Solving Supersymmetric Hirota–Satsuma Equation by a Direct Bosonization Approach

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The $\mathcal{N}=1$ supersymmetric Hirota–Satsuma equation is transformed into systems of coupled bosonic equations by expanding fermionic superfield in terms of 2, 3, and $n\,(n \geq 4)$ Grassmann parameters, respectively. Taking advantage of the resulting coupled bosonic systems being linear in the undetermined variables, the supersymmetric Hirota–Satsuma equation is solved out by using the mapping and deformation method. Besides, the richness of the localized excitations of the supersymmetric integrable system is discovered.

Key words: Supersymmetric Hirota–Satsuma Equation; Bosonization Approach; Mapping and Deformation Method; Exact Solution.
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1. Introduction

The theory of supersymmetric which was originally introduced and developed for applications in elementary particle physics [1 – 3], has been extensively studied in the past thirty years. Of particular interest in recent times is the class of supersymmetric integrable systems such as the Sine–Gordon equation, the Kadomtsev–Petviashvili (KP) hierarchy, the Korteweg–de Vries (KdV) hierarchy, the Boussinesq equation, and a number of other systems [4 – 11], which were all established with the supersymmetrization of corresponding bosonic integrable models. These supersymmetric equations involve Grassmann variables including both even (commuting or bosonic) and odd (anticommuting or fermionic) variables.

Among the various techniques which have been applied to generate soliton solutions of supersymmetric integrable systems [12 – 17], a simple bosonization method was recently proposed [18, 19]. This method gives a proper bosonization procedure in the complex fermionic fields in the usual quantum field theory and has the advantage that it can effectively avoid difficulties caused by intractable fermionic fields which are anticommuting. In this paper, we will use this method to study the supersymmetric Hirota–Satsuma equation and find new exact solutions of the supersymmetric integrable systems.

As we have known, the Hirota–Satsuma equation

$$u_{xxxt} + u_{xt}u_x + u_{xx}u_t - u_{xt} - u_{xx} = 0 \tag{1}$$

is proposed to describe interactions of two long waves with different dispersion relations [20]. It is found that the Hirota–Satsuma equation is just an example of many integrable systems arose from the Drinfeld–Sokolov theory [21, 22]. Some significant properties of the equation have been revealed in the past years. For instance, the Hirota–Satsuma equation possesses bilinear form [23], Lax pair [24 – 26], Bäcklund transformations [27], Darboux transformations [28 – 30], Painlevé property [25, 26], infinitely many symmetries and conservation laws [31] etc. The $\mathcal{N}=1$ supersymmetric Hirota–Satsuma (sHS) equation is established by extending the classical spacetime (x,t) to a superspacetime (θ,x,t), where θ is a Grassmann variable, and the field u to a fermionic superfield

$$\Phi(\theta,x,t) = \xi(x,t) + \theta u(x,t)\,. \tag{2}$$

Then, we get the nontrivial fermionic extension result

$$D^4\Phi_t + \Phi_t D^3\Phi + 2D^2\Phi D\Phi_t - D^2\Phi - \Phi_t = 0\,, \tag{3}$$

where $D = \partial_\theta + \theta\partial_x$ is the covariant derivative satisfying $D^2 = \partial_x$. The component version of (3) reads as

$$3u_xu_t + 2\xi_x\xi_{tx} + \xi_t\xi_{xx} + u_xu_{txx} - u_t - u_x = 0\,, \tag{4a}$$
$$\xi_{txx} + 2\xi_xu_t + \xi_tu_x - \xi_x - \xi_t = 0\,, \tag{4b}$$

where u and ξ are bosonic and fermionic component fields, respectively.

In this paper, we concentrate on the bosonization of the supersymmetric Hirota–Satsuma (sHS) equation based on (4). The detailed content is organized as follows: In Section 2 and 3, the sHS equation is bosonized into a coupled bosonic-looking system by expanding the supperfields with respect to two and three fermionic parameters, respectively. The general solutions of the model are found by using the mapping and deformation method, meanwhile some special types of nontravelling wave solutions are also obtained. In the last section, we extend the bosonization approach of the sHS system into the case of n fermionic parameters and get the exact solution in the general form. Final section contains a brief discussion.

