Computing the Edge-Neighbour-Scattering Number of Graphs

Zongtian Wei[a], Nannan Qi[b], and Xiaokui Yue[c]

[a] School of Science, Xi'an University of Architecture and Technology, Xi'an, Shaanxi 710055, P.R. China

[b] Science and Technology on EO-Control Laboratory, Luoyang Institute of Electro-Optic Equipment, Luoyang, Henan 471009, P.R. China

[c] School of Astronautics, Northwestern Polytechnical University, Xi'an, Shaanxi 710072, P.R. China

Reprint requests to Z. W.; E-mail: wzt6481@163.com

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A set of edges X is subverted from a graph G by removing the closed neighbourhood $N[X]$ from G. We denote the survival subgraph by G/X. An edge-subversion strategy X is called an edge-cut strategy of G if G/X is disconnected, a single vertex, or empty. The edge-neighbour-scattering number of a graph G is defined as $ENS(G) = \max\{\omega(G/X) - |X| : X$ is an edge-cut strategy of $G\}$, where $\omega(G/X)$ is the number of components of G/X. This parameter can be used to measure the vulnerability of networks when some edges are failed, especially spy networks and virus-infected networks. In this paper, we prove that the problem of computing the edge-neighbour-scattering number of a graph is **NP**-complete and give some upper and lower bounds for this parameter.

Key words: Vulnerability of Networks; Edge-Neighbour-Scattering Number; Computational Complexity; **NP**-Complete; Bounds.

1. Introduction

In this paper, we use [1] and [2] for terminology and notations not defined here and consider finite, simple, and undirected graphs only.

The concept of spy network was introduced by Gunther and Hartnell [3, 4]. They modelled a spy network by a graph whose vertices represent the stations and whose edges represent the lines of communication. The most important property of spy networks is that, if a station is destroyed, the adjacent stations will be betrayed and so the betrayed stations become useless to the network as a whole. Therefore, instead of considering the vulnerability or invulnerability of a network in the classic sense, a number of other related parameters were introduced to deal with this circumstance, including vertex-neighbour connectivity [4], edge-neighbour connectivity [5], vertex-neighbour integrity [6], edge-neighbour integrity [7], vertex-neighbour-scattering number [8], and edge-neighbour-scattering number [9]. The common ground of these parameters is that, when removing some vertices (or edges) from a graph, all of their adjacent vertices (or edges) are removed. It is shown that these parameters have theoretical as well as applied significance in the design and analysis of networks such as spy networks and virus-infected networks, see [8, 9].

Let $G = (V,E)$ be a graph and $e = uv$ be an edge of G. The edge e is said to be *subverted* if the edge e, all of its incident edges, and the two ends of e, u and v, are removed from G [10]. A set of edges $X \subseteq E$ is called an *edge-subversion strategy* of G if each of the edges in X has been subverted. The survival subgraph is denoted by G/X. An edge-subversion strategy X is called an *edge-cut strategy* of G if G/X is disconnected, a single vertex, or empty.

Let G be a graph. The *edge-neighbour connectivity* of G, denoted by $\Lambda(G)$, is the minimum size of all edge-cut strategies of G. An *edge-dominating set* D of G is a set of edges such that every edge not in D is adjacent to an edge in D. The *edge-domination number* of G is defined to be $\gamma'(G) = min\{|D| : D$ is an edge-dominating set of $G\}$.

The *edge-neighbour-scattering number* of a graph G is defined as

$$ENS(G) = \max\{\omega(G/X) - |X| : X \text{ is an edge-cut strategy of } G\},$$

where $\omega(G/X)$ stands for the number of components of G/X. We call $X^* \subseteq E(G)$ an *edge-neighbour-scattering set* (*ENS*-set) of G if $ENS(G) = \omega(G/X^*) - |X^*|$.

