A Numerical Study for the Solution of Time Fractional Nonlinear Shallow Water Equation in Oceans

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In this paper, an analytical solution for the coupled one-dimensional time fractional nonlinear shallow water system is obtained by using the homotopy perturbation method (HPM). The shallow water equations are a system of partial differential equations governing fluid flow in the oceans (sometimes), coastal regions (usually), estuaries (almost always), rivers and channels (almost always). The general characteristic of shallow water flows is that the vertical dimension is much smaller than the typical horizontal scale. This method gives an analytical solution in the form of a convergent series with easily computable components, requiring no linearization or small perturbation. A very satisfactory approximate solution of the system with accuracy of the order 10^{-4} is obtained by truncating the HPM solution series at level six.

Key words: Nonlinear Shallow Water System; Approximate Analytical Solution; Homotopy Perturbation Method; Caputo Derivatives.

1. Introduction

In the past few decades, fractional differential equations and partial differential equations have been the centre of many studies due to their frequent applications in fluid mechanics, viscoelasticity, biology, physics, electrical network, control theory of dynamical systems, optics, and signal processing, as these can be modelled by linear and nonlinear fractional order differential equations as proposed by Oldham and Spanier [1]. Some fundamental results related to solving fractional differential equations may be found in Miller and Ross [2], Podlubny [3], Kilbas et al. [4], Diethelm and Ford [5], and Diethelm [6].

The shallow water equations (SWEs) are a system of partial differential equations governing fluid flow in the oceans, coastal regions, estuaries, rivers and channels. The general characteristic of shallow water flows is that the vertical dimension is much smaller than the typical horizontal scale. In this case, we can average over the depth to get rid of the vertical dimension. The SWEs can be used to predict tides, storm surge levels and coastline changes from hurricanes, ocean currents, and to study dredging feasibility. SWEs also arise in atmospheric flows and debris flows. Many geophysical flows are modelled by the variants of the SWEs. One form of the SWEs may be derived from Benney system.

The Benney equations [7], which are derived from the two-dimensional and time-dependent motion of an inviscid homogeneous fluid in a gravitational field by assuming the depth of the fluid to be small compared to the horizontal wave lengths considered, are expressed as

$$\begin{aligned}
&\frac{\partial u(x,y,t)}{\partial t} + u(x,y,t)\frac{\partial u(x,y,t)}{\partial x} - \frac{\partial u(x,y,t)}{\partial y} \\
&\cdot \int_0^y \frac{\partial u(x,\tau,t)}{\partial x}\,\mathrm{d}\tau + \frac{\partial h(x,t)}{\partial x} = 0\,, \qquad (1)\\
&\frac{\partial h(x,t)}{\partial t} + \frac{\partial}{\partial x}\int_0^h u(x,\tau,t)\,\mathrm{d}\tau = 0\,,
\end{aligned}$$

where y is the rigid bottom, $y = h(x,t)$ is the free surface, and $u(x,y,t)$ is the horizontal velocity component. If the horizontal velocity component u is independent of the height h, system (1) reduces to the equation system in the classical water theory corresponding to the case of irrational motion. The corresponding wave motion is determined by the coupled one-dimensional nonlinear shallow water system:

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$$D_t h(x,t) + u(x,t)D_x h(x,t) + h(x,t)D_x u(x,t) = 0\,,$$
$$D_t u(x,t) + u(x,t)D_x u(x,t) + D_x h(x,t) = 0\,. \quad (2)$$

The aim of this paper is to obtain an analytical solution of the system described by (2) by using the homotopy perturbation method (HPM). This method was first proposed by He [8] and was successfully applied to solve nonlinear wave equations [9]. The essential idea of this method is to introduce a homotopy parameter, say p, which takes values from 0 to 1, when $p = 0$, the system of equations usually reduces to a sufficiently simplified form, which normally admits a rather simple solution. As p gradually increases to 1, the system goes through a sequence of deformations, the solution for each of which is close to that of the previous stage of deformation. Eventually at $p = 1$, the system takes the original form of the equation and the final stage of deformation gives the desired solution. One of the most remarkable features of HPM is that usually just few perturbation terms are sufficient for obtaining a reasonably accurate solution. In recent years, the application of the homotopy perturbation method in nonlinear problems has been devoted by scientists and engineers, because this continuously deforms a simple problem easy to solve into the difficult problem under study. Many authors [10 – 17] applied HPM to solve a variety of nonlinear problems of physical and engineering interests. Recently, Wei et al. [18 – 20] have applied to obtain the solutions of the fractional partial differential equation in physics by using the implicit fully discrete local discontinuous Galerkin method. Recently, Younesian et al. [21 – 23] and Yıldırım et al. [24] have solved many physical models by using different methods.