2. Two Fermionic Parameters Bosonization and its Solutions

In order to get rid of the trouble caused by the anticommutative fermionic field of the supersymmetric equations, let us first expand the component fields ξ and u by two fermionic parameters as the following form [19, 20]:

$$\xi(x,t) = p_1\zeta_1 + p_2\zeta_2\,, \tag{5a}$$
$$u(x,t) = u_0 + u_{12}\zeta_1\zeta_2\,, \tag{5b}$$

where ζ_1 and ζ_2 are two Grassmann parameters, while the coefficients $p_1 = p_1(x,t)$, $p_2 = p_2(x,t)$, $u_0 = u_0(x,t)$, and $u_{12} = u_{12}(x,t)$ are four classical real or complex functions with respect to the spacetime variables x and t. Hence from (4), we obtain the system for the components of $u(x,t)$ and $\xi(x,t)$:

$$3u_{0,x}u_{0,t} + u_{0,txx} - u_{0,x} - u_{0,t} = 0\,, \tag{6a}$$
$$p_{1,txx} + 2p_{1,x}u_{0,t} + p_{1,t}u_{0,x} - p_{1,x} - p_{1,t} = 0\,, \tag{6b}$$
$$p_{2,txx} + 2p_{2,x}u_{0,t} + p_{2,t}u_{0,x} - p_{2,x} - p_{2,t} = 0\,, \tag{6c}$$
$$\begin{aligned}&3u_{12,x}u_{0,t} + 3u_{12,t}u_{0,x} - u_{12,x} + u_{12,txx} - u_{12,t}\\ &= 2p_{2,x}p_{1,xt} - 2p_{1,x}p_{2,xt} + p_{2,t}p_{1,xx} - p_{1,t}p_{2,xx}\,,\end{aligned} \tag{6d}$$

that is just the bosonic system of the original sHS equation (3) bosonized with two fermionic parameters. Equation (6a) is the integrated form of the Hirota–Satsuma equation which has been widely studied. Equations (6b) and (6c) are linear homogeneous in p_1 and p_2, respectively, and (6d) is linear nonhomogeneous in u_{12}. Therefore these pure bosonic equations can be solved out one after another in principle. So the bosonization approach can be used to get exact solutions of supersymmetric systems without too much difficulty and that is a big advantage of this method.

Now let us consider the travelling wave solutions of the bosonic system (6). Introducing the travelling wave variable $X = kx + \omega t + c_0$ with constants k, ω, and c_0, then (6) is transformed to the ordinary differential equations (ODEs)

$$k^2\omega u_{0,XXX} + 3k\omega u_{0,X}^2 - (k+\omega)u_{0,X} = 0\,, \tag{7a}$$
$$k^2\omega p_{1,XXX} + 3k\omega p_{1,X}u_{0,X} - (k+\omega)p_{1,X} = 0\,, \tag{7b}$$
$$k^2\omega p_{2,XXX} + 3\omega k p_{2,X}u_{0,X} - (k+\omega)p_{2,X} = 0\,, \tag{7c}$$
$$\begin{aligned}&k^2\omega u_{12,XXX} + 6k\omega u_{12,X}u_{0,X} - (k+\omega)u_{12,X}\\ &= 3k^2\omega(p_{2,X}p_{1,XX} - p_{1,X}p_{2,XX})\,.\end{aligned} \tag{7d}$$

Notation: The travelling waves in the superspace, $\Phi(x,t,\phi) = \Phi(kx + \omega t + c_0 + \zeta\theta)$, with Grassmann constant ζ are different from those in the usual spacetime $\{x,t\}$. Hereafter, the travelling waves be discussed in this paper are only in the usual spacetime $\{x,t\}$ not in the superspace $\{x,t,\theta\}$.

It is obviously that (7a) is the travelling wave reduction of the Hirota–Satsuma equation of which the solutions have been studied widely. So we try to build the mapping and deformation relation between the travelling wave solutions of (7a) and (7b)–(7d), and then to construct the exact solutions of the sHS equation by using the known solutions of the Hirota–Satsuma equation.