The concept of edge-neighbour-scattering number was introduced in [9]. Some properties of this parameter as well as some of its applications were discussed there when it is used to measure the vulnerability of networks. In this paper, we prove that the problem of computing the edge-neighbour-scattering number of a graph is **NP**-complete and give some upper and lower bounds of edge-neighbour-scattering number via some other well-known graphic parameters.

2. Computing Edge-Neighbour-Scattering Number is NP-Complete

It is of prime importance to determine the edge-neighbour-scattering number of a graph. In this section, we will investigate the complexity for computing the edge-neighbour-scattering number of a graph.

Problem 1. **EDGE-NEIGHBOUR-SCATTERING NUMBER**
Instance: A graph G; and an integer k.
Question: Does there exist an edge-cut strategy X of G such that $\omega(G/X) - |X| \geq k$?

We solve this complexity problem by considering the following

Problem 2. **EDGE-DOMINATION NUMBER**
Instance: A bipartite graph G; and a positive integer d.
Question: Does there exist an edge-dominating set D of G such that $|D| \leq d$?

It was proved by Yannakakis and Gavril [11] that the problem EDGE-DOMINATION NUMBER is **NP**-complete. Based on this conclusion, we prove that the problem EDGE-NEIGHBOUR-SCATTERING NUMBER is also **NP**-complete.

Theorem 1. *EDGE-NEIGHBOUR-SCATTERING NUMBER is **NP**-complete.*

Proof. Let $G = (V,E)$ be a bipartite graph with order n. Denote $V = \{v_1, v_2, \ldots, v_n\}$. Replace each vertex $v_i \in V$ by a copy of a complete graph K_n, and denote this copy by G_i. Select a vertex from G_i, and denote it by v_i^* $(i = 1, 2, \ldots, n)$. Add edges $v_i^* v_j^*$ if $v_i v_j \in E$. Denote the resulting graph by G^* (An example of G and G^* in case $n = 5$ is shown in Figure 1).

For convenience, denote the subgraph induced by $\{v_1^*, v_2^*, \ldots, v_n^*\}$ in G^* as G'. Obviously, $G' \cong G$. Assume that X^* is an *ENS*-set of G^*, i.e., $ENS(G^*) = \omega(G^*/X^*) - |X^*|$, and D is a smallest edge dominating set of G.

Clearly, EDGE-NEIGHBOUR-SCATTERING NUMBER is in the class **NP**. We now prove that $ENS(G^*) = n - |D|$. By the construction of G^*, and the **NP**-completeness of EDGE-DOMINATION NUMBER, this is sufficient for the conclusion.

Claim 1. If e is an edge in G_i which is not incident with v_i^*, then $e \notin X^*$, $i = 1, 2, \ldots, n$.

Proof. Otherwise, denote $X^{**} = X^* \setminus \{e\}$. Notice that $G_i/\{e\} = K_{n-2}$ and v_i^* is still in $G_i/\{e\}$. So we have $\omega(G^*/X^*) = \omega(G^*/X^{**})$. But $|X^{**}| = |X^*| - 1$, thus $\omega(G^*/X^{**}) - |X^{**}| > \omega(G^*/X^*) - |X^*|$. This is contradictory to that X^* is an *ENS*-set of G^*. □

Claim 2. Let $E_i^* = \{e : e \in E(G_i)$ and e is incident with $v_i^*\}$. Then $|E_i^* \cap X^*| \leq 1$ for $i = 1, 2, \ldots, n$.

Proof. Suppose that, for some i, $|E_i^* \cap X^*| \geq 2$. Without loss of generality, assume that $e, f \in E_i^* \cap X^*$ and $e \neq f$. Denote $X^{**} = X^* \setminus \{e\}$. Since $G_i/\{e, f\} = K_{n-3}$ and $G_i/\{f\} = K_{n-2}$, we have $\omega(G^*/X^*) \leq \omega(G^*/X^{**})$. But $|X^{**}| = |X^*| - 1$, thus we have $\omega(G^*/X^{**}) - |X^{**}| > \omega(G^*/X^*) - |X^*|$. This is contradictory to that X^* is an *ENS*-set of G^*. □