To illustrate the basic ideas of HPM for fractional differential equations, we consider the following problem:

$$D^{n\alpha}_{*t} u(x,t) = v(x,t) - Lu(x,t) - Nu(x,t)\,,$$
$$n-1 < n\alpha \le n\,,\ n \in \mathbb{N}\,,\ t \ge 0\,,\ x \in \mathbb{R}^n\,, \quad (3)$$

subject to the initial and boundary conditions

$$u^{(i)}(0,0) = c_i\,,\ B\left(u, \frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial t}\right) = 0\,,$$
$$i = 0,1,2,\ldots,m-1\,,\ \ j = 1,2,3,\ldots,n\,, \quad (4)$$

where L is a linear operator, while N is a nonlinear operator, v is a known analytical function, and D^{α}_{*t} denotes the fractional derivative in the Caputo sense [3]. u is assumed to be a causal function of time, i. e., vanishing for $t < 0$. Also $u^{(i)}(x,t)$ is the ith derivative of u. c_i, $i = 0,1,2,\ldots,m-1$ are the specified initial conditions, and B is a boundary operator.

We construct the following homotopy:

$$(1-p)D^{n\alpha}_{*t} u(x,t) + p\big(D^{n\alpha}_{*t} u(x,t) + Lu(x,t)$$
$$+ Nu(x,t) - v(x,t)\big) = 0\,,\ \ p \in [0,1]\,, \quad (5)$$

which is equivalent to

$$D^{n\alpha}_{*t} u(x,t) + p\big(Lu(x,t) + Nu(x,t)$$
$$- v(x,t)\big) = 0\,,\ \ p \in [0,1]\,. \quad (6)$$

The homotopy parameter p always changes from zero to unity. In case $p = 0$, (6) becomes

$$D^{n\alpha}_{*t} u(x,t) = 0\,, \quad (7)$$

when $p = 1$, (6) turns out to be the original fractional differential equation. The homotopy parameter p is used to expand the solution in the form

$$u(x,t) = u_0(x,t) + pu_1(x,t) + p^2 u_2(x,t)$$
$$+ pu_3(x,t) + \ldots\,. \quad (8)$$

For nonlinear problems, we set $Nu(x,t) = S(x,t)$. Substituting (8) into (6) and equating the terms with identical power of p, we obtain a sequence of equations of the form

$$p^0 : D^{n\alpha}_{*t} u_0(x,t) = 0\,,$$
$$p^1 : D^{n\alpha}_{*t} u_1(x,t) = -Lu_0(x,t) - S_0(u_0(x,t)) + v(x,t)\,,$$
$$p^2 : D^{n\alpha}_{*t} u_2(x,t) = -Lu_1(x,t) - S_1(u_0(x,t), u_1(x,t))\,,$$
$$p^j : D^{n\alpha}_{*t} u_j(x,t) = -Lu_{j-1}(x,t) - S_{j-1}(u_0(x,t), u_1(x,t),$$
$$u_2(x,t),\ldots,u_{j-1}(x,t))\,,$$
$$j = 2,3,4,\ldots\,. \quad (9)$$

The functions $S_0, S_1, S_2, \ldots$ satisfy the equation

$$S\Big(u_0(x,t) + pu_1(x,t) + p^2 u_2(x,t) + p^3 u_3(x,t) + \ldots\Big)$$
$$= S_0(u_0(x,t)) + pS_1\Big(u_0(x,t), u_1(x,t)\Big) \quad (10)$$
$$+ p^2 S_2\Big(u_0(x,t), u_1(x,t), u_2(x,t)\Big) + \ldots\,.$$

Applying the inverse operator J^{α}_t where $J^{\alpha}_t f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau$, $(\alpha > 0, t > 0)$, on both sides of (9) and considering the initial and boundary conditions, the various components of the series solution are given by

$$u_0(x,t) = \sum_{i=0}^{n-1} c_i \frac{t^i}{i!}, \tag{11}$$
$$u_1(x,t) = -J_t^{n\alpha}(Lu_0(x,t)) - J_t^{n\alpha} S_0(u_0(x,t)) + J_t^{n\alpha} v(x,t),$$
$$u_j(x,t) = -J_t^{n\alpha}(Lu_{j-1}(x,t)) - J_t^{n\alpha} S_{j-1}\Big(u_0(x,t), u_1(x,t), u_2(x,t), \ldots, u_{j-1}(x,t)\Big), \quad j = 2,3,4,\ldots.$$

