phi(t) = \frac{1}{2\alpha}\ln\left[-\frac{a_2^{\alpha+2}}{2\lambda\alpha(t+a_7)}\right],$$

$$a_2^2\Big[c_1^2\big(\psi(x)\big)^{\alpha}\Big(a_2^2\omega''(x) + 2a_2\psi'(x)\omega'(x) - a_3\big(\psi'(x)\big)^2\Big) + a_2^{\alpha+3}\Big(F_1\Big(\frac{\psi'(x)}{a_2}\Big) + c_3\Big)\Big] \cdot \mathrm{e}^{2\alpha(\psi(x)-a_1)/a_2} + a_2^3\lambda\alpha\big(\psi'(x)\big)^{-1}\omega'(x) + a_2\lambda\alpha\Big[2a_2(\omega(x)-a_4) - 2a_3(\psi(x)-a_1) - a_2a_3\Big] + \lambda_1 = 0,$$

$$a_2^{\alpha+4}\big(\theta'(t) - a_2c_2\big)\,\mathrm{e}^{2\alpha\phi(t)/a_2} - 2a_2\alpha\lambda\big(a_2\theta(t) - a_3\phi(t)\big) + \lambda_1 = 0,$$

where

$$H(\xi) = \mathrm{WhittakerM}\Big(-\frac{\alpha}{2}, \frac{1-\alpha}{2}, \frac{2(\alpha-1)\xi}{a_2}\Big),$$

where α, λ, λ_1, a_i, and b_j $(i = 1,\dots,7, j = 1,\dots,4)$ are arbitrary constants in their sets of definition when not specified. where and hereafter the prime denotes the corresponding order derivative with respect to x or t.

Example 2. Some AFSSs to (14) is given by (4), with

$$f(u) = d_2u + d_1, \quad d_2 \neq 0,$$
$$g(u) = \frac{1}{6}c_2d_2u^3 + \frac{1}{2}c_3d_2u^2 + d_3u + d_4,$$

where $\psi(x)$, $\phi(t)$, $\omega(x)$, and $\theta(t)$ are expressed by

(1) $\alpha \neq -1, -2$, $c_1c_2\alpha\rho \neq 0$.

$$\psi(x) = \Big(d_2^{\alpha/(\alpha+1)}\rho^{1/(\alpha+1)}\Big[(\alpha+1) \cdot (x-d_5)\Big]^{(\alpha+2)/(\alpha+1)}\Big)\Big(c_1^{1/(\alpha+1)}(\alpha+2)\Big)^{-\frac{1}{2}} + \frac{\rho_2}{2c_2d_2\alpha\rho(\alpha+2)} + \frac{c_4d_2^2}{c_2\alpha\rho} - \frac{d_2c_3}{c_2} + d_1,$$

$$\phi(t) = \rho t + d_6,$$

$$2c_1d_2^{1-\alpha}(\psi'(x))^{\alpha}\Big[d_2(\alpha+2)\Big(d_2\omega''(x) - c_3(\psi'(x))^2\Big) - c_2\alpha(d_1 - \psi(x))(\psi'(x))^2\Big] + 2d_2^3(\alpha+2)\Big[d_2F_2\Big(\frac{\psi'(x)}{d_2}\Big) + \rho\alpha(\psi'(x))^{-1}\omega''(x)\Big] - c_2d_2\alpha\rho(\alpha+2)\psi^2(x) + 2d_2(\alpha+2)\Big[\alpha\rho(c_2d_1 - c_3d_2) + c_4d_2^2\Big]\psi(x) - d_2(\alpha+2)\Big[d_1(c_2d_1\alpha\rho + 2c_4d_2^2) - 2d_2\rho\alpha(d_1c_3 - d_3)\Big] + \rho_1 = 0,$$

$$\theta(t) = -\frac{c_2\alpha\rho^3t^3}{6d_2^2} - \frac{\rho\,[2c_2d_2d_6\rho\alpha(\alpha+2) + \rho_2]\,t^2}{4d_2^3(\alpha+2)} - \frac{[c_2d_2d_6^2\rho\alpha(\alpha+2) + d_6\rho_2 + \rho_1]\,t}{2d_2^3(\alpha+2)} + d_7.$$

(2) $\alpha = -1$, $c_1c_2\rho \neq 0$.