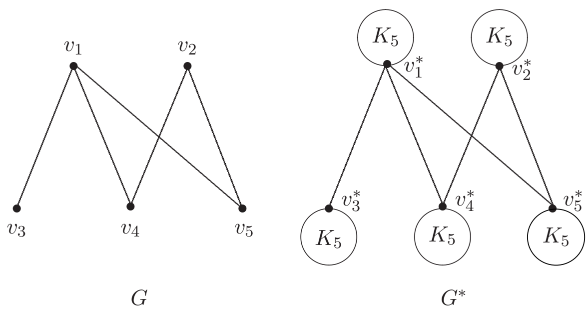

Fig. 1. Graphs G and G^*.

Claim 3. There exists an *ENS*-set X of G^* such that $E(G^*/X) \cap E(G') = \emptyset$ and $X \subseteq E(G')$.

Proof. Suppose that X^* is an *ENS*-set of G^* such that $E(G^*/X^*) \cap E(G') \neq \emptyset$. Without loss of generality, we assume $v_i^* v_j^* \in E(G^*/X^*) \cap E(G')$. Then any edge which is incident with v_i^* or v_j^* is not in X^*. By Claim 1, any edge of $E(G_i) \cup E(G_j)$ is not in X^*. Therefore, G_i, G_j and $v_i^* v_j^*$ belong to one component of G^*/X^*. Let $X^{**} = X^* \cup \{v_i^* v_j^*\}$. Then we have

$$\omega(G^*/X^{**}) \geq \omega(G^*/X^*) + 1$$

and

$$\omega(G^*/X^{**}) - |X^{**}| \geq \omega(G^*/X^*) + 1 - (|X^*| + 1) = \omega(G^*/X^*) - |X^*|\,.$$

On the other hand, since X^{**} is an edge-cut strategy of G^*, we have

$$\omega(G^*/X^{**}) - |X^{**}| \leq \omega(G^*/X^*) - |X^*|\,.$$

Thus

$$\omega(G^*/X^{**}) - |X^{**}| = \omega(G^*/X^*) - |X^*| = ENS(G^*)\,.$$

This implies that X^{**} is also an *ENS*-set of G^*. Therefore, if we add all the edges of $E(G^*/X) \cap E(G')$ to X, we then get an *ENS*-set X of G^* such that $E(G^*/X) \cap E(G') = \emptyset$. In other words, there always exists an *ENS*-set X of G^* such that all the edges of G' are in X or adjacent to some edges of X.

Let X^* be an *ENS*-set of G^*. By Claims 1 and 2, we then need only to prove that $E_i^* \cap X^* = \emptyset$ for $i = 1, 2, \ldots, n$. Suppose that $E_i^* \cap X^* \neq \emptyset$ for some i. Assume $e_i \in E_i^* \cap X^*$. Then any edge in $E(G')$ which is incident with v_i^* must be not in X^*. Otherwise, let $X^{**} = X^* \setminus \{e_i\}$. Then we have

$$\omega(G^*/X^*) = \omega(G^*/X^{**}), |X^{**}| = |X^*| - 1$$

and

$$\omega(G^*/X^{**}) - |X^{**}| > \omega(G^*/X^*) - |X^*|\,.$$

This is contradictory to that X^* is an *ENS*-set of G^*. □

Claim 3 implies that, there exists an *ENS*-set X^* of G^* such that X^* is also an edge dominating set of G'. Thus we have $\omega(G^*/X^*) = n$, i.e., $ENS(G^*) = \omega(G^*/X^*) - |X^*| = n - |X^*|$.