Hence, we get the HPM solution $u(x,t)$ as

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t). \tag{12}$$

We consider the following fractional version of the standard nonlinear shallow water system (2):

$$\begin{aligned} &D_t^\alpha h(x,t) + u(x,t) D_x h(x,t) + h(x,t) D_x u(x,t) = 0, \\ &\quad 0 < \alpha \le 1, \\ &D_t^\beta u(x,t) + u(x,t) D_x u(x,t) + D_x h(x,t) = 0, \\ &\quad 0 < \beta \le 1, \end{aligned} \tag{13}$$

with initial conditions

$$h(x,0) = \frac{1}{9}(x^2 - 2x + 1) \text{ and } u(x,0) = \frac{2}{3}(1-x), \tag{14}$$

where the fractional derivatives $D_t^\alpha = \frac{\partial}{\partial t^\alpha}$, $D_t^\beta = \frac{\partial}{\partial t^\beta}$ are in the Caputo sense [1–6]. The nonlinear shallow water system (13) has the exact solutions $h(x,t) = \frac{(x-1)^2}{9(t-1)^2}$ and $u(x,t) = \frac{2(x-1)}{3(t-1)}$, [7] for $\alpha = \beta = 1$.

2. Basic Definitions of the Fractional Calculus

In this section, we give some definitions and properties of the fractional calculus which are used further in this paper.

Definition 1. A real function $f(x)$, $x > 0$, is said to be in the space $\mathbb{C}_\mu$, $\mu \in \mathbb{R}$, if there exists a real number $p(> \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in \mathbb{C}[0,\infty)$, and it is said to be in the space $\mathbb{C}_\mu^m$ if and only if $f^{(m)} \in \mathbb{C}_\mu$, $m \in \mathbb{N}$.

Definition 2. The Riemann–Liouville fractional integral operator (J^α) of order $\alpha \ge 0$ of the function $f \in \mathbb{C}_\mu$, $\mu \ge -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} \,\mathrm{d}t, \quad \alpha > 0, \quad x > 0,$$
$$J^0 f(x) = f(x).$$

Properties of the operator J^α, can be found in [1–4]; we mention only the following. For $f \in \mathbb{C}_\mu$, $\mu \ge -1$, $\alpha, \beta \ge 0$, and $\gamma \ge -1$:
1. $\left(J^\alpha J^\beta\right) f(x) = J^{\alpha+\beta} f(x)$,
2. $\left(J^\alpha J^\beta\right) f(x) = \left(J^\beta J^\alpha\right) f(x)$,
3. $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}$.

The Riemann–Liouville derivative has certain disadvantages when trying to model real world phenomena with fractional differential equations. Podlubny [3] and Gorenflo et al. [25] have pointed out that the Caputo fractional derivative represents a short of regularization in the time origin for the Riemannian–Liouville fractional derivative and satisfies the requirements of being zero when applied to a constant. Besides, the Caputo definition does not use the fractional order derivative in the initial condition, thus is convenient in physical and engineering applications where the initial conditions are usually given in terms of the integer-order derivatives.

Definition 3. The fractional derivatives D^α of $f(x)$ in the Caputo's sense is defined as

$$\begin{aligned} D^\alpha f(x) &= J^{m-\alpha} D^m f(x) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha+1-m}} \,\mathrm{d}t, \\ &\quad \alpha > 0, \ x > 0, \end{aligned}$$

for $m - 1 < \mathrm{Re}(\alpha) \le m$, $m \in \mathbb{N}$, $f \in \mathbb{C}_{-1}^m$.

The following are two basic properties of the Caputo's fractional derivative:

Lemma 1. *If $m-1 < \alpha \le m$, $m \in \mathbb{N}$ and $f \in \mathbb{C}_\mu^n$, $\mu \ge -1$, then*

$$(D^\alpha J^\alpha) f(x) = f(x),$$
$$(J^\alpha D^\alpha) f(x) = f(x) - \sum_{i=0}^{m-1} f^i(0^+) \frac{x^i}{i!}.$$

The Caputo fractional derivatives are considered here because it allows traditional initial conditions to be included in the formulation of the problem.