To this end, we first solve out $u_{0,X}$ from (7a). The result reads as

$$u_{0,X} = -\frac{u_0^2}{2k} + \frac{\omega + k}{2k\omega}\,. \tag{8}$$

In order to get the mapping relations of p_1, p_2, and u_{12}, we introduce the variable transformations as follows:

$$\begin{aligned}&p_1(X) = P_1(u_0(X))\,,\ \ p_2(X) = P_2(u_0(X))\,,\\ &u_{12}(X) = U_{12}(u_0(X))\,.\end{aligned} \tag{9}$$

Using the transformation (9) and vanishing $u_{0,X}$ via (8), the linear ODEs (7b)–(7d) are changed to

$$(-\omega u_0^2+\omega+k)\frac{d^3P_1(u_0)}{du_0^3} - 6\omega u_0\frac{d^2P_1(u_0)}{du_0^2}=0, \quad (10a)$$

$$(-\omega u_0^2+\omega+k)\frac{d^3P_2(u_0)}{du_0^3} - 6\omega u_0\frac{d^2P_2(u_0)}{du_0^2}=0, \quad (10b)$$

$$(-\omega u_0^2+\omega+k)\frac{d^3U_{12}(u_0)}{du_0^3}+6\omega\frac{dU_{12}(u_0)}{du_0} - 6u_0\omega\frac{d^2U_{12}(u_0)}{du_0^2}=F(u_0), \quad (10c)$$

where

$$F(u_0)=3(-\omega u_0^2+\omega+k)\left(\frac{dP_2(u_0)}{du_0}\frac{d^2P_1(u_0)}{du_0^2} - \frac{dP_1(u_0)}{du_0}\frac{d^2P_2(u_0)}{du_0^2}\right).$$

Solving the above equations, the mapping and deformation relations are constructed as

$$P_1(u_0)=\frac{A_1}{\omega(\omega+k)(\omega u_0^2-\omega-k)} - \frac{3A_1u_0\tanh^{-1}\left(\frac{\omega u_0}{\sqrt{\omega(\omega+k)}}\right)}{(\omega+k)^2\sqrt{\omega(\omega+k)}} + \frac{3A_1\ln(-(\omega+k))}{2(\omega+k)^2\omega}+A_2u_0+A_3, \quad (11a)$$

$$P_2(u_0)=\frac{A_4}{\omega(\omega+k)(\omega u_0^2-\omega-k)} - \frac{3A_4u_0\tanh^{-1}\left(\frac{\omega u_0}{\sqrt{\omega(\omega+k)}}\right)}{(\omega+k)^2\sqrt{\omega(\omega+k)}} + \frac{3A_4\ln(-(\omega+k))}{2(\omega+k)^2\omega}+A_5u_0+A_6, \quad (11b)$$

$$U_{12}(u_0)=\frac{A_7G(u_0)}{\omega u_0^2-\omega-k}+A_8(5\omega u_0^2-\omega-k) - \frac{15}{16\sqrt{\omega(\omega+k)}(\omega+k)^3((u_0^2-1)\omega-k)} \cdot\left[((-5u_0^2+1)\omega+k)((u_0^2-1)\omega-k)\int^{u_0}G(y)\widetilde{F}(y)\,dy + G(u_0)\int^{u_0}(5\omega y^2-\omega-k)(\omega(y^2-1)-k)\widetilde{F}(y)\,dy\right], \quad (11c)$$

where A_i, $(i=1,2,\ldots,8)$ are arbitrary constants and

$$G(u_0)=\left(\left(u_0^2-\frac{1}{5}\right)\omega-\frac{1}{5}k\right)((u_0^2-1)\omega-k) \cdot\tanh^{-1}\left(\frac{\sqrt{\omega(\omega+k)}u_0}{\omega+k}\right) - u_0\left(\omega u_0^2-\frac{13k}{15}-\frac{13\omega}{15}\right)\sqrt{\omega(\omega+k)},$$

$$\widetilde{F}(u_0)=\int^{u_0}F(y)\,dy+b_0$$

with an integral constant b_0.