Note that D is a smallest edge dominating set of G and $G' \cong G$. So, the edge set corresponding to D in G' is also a smallest edge dominating set of G'. Therefore, $|X^*| \geq |D|$. We have

$$ENS(G^*) = n - |X^*| \leq n - |D|\,.$$

On the other hand, since D is a smallest edge dominating set of G, the edge set corresponding to D in G' is an edge-cut strategy of G^* and $\omega(G^*/X^*) = n$. Thus we have

$$ENS(G^*) \geq \omega(G^*/D) - |D| = n - |D|\,.$$

Therefore, we have $ENS(G^*) = n - |D|$. The proof is complete. □

3. Lower and Upper Bounds for Edge-Neighbour-Scattering Number

In this section, we give some lower and upper bounds for edge-neighbour-scattering number in terms of other well-known graphic parameters.

Theorem 2. *Let G be a connected graph with order $n > 5$, and M be a maximum but not perfect matching of G. Denote the set of the unsaturated vertices on M as V^*, and assume that $\delta^* = \min_{v \in V^*}\{d_G(v)\}$. Then $ENS(G) \geq 2 - \delta^*$.*

Proof. Let w be a vertex in V^* such that $d(w) = \delta^*$. Denote $N(w) = \{u_1, u_2, \ldots, u_{\delta^*}\}$ and $|M| = m$. It is obvious that $m \geq 1$. Let $M^* = \{e : e \in M$ and e is incident with at least one vertex in $N(w)\}$. We then have $|M^*| \leq \delta^*$.

It is easy to know that $|V^*| \geq 1$. For any edge $uv \in M$ and $x, y \in V^*$, it is impossible that both of $xu \in E$ and $vy \in E$ hold at the same time. Otherwise, there exists an M-augmenting path $xuvy$ in G, i.e., $M' = M \setminus \{uv\} \cup \{xu, yv\}$, which is a matching of G greater than M, a contradiction.

On the other hand, no two vertices in V^* are adjacent. If not, let x and y be two vertices in V^* such that $xy \in E$. Then $M \cup \{xy\}$ is a matching of G greater than M, contradicting to the choice of M. In other words, every vertex of $N(w)$ is incident with one of the edges in M and M^* is an edge-cut strategy of G.

We distinguish two cases for V^* as follows.

Case 1. $|V^*| \geq 2$.

Obviously, $\omega(G/M^*) \geq 2$. So we have

$$ENS(G) \geq \omega(G/M^*) - |M^*| \geq 2 - \delta^*.$$

Case 2. $|V^*| = 1$.

Case 2.1. $\delta^* > m$.

It is not difficult to know that M is an edge-cut strategy of G, so we have

$$ENS(G) \geq \omega(G/M) - |M| = 1 - m \geq 2 - \delta^*.$$

Case 2.2. $\delta^* \leq m$.

Case 2.2.1. $M^* \neq M$.

Since $n > 5$, we have $\omega(G/M^*) \geq 2$ and

$$ENS(G) \geq \omega(G/M^*) - |M^*| \geq 2 - \delta^*.$$

Case 2.2.2. $M^* = M$**.**

In this case, every vertex of $N(w)$ is incident with exactly one edge of M, and vise versa. Therefore, $d(w) = \delta^* = m$. It follows from $n > 5$ that $d(w) \geq 3$. Denote $V' = V(G) \setminus N[w] = \{v_1, v_2, \ldots, v_m\}$ and the subgraph induced by V' in G as $G[V']$.

Case 2.2.2.1. $G[V']$ is not a complete graph.

There exist two edges in M, say u_iv_i and u_jv_j such that $u_i \in N(w), u_j \in N(w)$ and $v_iv_j \notin E$. Denote $M^{**} = (M \setminus \{u_iv_i, u_jv_j\}) \cup \{wu_i, wu_j\}$. Then we have $|M^{**}| = m$ and $\omega(G/M^{**}) = 2$. Thus $ENS(G) \geq 2 - \delta^*$ holds.

Case 2.2.2.2. $G[V']$ is a complete graph.