Definition 4. For m to be the smallest integer that exceed α, the Caputo time fractional derivatives operator of $\alpha > 0$ is defined as

$$D_t^\alpha u(x,t)=\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\begin{cases}\dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t(t-\tau)^{m-\alpha-1}\frac{\partial^m u(x,\tau)}{\partial t^m}, & \text{for } m-1<\alpha<m,\\ \dfrac{\partial^m u(x,t)}{\partial t^m}, & \text{for } \alpha=m\in\mathbb{N}.\end{cases}$$

3. Solution of the Given Problem by HPM

In this section, the application of the homotopy perturbation method for coupled one-dimensional time fractional nonlinear shallow water equations with initial condition is discussed. To do so, we construct the homotopy:

$$\begin{aligned}&D_t^\alpha h+p(uD_xh+uD_xh)=0,\ \ 0<\alpha\le 1,\\&D_t^\beta u+p(uD_xu+D_xh)=0,\ \ 0<\beta\le 1.\end{aligned}\tag{15}$$

Now applying the classical perturbation technique, we assume that the solutions $h(x,t)$ and $u(x,t)$ of (15) may be expressed as power series in p as follows:

$$h(x,t)=h_0(x,t)+ph_1(x,t)+p^2h_2(x,t)+p^3h_3(x,t)+\ldots,\tag{16}$$

$$u(x,t)=u_0(x,t)+pu_1(x,t)+p^2u_2(x,t)+p^3u_3(x,t)+\ldots.\tag{17}$$

Substituting (16)–(17) into (15) and equating the coefficients of like powers of p, we get the following sets of differential equations:

$$p^0: D_t^\alpha h_0(x,t)=0,\ \ D_t^\beta u_0(x,t)=0,\tag{18}$$

$$\begin{aligned}p^1:&\ D_t^\alpha h_1+u_0D_xh_0+h_0D_xu_0=0,\\&D_t^\beta u_1+u_0D_xu_0+D_xh_0=0,\end{aligned}\tag{19}$$

$$\begin{aligned}p^2:&\ D_t^\alpha h_2+(u_0D_xh_1+u_1D_xh_0)\\&+(h_0D_xu_1+h_1D_xu_0)=0,\\&D_t^\beta u_2+(u_0D_xu_1+u_1D_xu_0)+D_xh_1=0,\end{aligned}\tag{20}$$

$$\begin{aligned}p^3:&\ D_t^\alpha h_3+(u_0D_xh_2+u_1D_xh_1+u_2D_xh_0)\\&+(h_0D_xu_2+h_1D_xu_1+h_2D_xu_0)=0,\\&D_t^\beta u_3+(u_0D_xu_2+u_1D_xu_1+u_2D_xu_0)+D_xh_2=0,\\&\vdots\end{aligned}\tag{21}$$

$$\begin{aligned}p^n:&\ D_t^\alpha h_n+(u_0D_xh_{n-1}+u_1D_xh_{n-2}+u_2D_xh_{n-3}+\ldots\\&+u_{n-1}D_xh_0)+(h_0D_xu_{n-1}+h_1D_xu_{n-2}+h_2D_xu_{n-3}\\&+\ldots+h_{n-1}D_xu_0)=0,\\&D_t^\beta u_n+(u_0D_xu_{n-1}+u_1D_xu_{n-2}+u_2D_xu_{n-3}+\ldots\\&+u_{n-1}D_xu_0)+D_xh_{n-1}=0.\end{aligned}\tag{22}$$

The above system of nonlinear equations can be easily solved by applying the operator J_t^α to (18)–(22) to obtain the various components $h_n(x,t)$ and $u_n(x,t)$, thus enabling the series solution to be entirely determined. The first few components of the homotopy perturbation solutions for (13) with the initial conditions (14) are as follows:

$$h_0(x,t)=h(x,0)=\frac{1}{9}(x^2-2x+1),$$

$$h_1(x,t)=\frac{2}{9}(x-1)^2\frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$h_2(x,t)=\frac{4(x-1)^2}{9}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{2(x-1)^2}{9}\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)},$$

$$\begin{aligned}h_3(x,t)=&\frac{8(x-1)^2}{9}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{4(x-1)^2}{9}\\&\cdot\left(\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{4}{9}\right)\frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)}\\&+\frac{8(x-1)^2}{27}\frac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)},\ldots,\end{aligned}$$