Thus, we get the general two-fermionic parameter travelling wave solutions of the sHS equation from (5), (9), and (11) as following:

$$u=u_0+\left\{\frac{A_7G(u_0)}{\omega u_0^2-\omega-k}+A_8(5\omega u_0^2-\omega-k) - \frac{15}{16\sqrt{\omega(\omega+k)}(\omega+k)^3((u_0^2-1)\omega-k)}\left[((-5u_0^2+1) \cdot\omega+k)((u_0^2-1)\omega-k)\int^{u_0}G(y)\widetilde{F}(y)\,dy+G(u_0) \cdot\int^{u_0}(5\omega y^2-\omega-k)(\omega(y^2-1)-k)\widetilde{F}(y)\,dy\right]\right\}\zeta_1\zeta_2, \quad (12a)$$

$$\xi=\zeta_1\left[\frac{A_1}{\omega(\omega+k)(\omega u_0^2-\omega-k)} - \frac{3A_1u_0\tanh^{-1}\left(\frac{\omega u_0}{\sqrt{\omega(\omega+k)}}\right)}{(\omega+k)^2\sqrt{\omega(\omega+k)}}+\frac{3A_1\ln(-(\omega+k))}{2(\omega+k)^2\omega} +A_2u_0+A_3\right]+\zeta_2\left[\frac{A_4}{\omega(\omega+k)(\omega u_0^2-\omega-k)} - \frac{3A_4u_0\tanh^{-1}\left(\frac{\omega u_0}{\sqrt{\omega(\omega+k)}}\right)}{(\omega+k)^2\sqrt{\omega(\omega+k)}}+\frac{3A_4\ln(-(\omega+k))}{2(\omega+k)^2\omega} +A_5u_0+A_6\right] \quad (12b)$$

with the known solution u_0 of the usual Hirota–Satsuma equation.

Taking $A_1=A_3=A_4=A_6=A_7=b_0=0$ in (12), we get a special form of travelling wave solution

$$P_1(u_0)=A_2u_0, \quad (13a)$$

$$P_2(u_0)=A_5u_0, \quad (13b)$$

$$U_{12}(u_0)=A_9u_{0,X}, \quad (13c)$$

where $A_9 = -\frac{1}{10k^2}(\omega = -k)$. More generally, for any given $u_0(x,t)$ being a solution of the usual Hirota–Satsuma equation, a certain type of solutions of the bosonic equation (7) can be constructed as

$$p_1 = A_2 u_0 \,, \tag{14a}$$
$$p_2 = A_5 u_0 \,, \tag{14b}$$
$$u_{12} = \sigma(u_0) \,, \tag{14c}$$

where $\sigma(u_0)$ represents any symmetry of the usual Hirota–Satsuma equation.

In (14), u_0 can be chosen as any solution of the usual Hirota–Satsuma equation, so we have much freedom to choose u_0 so as to construct solutions of the sHS equation. It is easily verified that the first three equations of the bosonic-looking equations (6) are satisfied automatically if p_1 and p_2 are taken as the form of (14a)–(14b). Meanwhile, the right hand side of the nonhomogeneous equation (6d) vanishes after we substitute p_1 and p_2 into it. This means that u_{12} from (6d) exactly satisfies the symmetry equation of the integrated form of the Hirota–Satsuma equation (6a). As mentioned before, the Hirota–Satsuma equation possesses infinitely many symmetries, so infinitely many u_{12} can be generated. Furthermore, we can construct not only travelling wave solutions but also many other new types of solutions of the sHS equation using the solutions and infinitely many symmetries of the Hirota–Satsuma equation.

3. Three Fermionic Parameters Bosonization

The component fields ξ and u can also be expanded by three Grassmann parameters ζ_1, ζ_2, and ζ_3 as following:

$$\xi(x,t) = p_1\zeta_1 + p_2\zeta_2 + p_3\zeta_3 + p_{123}\zeta_1\zeta_2\zeta_3 \,, \tag{15a}$$
$$u(x,t) = u_0 + u_{12}\zeta_1\zeta_2 + u_{23}\zeta_2\zeta_3 + u_{31}\zeta_3\zeta_1 \,, \tag{15b}$$