Denote $M = \{u_1v_1, u_2v_2, \ldots, u_mv_m\}$. If there exist two vertices u_i and v_j such that $i \neq j$ and $u_iv_j \in E$. Let $M' = M \cup \{u_iv_j\} \setminus \{u_iv_i\}$. Then $|M'| = \delta^*$ and G/M' is a subgraph of G which consists of two isolated vertices v_i and w. So we have

$$ENS(G) \geq \omega(G/M') - |M'| = 2 - m = 2 - \delta^*.$$

If for any $i \neq j$, $u_iv_j \notin E$. Suppose that there exist two vertices in $N(w)$, say u_i and u_j, such that $u_iu_j \notin E$. Let $X' = M \setminus \{u_iv_i, u_jv_j\} \cup \{v_iv_j, wu_k\}$, where $k \neq i$ and $k \neq j$. Then $|X'| = m$ and G/X' is a subgraph of G which consists of two isolated vertices u_i and u_j. Therefore, we have

$$ENS(G) \geq \omega(G/X') - |X'| = 2 - m = 2 - \delta^*.$$

Suppose that any two vertices in $N(w)$ are adjacent. Denote $\lfloor \frac{m}{2} \rfloor = k$. When m is even, let $X'' = \{u_1u_2, u_3u_4, \ldots, u_{m-1}u_m\}$. We have $|X''| = k = \frac{m}{2}$ and $\omega(G/X'') = 2$. Therefore,

$$ENS(G) \geq \omega(G/X'') - |X''| = 2 - k > 2 - \delta^*.$$

When m is odd, let $X'' = \{u_1u_2, u_3u_4, \ldots, u_{2k-1}u_{2k}, u_mv_m\}$. We have $\omega(G/X'') = 2$ and $|X''| < m$. Therefore,

$$ENS(G) \geq \omega(G/X'') - |X''| > 2 - m > 2 - \delta^*.$$

The proof is complete. □

Remark 1. The lower bound in Theorem 2 is best possible. For example, when $n \geq 7$ and n is odd, we have $\delta^* = 2$ and $ENS(C_n) = 0$.

Theorem 3. *Let G be a graph with order $n \geq 3$ and $\gamma'(G)$ be the edge domination number of G. Then $ENS(G) \geq \max\{1 - \gamma'(G), n - 3\gamma'(G)\}$.*

Proof. The cases $n = 3$, 4 are trivial. Suppose that $n \geq 5$ and D is a smallest edge dominating set of G. Obviously, D is an edge-cut strategy of G. Let $G[D]$ be the subgraph induced by D in G. Then $|V(G[D])| \leq 2\gamma'(G)$. By the definition of edge-dominating set, we know that G/D is empty or consists of isolated vertices.

Case 1. $G/D \neq \emptyset$.

It is easy to see that there are at least $n - 2\gamma'(G)$ isolated vertices in G/D. So we have

$$ENS(G) \geq \omega(G/D) - |D| \geq n - 2\gamma'(G) - \gamma'(G) = n - 3\gamma'(G).$$

On the other hand, if $G/D \neq \emptyset$, then $\omega(G/D) \geq 1$. Thus

$$ENS(G) \geq \omega(G/D) - |D| \geq 1 - \gamma'(G).$$

So we have

$$ENS(G) \geq \max\{1 - \gamma'(G), n - 3\gamma'(G)\}.$$

Case 2. $G/D = \emptyset$.

Each vertex of $V(G)$ is incident with an edge of D. Assume that $uv \in D$. Then it is impossible that both u and v are incident with some edges of D except uv. Otherwise, $D \setminus \{uv\}$ is an edge-dominating set of G

smaller than D, a contradiction. Since $n \geq 3$, there must exist an edge $e \in D$ such that $N(e) \cap (E(G) \backslash D) \neq \emptyset$. Let $D^* = (D \setminus \{e\}) \cup \{f\}$, where f is an arbitrary edge of $N(e) \cap (E(G) \backslash D)$. Then $|D^*| = \gamma'(G)$ and G/D^* is an isolated vertex. So we have