$$u_0(x,t)=u(x,0)=\frac{2}{3}(1-x),$$

$$u_1(x,t)=\frac{2}{3}(1-x)\frac{t^\beta}{\Gamma(\beta+1)},$$

$$u_2(x,t)=\frac{8(1-x)}{9}\frac{t^{2\beta}}{\Gamma(2\beta+1)}+\frac{4(1-x)}{9}\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)},$$

$$\begin{aligned}u_3(x,t)=&\frac{8(1-x)}{9}\frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)}+\frac{28(1-x)}{27}\\&\cdot\frac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)}+\frac{4(1-x)}{9}\left(\frac{\Gamma(2\beta+1)}{(\Gamma(\beta+1))^2}\right.\\&\left.+\frac{8}{3}\right)\frac{t^{3\beta}}{\Gamma(3\beta+1)},\ldots.\end{aligned}$$

In this manner, the rest of components of the homotopy perturbation solution can be obtained. Thus the solu-

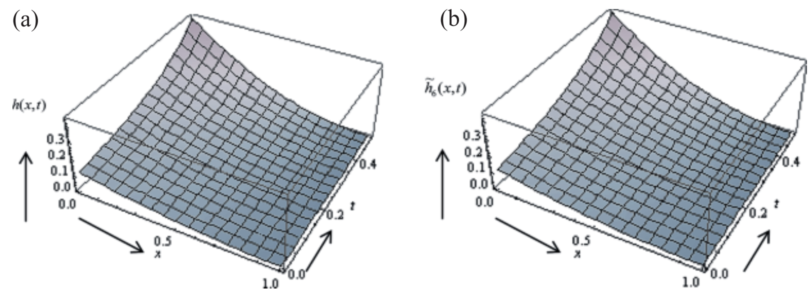

Fig. 1 (colour online). Comparison between exact solution $h(x,t)$ and approximate solution $\tilde{h}_6(x,t)$ obtained by HPM.

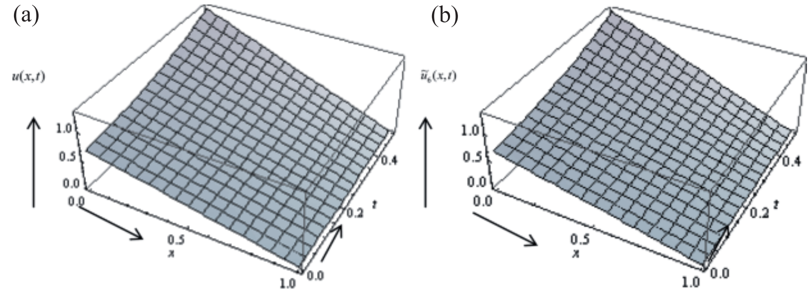

Fig. 2 (colour online). Comparison between exact solution $u(x,t)$ and approximate solution $\tilde{u}_6(x,t)$ obtained by HPM.

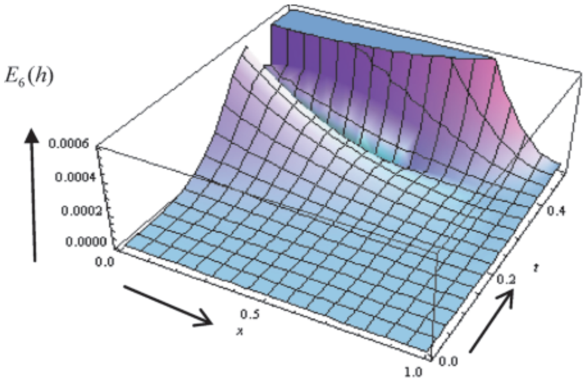

Fig. 3 (colour online). Absolute error $E_6(h)$ for $\alpha = 1$.

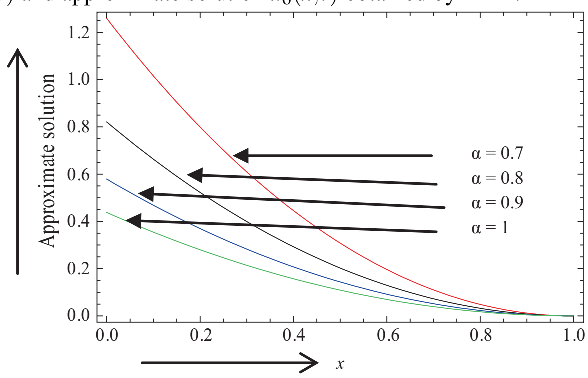

Fig. 5 (colour online). Approximate solutions $\tilde{h}_6(x,t)$ for different values of α at $t = 0.5$ and $\beta = 1$.