where the coefficients u_0, u_{12}, u_{23}, u_{31}, p_{123}, $p_i = p_i(x,t)$ $(i = 1,2,3)$ are eight usual real or complex functions with respect to the spacetime variables x and t. Substituting the above equations into the sHS system (4), we get the bosonized form

$$3u_{0,x}u_{0,t} + u_{0,txx} - u_{0,x} - u_{0,t} = 0 \,, \tag{16a}$$
$$p_{1,txx} + 2p_{1,x}u_{0,t} + p_{1,t}u_{0,x} - p_{1,x} - p_{1,t} = 0 \,, \tag{16b}$$
$$p_{2,txx} + 2p_{2,x}u_{0,t} + p_{2,t}u_{0,x} - p_{2,x} - p_{2,t} = 0 \,, \tag{16c}$$
$$p_{3,txx} + 2p_{3,x}u_{0,t} + p_{3,t}u_{0,x} - p_{3,x} - p_{3,t} = 0 \,, \tag{16d}$$
$$\begin{aligned} &3u_{12,x}u_{0,t} + 3u_{12,t}u_{0,x} - u_{12,x} + u_{12,txx} - u_{12,t} \\ &= 2p_{2,x}p_{1,xt} - 2p_{1,x}p_{2,xt} + p_{2,t}p_{1,xx} - p_{1,t}p_{2,xx} \,, \end{aligned} \tag{16e}$$
$$\begin{aligned} &3u_{23,x}u_{0,t} + 3u_{23,t}u_{0,x} - u_{23,x} + u_{23,txx} - u_{23,t} \\ &= 2p_{3,x}p_{2,xt} - 2p_{2,x}p_{3,xt} + p_{3,t}p_{2,xx} - p_{2,t}p_{3,xx} \,, \end{aligned} \tag{16f}$$
$$\begin{aligned} &3u_{31,x}u_{0,t} + 3u_{31,t}u_{0,x} - u_{31,x} + u_{31,txx} - u_{31,t} \\ &= 2p_{1,x}p_{3,xt} - 2p_{3,x}p_{1,xt} + p_{1,t}p_{3,xx} - p_{3,t}p_{1,xx} \,, \end{aligned} \tag{16g}$$
$$\begin{aligned} &p_{123,txx} + 2p_{123,x}u_{0,t} + p_{123,t}u_{0,x} - p_{123,x} - p_{123,t} \\ &= -2(p_{1,x}u_{23,t} + p_{2,x}u_{31,t} + p_{3,x}u_{12,t}) \\ &\quad - p_{1,t}u_{23,x} - p_{2,t}u_{31,x} - p_{3,t}u_{12,x} \,. \end{aligned} \tag{16h}$$

Similar to the previous two fermionic parameters expansion case, except (16a) which is just the integrated form of the Hirota–Satsuma equation, the rest of the equations of bosonic equation system (16) are linear in p_i $(i = 1,2,3,123)$ and u_l $(l = 12,23,31)$, respectively.

To solve the equation system (16) in the travelling wave solution form, let us introduce the travelling wave variable $X = kx + \omega t + c_0$ with constants k, ω, and c_0, then the system is transformed to the OEDs

$$k^2\omega u_{0,XXX} + 3k\omega u_{0,X}^2 - (k+\omega)u_{0,X} = 0 \,, \tag{17a}$$
$$k^2\omega p_{1,XXX} + 3k\omega p_{1,X}u_{0,X} - (k+\omega)p_{1,X} = 0 \,, \tag{17b}$$
$$k^2\omega p_{2,XXX} + 3k\omega p_{2,X}u_{0,X} - (k+\omega)p_{2,X} = 0 \,, \tag{17c}$$
$$k^2\omega p_{3,XXX} + 3k\omega p_{3,X}u_{0,X} - (k+\omega)p_{3,X} = 0 \,, \tag{17d}$$
$$\begin{aligned} &k^2\omega u_{12,XXX} + 6k\omega u_{12,X}u_{0,X} - (k+\omega)u_{12,X} \\ &= 3k^2\omega(p_{2,X}p_{1,XX} - p_{1,X}p_{2,XX}) \,, \end{aligned} \tag{17e}$$
$$\begin{aligned} &k^2\omega u_{23,XXX} + 6k\omega u_{23,X}u_{0,X} - (k+\omega)u_{23,X} \\ &= 3k^2\omega(p_{3,X}p_{2,XX} - p_{2,X}p_{3,XX}) \,, \end{aligned} \tag{17f}$$
$$\begin{aligned} &k^2\omega u_{31,XXX} + 6k\omega u_{31,X}u_{0,X} - (k+\omega)u_{31,X} \\ &= 3k^2\omega(p_{1,X}p_{3,XX} - p_{3,X}p_{1,XX}) \,, \end{aligned} \tag{17g}$$
$$\begin{aligned} &k^2\omega p_{123,XXX} + 3k\omega p_{123,X}u_{0,X} - (\omega+k)p_{123,X} \\ &= -3k\omega(p_{1,X}u_{23,X} + p_{2,X}u_{31,X} + p_{3,X}u_{12,X}) \,. \end{aligned} \tag{17h}$$