$$\psi(x) = k_1\,\mathrm{e}^{\rho x/(c_1d_2)} - \frac{2d_2^2(d_2c_4 + c_3\rho) + \rho_2}{c_2d_2\rho} + d_1, \quad (k_1 \neq 0)$$

$$\phi(t) = \rho t + k_2,$$

$$2d_2^2\Big[\big(\psi'(x)\big)^{-1}\big(c_1d_2\omega''(x) - \rho\omega'(x)\big) + d_2F_2 \cdot \Big(\frac{\psi'(x)}{d_2}\Big)\Big] + c_2\rho\big(\psi(x)\big)^2 + 2c_1d_2\big(c_2d_1 - d_2c_3 - c_2\psi(x)\big)\psi'(x) + 2\Big(c_3d_2\rho - c_2d_1\rho + c_4d_2^2\Big)\psi(x) - 2d_2(c_4d_1d_2 - d_3\rho + c_3d_1\rho) + c_1d_1^2 + d_2^{-1}\rho_1 = 0,$$

$$\theta(t) = \frac{c_2\rho^3t^3}{6d_2^2} + \Big(\rho\big(2c_2d_2k_2\rho - \rho_2\big)t^2 - 2\Big[\rho_1 + k_2\big(\rho_2 - c_2d_2k_2\rho\big)\Big]t\Big)\Big(4d_2^3\Big)^{-1} + k_3.$$

(3) $\alpha \neq -1, -2$, $c_2 = 0$, $c_1\rho \neq 0$.

$$\psi(x) = \Big(d_2^{\frac{\alpha}{(\alpha+1)}}\rho^{\frac{1}{(\alpha+1)}}\Big[(\alpha+1)(x+r_1)\Big]^{\frac{(\alpha+2)}{(\alpha+1)}}\Big) \cdot \Big(c_1^{1/(\alpha+1)}(\alpha+2)\Big)^{-1} + r_2,$$

$$\phi(t) = \rho t + r_3,$$

$$2c_1d_2^{2-\alpha}(\alpha+2)(\psi'(x))^{\alpha}\big(d_2\omega''(x) - c_3(\psi'(x))^2\big) + 2d_2^3(\alpha+2)\Big[d_2F_2\Big(\frac{\psi'(x)}{d_2}\Big) + \rho\alpha(\psi'(x))^{-1} \cdot \omega''(x)\Big] + 2d_2^2(\alpha+2)(c_4d_2 - \alpha\rho c_3)\psi(x) - 2d_2^2(\alpha+2)\,[c_4d_1d_2 - \rho\alpha(d_1c_3 - d_3)] + \rho_1 = 0,$$

$$\theta(t) = \frac{\rho(d_2c_4 - c_3\alpha\rho)t^2}{2d_2}$$

$$-\frac{\left[2r_3d_2^2(\alpha+2)(c_3\rho\alpha-c_4d_2)+\rho_1\right]t}{2d_2^3(\alpha+2)}+r_4.$$

(4) $\alpha=-1,\ c_2=0,\ c_1\rho\neq 0.$

$$\psi(x)=h_2\,\mathrm{e}^{\rho x/(c_1d_2)}+h_1,\ (h_2\neq 0)$$
$$\phi(t)=\rho t+h_3,$$
$$d_2\left(\psi'(x)\right)^{-1}\left(c_1d_2\omega''(x)-\rho\omega'(x)\right)-d_2\left[c_1c_3\psi'(x)-d_2F_2\left(\frac{\psi'(x)}{d_2}\right)\right]+(c_4d_2+c_3\rho)(\psi(x)-d_1)+\rho d_3+2^{-1}d_2^{-2}\rho_1=0,$$
$$\theta(t)=\frac{\rho(c_3\rho+d_2c_4)t^2}{2d_2}+\frac{\left[2d_2^2h_3(c_4d_2+c_3\rho)-\rho_1\right]t}{2d_2^3}+h_4,$$

where α, ρ, ρ_1, d_i, k_j, r_n, and h_n $(i=1,\dots,7, j=1,\dots,3, n=1,\dots,4)$ are arbitrary constants in their sets of definition.

Example 3. Some AFSSs to (16) is determined by (4), with

$$f(u)=s_2\ln(u)+s_1,\ s_2\neq 0,$$
$$g(u)=\frac{c_2s_2u^{2-\alpha}}{(\alpha-2)^2}+s_3\ln(u)+s_4,$$