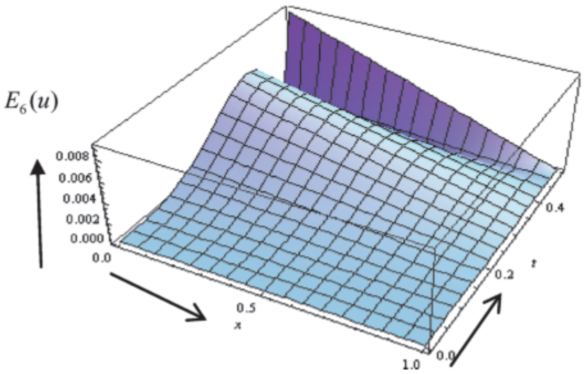

Fig. 4 (colour online). Absolute error $E_6(u)$ for $\alpha = 1$.

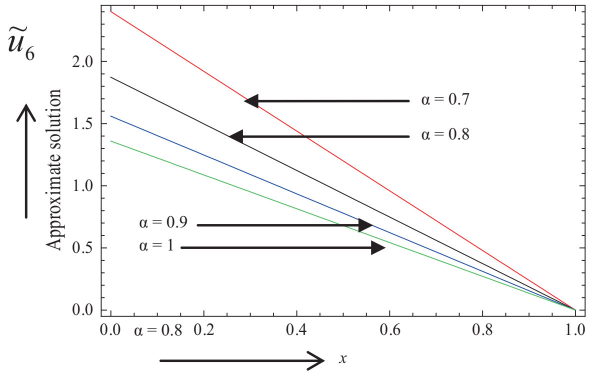

Fig. 6 (colour online). Approximate solutions $\tilde{u}_6(x,t)$ for different values of β at $t = 0.5$ and $\alpha = 1$.

tions $h(x,t)$ and $u(x,t)$ of the system described by (13) with the given initial conditions (14) is given by

$$h(x,t) = \lim_{N\to\infty} \sum_{n=0}^{N} h_n(x,t) \quad \text{and} \qquad (23)$$
$$u(x,t) = \lim_{N\to\infty} \sum_{n=0}^{N} u_n(x,t).$$

The series solution converges very rapidly. The rapid convergence means only few terms are required to get the analytic function.

The comparison between the exact solution and the approximate solution obtained by HPM is depicted through Figure 1 and 2. It can be seen from these figures that the analytical solution obtained by the present method is nearly identical to the exact solution of the standard gas dynamics, i. e. for the standard motion $\alpha, \beta = 1$.

4. Numerical Result and Discussion

The simplicity and accuracy of the proposed method is illustrated by computing the absolute errors $E_{h_6}(x,t) = |h(x,t) - \tilde{h}_6(x,t)|$ and $E_{u_6}(x,t) = |u(x,t) - \tilde{u}_6(x,t)|$, where $h(x,t)$ and $u(x,t)$ are the exact solutions and $\tilde{h}_6(x,t)$ and $\tilde{u}_6(x,t)$ are the approximate solutions of (13) obtained by truncating the respective solutions series (16) and (17) at level $N = 6$. Figures 3 and 4 represent the absolute error between exact and approximate solutions for height $h(x,t)$ and horizontal velocity $u(x,t)$ and their associated absolute errors. Mathematica (Version 7.0) software is used in computing and drawing the figures.

Figures 5 and 6 show the behaviour of the approximate solution $h(x,t)$ and $u(x,t)$ for different values $\alpha = 0.7, 0.8, 0.9$ and for standard shallow water equations, i. e. at $\alpha = 1$ for (13). It is seen from Figures 5 and 6 that the solution obtained by the present method decreases very rapidly with the increase of x. The accuracy of the result can be improved by introducing more terms of the approximate solutions.

5. Concluding Remarks

In this paper, the homotopy perturbation method is applied to obtain an approximate solution of the time fractional nonlinear shallow water equation. In HPM, a homotopy with an embedding parameter $p \in [0,1]$ is constructed, and the embedding parameter is considered as a 'small parameter', which can take full advantages of the traditional perturbation methods and homotopy techniques. This method contains the homotopy parameter p, which provides us with a simple way to control the convergence region of solution series for large values of t. The obtained results demonstrate the reliability of the algorithm and its wider applicability to nonlinear fractional partial differential equations.

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