It can be found that (17) is similar to (7) in form. To be described in a specific manner, (17a) is the same as (7a), while (17b)–(17d) have an analogy with (7b)–(7c) and (17e)–(17g) with (7d). Coefficients of the left-hand side of the last equation (17h) is the same as (17b)–(17d), but its right-hand side is related to p_l and u_j $(l = 1,2,3;\ j = 12,23,31)$, not equal to zero usually.

To solve the ODE system (17a)–(17h) by mapping and deformation method adopted in the previous section, we first use the following variable transformations:

$$p_i(X) = P_i(u_0(X)),\ (i=1,2,3),$$
$$p_{123}(X) = P_{123}(u_0(X)),$$
$$u_{12}(X) = U_{12}(u_0(X)),\ u_{23}(X) = U_{23}(u_0(X)),$$
$$u_{31}(X) = U_{31}(u_0(X))$$

and then eliminate u_{0X} by (8). The linear ODEs (17b)–(17h) are then changed to

$$(-\omega u_0^2+\omega+k)\frac{\mathrm{d}^3P_i(u_0)}{\mathrm{d}u_0^3} - 6\omega u_0\frac{\mathrm{d}^2P_i(u_0)}{\mathrm{d}u_0^2} = 0,\ (i=1,2,3), \tag{18a}$$

$$(-\omega u_0^2+\omega+k)\frac{\mathrm{d}^3U_{12}(u_0)}{\mathrm{d}^3u_0^3}+6\omega\frac{\mathrm{d}U_{12}(u_0)}{\mathrm{d}u_0} - 6u_0\omega\frac{\mathrm{d}^2U_{12}(u_0)}{\mathrm{d}u_0^2} = F_1(u_0), \tag{18b}$$

$$(-\omega u_0^2+\omega+k)\frac{\mathrm{d}^3U_{23}(u_0)}{\mathrm{d}^3u_0^3}+6\omega\frac{\mathrm{d}U_{23}(u_0)}{\mathrm{d}u_0} - 6u_0\omega\frac{\mathrm{d}^2U_{23}(u_0)}{\mathrm{d}u_0^2} = F_2(u_0), \tag{18c}$$

$$(-\omega u_0^2+\omega+k)\frac{\mathrm{d}^3U_{31}(u_0)}{\mathrm{d}^3u_0^3}+6\omega\frac{\mathrm{d}U_{31}(u_0)}{\mathrm{d}u_0} - 6u_0\omega\frac{\mathrm{d}^2U_{31}(u_0)}{\mathrm{d}u_0^2} = F_3(u_0), \tag{18d}$$

$$(-\omega u_0^2+\omega+k)\frac{\mathrm{d}^3U_{123}(u_0)}{\mathrm{d}u_0^3} - 6\omega u_0\frac{\mathrm{d}^2U_{123}(u_0)}{\mathrm{d}u_0^2} = F_4(u_0), \tag{18e}$$