where $\psi(x)$, $\phi(t)$, $\omega(x)$, and $\theta(t)$ are determined by

(1) $\alpha\neq 0,1,2,\ c_1c_2\lambda\neq 0.$

$$\int^{\psi(x)}\left\{\left(c_1c_2s_2\alpha(\alpha-2)\,\mathrm{e}^{(\xi-s_1)/s_2}\right)\left(c_1c_2\alpha(2-\alpha)\left[2c_2s_2^2\alpha\lambda\exp\left(\frac{(s_1-\xi)(\alpha-2)}{s_2}\right)+\lambda_2\right]\right)^{-\frac{1}{2}}\right\}\mathrm{d}\xi=\pm x+s_5,$$
$$\phi(t)=\frac{s_2}{\alpha}\ln\left[-\frac{s_2^2}{\alpha\lambda(t+s_6)}\right],$$
$$2s_2\left[s_2^2\left(c_1\omega''(x)+s_2F_1\left(\frac{\psi'(x)}{s_2}\right)\right)+c_1\left(2s_2\omega'(x)-s_3\psi'(x)\right)\psi'(x)+c_4s_2^3\right]\mathrm{e}^{\alpha(\psi(x)-s_1)/s_2}+2\alpha\lambda\left[s_2\cdot(\omega(x)-s_4)-s_3(\psi(x)-s_1)\right]-\lambda_1(\alpha-2)^{-2}=0,$$
$$\theta(t)=\frac{s_3}{\alpha}\ln\left[-\frac{s_2^2}{\alpha\lambda(t+s_6)}\right]+\frac{\lambda_2s_2^{\frac{4-3\alpha}{\alpha}}\left[\alpha(t+s_6)\right]^{\frac{\alpha-2}{\alpha}}}{4(-\lambda)^{\frac{2}{\alpha}}(\alpha-1)(\alpha-2)^2}+\frac{c_3s_2t^2}{2(t+s_6)}+\frac{s_3}{\alpha}+\frac{\left[2c_3s_2^2\lambda\alpha(\alpha-2)^2s_6-\lambda_1\right]t+2s_2s_7\lambda\alpha(\alpha-2)^2}{2s_2\lambda\alpha(\alpha-2)^2(t+s_6)}.$$

(2) $\alpha=0,\ c_2\neq 0.$

(i) $c_1\lambda>0.$

$$\psi(x)=-\sqrt{\frac{\lambda}{c_1}}x+\frac{1}{2}s_2\cdot\ln\left[\frac{c_1}{4\lambda}\left(k_1\exp\left(\frac{2}{s_2}\sqrt{\frac{\lambda}{c_1}}x\right)+k_2\right)^2\right],$$
$$\phi(t)=\frac{\lambda t}{s_2}+k_3,$$
$$s_2^2\left[c_1\omega''(x)+s_2F_1\left(\frac{\psi'(x)}{s_2}\right)\right]+c_1\left(2s_2\omega'(x)-s_3\psi'(x)\right)\psi'(x)+c_4s_2^3-8^{-1}s_2^{-1}\lambda_1=0,$$
$$\theta(t)=\frac{(8c_3s_2^4+\lambda_1)t}{8s_2^3}+k_4;$$

(ii) $c_1\lambda<0.$

$$\psi(x)=\frac{1}{2}s_2\ln\left\{-\frac{c_1}{\lambda}\left[h_1\sin\left(\frac{\sqrt{-c_1^{-1}\lambda x}}{s_2}\right)-h_2\cos\left(\frac{\sqrt{-c_1^{-1}\lambda x}}{s_2}\right)\right]^2\right\},$$
$$\phi(t)=\frac{\lambda t}{s_2}+h_3,$$
$$s_2^2\left[c_1\omega''(x)+s_2F_1\left(\frac{\psi'(x)}{s_2}\right)\right]+c_1\left(2s_2\omega'(x)-s_3\psi'(x)\right)\psi'(x)+c_4s_2^3-8^{-1}s_2^{-1}\lambda_1=0,$$
$$\theta(t)=\frac{(8c_3s_2^4+\lambda_1)t}{8s_2^3}+h_4.$$

(3) $\alpha\neq 0,2,\ c_2=0,\ c_1\lambda\neq 0.$

$$\int^{\psi(x)}\left\{\left(c_1(\alpha-2)\right)\left(c_1(\alpha-2)\left[b_2^{1/s_2}\,\mathrm{e}^{-2\xi/s_2}-2\lambda\,\mathrm{e}^{\alpha(s_1-\xi)/s_2}\right]\right)^{-\frac{1}{2}}\right\}\mathrm{d}\xi=\pm x+b_3,$$
$$\phi(t)=\frac{s_2}{\alpha}\ln\left[-\frac{s_2^2}{\alpha\lambda(t+b_1)}\right],$$
$$2s_2\left[s_2^2\left(c_1\omega''(x)+s_2F_1\left(\frac{\psi'(x)}{s_2}\right)\right)+c_1\left(2s_2\omega'(x)\right.\right.$$