$$ENS(G) \geq \omega(G/D^*) - |D^*| = 1 - \gamma'(G).$$

On the other hand, since $G[D]$ is a spanning subgraph of G, we have $|V(G[D])| = n \leq 2\gamma'(G)$. Therefore

$$1 - \gamma'(G) - (n - 3\gamma'(G)) = 2\gamma'(G) - n + 1 > 0.$$

Therefore,

$$ENS(G) \geq 1 - \gamma'(G) = max\{1 - \gamma'(G), n - 3\gamma'(G)\}.$$

The proof is complete. □

Remark 2. The lower bound $n - 3\gamma'(G)$ in Theorem 3 is best possible. Let C_n be the cycle with order n (≥ 6) and $n \equiv 0$ $(mod\ 3)$. Then we have $\gamma'(C_n) = \frac{n}{3}$ and $ENS(C_n) = 0 = n - 3\gamma'(C_n)$. On the other hand, although C_3 attains the bound $1 - \gamma'(G)$, we have not found general examples to illustrate that this bound is best possible.

Theorem 4. *Let G be a connected graph with order n and $\alpha'(G)$ be the matching number of G. Then $ENS(G) \geq n - 3\alpha'(G)$.*

Proof. Assume that M is a maximum matching of G. Then M is an edge-cut strategy of G. If G has a perfect matching, then we have

$$|M| = \alpha'(G) = \frac{n}{2}, \ \ G/M = \emptyset,$$

and

$$ENS(G) \geq \omega(G/M) - |M| = -\alpha'(G) = -\frac{n}{2}.$$

The conclusion holds.

If G has no perfect matchings, then G/M consists of only isolated vertices, and $\omega(G/M) = n - 2\alpha'(G)$. We have

$$\begin{aligned} ENS(G) \geq \omega(G/M) - |M| &= n - 2\alpha'(G) - \alpha'(G) \\ &= n - 3\alpha'(G). \end{aligned}$$

The proof is complete. □

Remark 3. The lower bound in Theorem 4 is best possible. For example, the complete graphs with odd order achieve this bound.

In the following, we give two upper bounds for the edge-neighbour-scattering number.

Theorem 5. *Let G be a connected graph with order n and $\Lambda(G)$ be the edge-neighbour-connectivity of G. Then $ENS(G) \leq n - 1 - 2\Lambda(G)$.*

Proof. Let X be an edge-cut strategy of G. Since X subverted from G means deleting at least $|X| + 1$ vertices of G, we have

$$ENS(G) \leq n - (|X| + 1) - |X| \leq n - 1 - 2\Lambda(G).$$

□

Corollary 1. *Let G be a connected graph with order $n \geq 3$. Then $ENS(G) \leq n - 3$.*

Remark 4. The upper bound in Theorem 5 is best possible. The stars and double stars can achieve this bound.

4. Conclusions and Future Research

In this paper, we prove that the problem of computing the edge-neighbour-scattering number of a graph is **NP**-complete and give some upper and lower bounds for this parameter. Here we list some other related interesting research problems.

Harary [12] determined the maximum and minimum connectivity of graphs with given order and size and also constructed corresponding extremal graphs, which are now widely known as the Harary graphs. Since then, finding the maximum or minimum value of graphic parameters with given order and size has become an attractive topic in graph theory. The opposite problem is finding the maximum or minimum size (order) when some parameters are given. It is natural to consider these two types of problems for the edge-neighbour-scattering number.

A Nordhaus–Gaddum type result is a (tight) lower or upper bound for the sum or product of the values of a parameter for a graph and its complement. Since Nordhaus and Gaddum [13] got the first result of this type on the chromatic number of graphs, many other similar results have been obtained (see [14] for a survey). It is interesting to investigate the Nordhaus–

Gaddum type result for the edge-neighbour-scattering number.

As we have shown, the problem of computing the edge-neighbour-scattering number of a graph is **NP**-complete, so it is interesting to consider whether we can find polynomial algorithms for computing this parameter of some special classes of graphs.

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