where

$$F_1(u_0) = 3(-\omega u_0^2+\omega+k) \cdot\left(\frac{\mathrm{d}P_2(u_0)}{\mathrm{d}u_0}\frac{\mathrm{d}^2P_1(u_0)}{\mathrm{d}u_0^2} - \frac{\mathrm{d}P_1(u_0)}{\mathrm{d}u_0}\frac{\mathrm{d}^2P_2(u_0)}{\mathrm{d}u_0^2}\right), \tag{19a}$$

$$F_2(u_0) = 3(-\omega u_0^2+\omega+k) \cdot\left(\frac{\mathrm{d}P_3(u_0)}{\mathrm{d}u_0}\frac{\mathrm{d}^2P_2(u_0)}{\mathrm{d}u_0^2} - \frac{\mathrm{d}P_2(u_0)}{\mathrm{d}u_0}\frac{\mathrm{d}^2P_3(u_0)}{\mathrm{d}u_0^2}\right), \tag{19b}$$

$$F_3(u_0) = 3(-\omega u_0^2+\omega+k) \cdot\left(\frac{\mathrm{d}P_1(u_0)}{\mathrm{d}u_0}\frac{\mathrm{d}^2P_3(u_0)}{\mathrm{d}u_0^2} - \frac{\mathrm{d}P_3(u_0)}{\mathrm{d}u_0}\frac{\mathrm{d}^2P_1(u_0)}{\mathrm{d}u_0^2}\right), \tag{19c}$$

$$F_4(u_0) = -6\omega\left(\frac{\mathrm{d}U_{23}(u_0)}{\mathrm{d}u_0}\frac{\mathrm{d}P_1(u_0)}{\mathrm{d}u_0} + \frac{\mathrm{d}U_{31}(u_0)}{\mathrm{d}u_0} \cdot\frac{\mathrm{d}P_2(u_0)}{\mathrm{d}u_0} + \frac{\mathrm{d}U_{12}(u_0)}{\mathrm{d}u_0}\frac{\mathrm{d}P_3(u_0)}{\mathrm{d}u_0}\right). \tag{19d}$$

By repeating the processes in the last section, the general three fermionic parameters travelling wave solution of u can be written as

$$\begin{aligned} u = u_0 + \sum_{l=1}^{2}\Bigg\{ &\frac{h_l G(u_0)}{\omega u_0^2-\omega-k} + g_l(5\omega u_0^2-\omega-k) \\ &- \frac{15}{16\sqrt{\omega(\omega+k)}(\omega+k)^3((u_0^2-1)\omega-k)}\Bigg[((-5u_0^2+1) \\ &\cdot \omega+k)((u_0^2-1)\omega-k)\int^{u_0} G(y)\widetilde{F}_l(y)\,\mathrm{d}y + G(u_0) \\ &\cdot\int^{u_0}(5\omega y^2-\omega-k)(\omega(y^2-1)-k)\widetilde{F}_l(y)\,\mathrm{d}y\Bigg]\Bigg\}\zeta_l\zeta_{l+1} \\ &+\Bigg\{\frac{c_7G(u_0)}{\omega u_0^2-\omega-k} + c_8(5\omega u_0^2-\omega-k) \\ &- \frac{15}{16\sqrt{\omega(\omega+k)}(\omega+k)^3((u_0^2-1)\omega-k)}\Bigg[((-5u_0^2+1) \\ &\cdot \omega+k)((u_0^2-1)\omega-k)\int^{u_0} G(y)\widetilde{F}_3(y)\,\mathrm{d}y + G(u_0) \\ &\cdot\int^{u_0}(5\omega y^2-\omega-k)(\omega(y^2-1)-k)\widetilde{F}_3(y)\,\mathrm{d}y\Bigg]\Bigg\}\zeta_3\zeta_1, \end{aligned} \tag{20}$$

$$\begin{aligned} \xi = \sum_{l=1}^{3}\zeta_l\Bigg[&\frac{r_l}{\omega(\omega+k)(\omega u_0^2-\omega-k)} \\ &- \frac{3r_lu_0\tanh^{-1}\left(\frac{\omega u_0}{\sqrt{\omega(\omega+k)}}\right)}{(\omega+k)^2\sqrt{\omega(\omega+k)}} + \frac{3r_l\ln(-(\omega+k))}{2(\omega+k)^2\omega} + s_lu_0 \\ &+\alpha_l\Bigg] + \zeta_1\zeta_2\zeta_3 \end{aligned} \tag{21}$$