$$-s_3\psi'(x)\big)\psi'(x)+c_4s_2^3\Big]\mathrm{e}^{\alpha(\psi(x)-s_1)/s_2}+2\alpha\lambda\Big[s_2$$
$$\cdot(\omega(x)-s_4)-s_3(\psi(x)-s_1)\Big]-\lambda_1(\alpha-2)^{-2}=0,$$
$$\theta(t)=\frac{s_3}{\alpha}\ln\left[-\frac{s_2^2}{\alpha\lambda(t+b_1)}\right]+\frac{s_2c_3t^2}{2(t+b_1)}+\frac{s_3}{\alpha}$$
$$+\frac{\big[2b_1c_3s_2^2\alpha\lambda(\alpha-2)^2-\lambda_1\big]t+2b_4s_2\lambda\alpha(\alpha-2)^2}{2s_2\lambda\alpha(\alpha-2)^2(t+b_1)}.$$

(4) $\alpha=1,\ c_1c_2\lambda\neq0.$

$$\int^{\psi(x)}\frac{c_1c_2s_2\,\mathrm{e}^{(\xi-s_1)/s_2}}{\sqrt{c_1c_2\big[2c_2s_2^2\lambda\,\mathrm{e}^{(\xi-s_1)/s_2}+\lambda_2\big]}}\mathrm{d}\xi=\pm x+a_1,$$
$$\phi(t)=s_2\ln\left[-\frac{s_2^2}{\lambda(t+a_2)}\right],$$
$$2s_2\left[s_2^2\left(c_1\omega''(x)+s_2F_1\left(\frac{\psi'(x)}{s_2}\right)\right)+c_1(2s_2\omega'(x)\right.$$
$$\left.-s_3\psi'(x))\psi'(x)+c_4s_2^3\right]\mathrm{e}^{(\psi(x)-s_1)/s_2}$$
$$+2\lambda\Big[s_2(\omega(x)-s_4)-s_3(\psi(x)-s_1)\Big]-\lambda_1=0,$$
$$\theta(t)=\frac{1}{2}\left(2s_3-\frac{s_2\lambda_2}{\lambda^2(t+a_2)}\right)\ln\left[-\frac{s_2^2}{\lambda(t+a_2)}\right]$$
$$+\frac{a_3}{t+a_2}+\frac{1}{2}s_2c_3(t+a_2)-\frac{\lambda_1}{2s_2\lambda}+s_3.$$

(5) $\alpha=1,\ c_2=0,\ c_1\lambda\neq0.$

$$\psi(x)=s_2\ln\left(\frac{\lambda\,\mathrm{e}^{s_1/s_2}x^2-2c_1s_2v_2x+2c_1s_2v_3}{2c_1s_2^2}\right),$$
$$\phi(t)=s_2\ln\left[-\frac{s_2^2}{\lambda(t+v_1)}\right],$$
$$2s_2\left[s_2^2\left(c_1\omega''(x)+s_2F_1\left(\frac{\psi'(x)}{s_2}\right)\right)+c_1\Big(2s_2\omega'(x)\right.$$
$$\left.-s_3\psi'(x)\Big)\psi'(x)+c_4s_2^3\right]\mathrm{e}^{(\psi(x)-s_1)/s_2}$$
$$+2\lambda\Big[s_2(\omega(x)-s_4)-s_3(\psi(x)-s_1)\Big]-\lambda_1=0,$$
$$\theta(t)=s_3\ln\left[-\frac{s_2^2}{\lambda(t+v_1)}\right]+s_3$$
$$+\frac{s_2^2c_3\lambda t^2+\big(2s_2^2c_3v_1\lambda-\lambda_1\big)t+2s_2v_4\lambda}{2s_2\lambda(t+v_1)},$$

where α, λ, λ_1, λ_2, s_i, k_j, b_j, v_j, h_j, and a_n ($i=1,\ldots,7,j=1,\ldots,4,n=1,\ldots,3$) are arbitrary constants in their sets of definition.

4. Conclusion and Discussion

In summary, by utilizing the AFVSA, we have classified the generalized diffusion equations with perturbation which admit AFSSs. AFSSs of some resulting equations are constructed. In general, these AFSSs cannot be obtained by other approximate methods. Other types of perturbed nonlinear evolution equations may be studied by the AFVSA, and some interesting results will be presented sooner or later.

There are still two interesting topics to be investigated later: (i) How to apply the AFVSA to other types of nonlinear evolution equations, such as the higher dimensional equations, and the system of equations with perturbation? (ii) How to extend the AFVSA so as to obtain more accurate approximate solutions?

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