$$\begin{aligned} \cdot\Bigg[&c_{11}u_0 + \frac{1}{2\sqrt{\omega(\omega+k)}((u_0^2-1)\omega-k)(\omega+k)^2} \\ &\cdot\left(-3((u_0^2-1)\omega-k)u_0\int^{u_0}H(y)\widetilde{F}_4(y)\,\mathrm{d}y + 3H(u_0)\right. \\ &\left.\cdot\int^{u_0} y\widetilde{F}_4(y)(\omega(y^2-1)-k)\,\mathrm{d}y\right) + \frac{c_{10}H(u_0)}{\omega u_0^2-\omega-k}\Bigg], \end{aligned}$$

where

$$\widetilde{F}_l = \int^{u_0} F_l(y)\,\mathrm{d}y + b_l \quad (l=1,2,3,4)$$

$$H(u_0) = \omega u_0((u_0^2-1)\omega - k)\tanh^{-1}\left(\frac{\sqrt{\omega(\omega+k)}u_0}{\omega+k}\right) - \left(\omega u_0^2 - \frac{2}{3}(\omega+k)\right)\sqrt{\omega(\omega+k)}$$

with integral constants b_l $(l=1,2,3,4)$.

Similar to the two fermionic parameters case, for nontravelling wave solutions (16), we just write down a special case with

$$p_i = s_i u_0 \ (i=1,2,3), \ \ p_{123} = s_4 u_0, \quad u_{12} = \sigma_{12}(u_0), \ \ u_{23} = \sigma_{23}(u_0), \quad u_{31} = d_1 u_{12} + d_2 u_{23}, \tag{22}$$

where d_i $(i=1,2,3,4)$ are constants, u_0 is an arbitrary solution of the Hirota–Satsuma equation while $\sigma_{12}(u_0)$ and $\sigma_{23}(u_0)$ are arbitrary symmetries of the Hirota–Satsuma equation. Therefore, the sHS system (4) possesses the following special solution:

$$u = u_0 + \sigma_{12}(u_0)\zeta_1\zeta_2 + \sigma_{23}(u_0)\zeta_2\zeta_3 + (d_1 u_{12} + d_2 u_{23})\zeta_3\zeta_1, \tag{23a}$$

$$\xi = (d_1\zeta_1 + d_2\zeta_2 + d_3\zeta_3 + d_4\zeta_1\zeta_2\zeta_3)u_0. \tag{23b}$$

When one of the Grassmann numbers ζ_i $(i=1,2,3)$ tends to zero, the solution (23) turns back to that of the last section for two fermionic parameters.

Actually, applying the similar procedure for any numbers of the fermionic parameters, one can obtain various exact solutions such as the general travelling wave solution and the special solutions like (23).

4. *N* Fermionic Parameters Bosonization

Generally, the fields u and ξ in sHS system (4) can be expanded by $N \geq 2$ fermionic parameters ζ_i $(i=1,2,\ldots,N)$ as following:

$$\xi(x,t) = \sum_{n=1}^{[\frac{N+1}{2}]} \cdot \sum_{1\le i_1<\cdots<i_{2n-1}\le N} p_{i_1 i_2\cdots i_{2n-1}}\zeta_{i_1}\zeta_{i_2}\cdots\zeta_{i_{2n-1}}, \tag{24a}$$

$$u(x,t) = u_0 + \sum_{n=1}^{[\frac{N+1}{2}]} \cdot \sum_{1\le i_1<\cdots<i_{2n}\le N} u_{i_1 i_2\cdots i_{2n}}\zeta_{i_1}\zeta_{i_2}\cdots\zeta_{i_{2n}}, \ \ (i=1,2,3), \tag{24b}$$

where the coefficients u_0, $u_{i_1 i_2\cdots i_{2n}}$ $(1\le i_1<\cdots<i_{2n}\le N)$ and $p_{i_1 i_2\cdots i_{2n-1}}$ $(1\le i_1<\cdots<i_{2n-1}\le N)$ are 2^N real or complex bosonic functions of classical space-time variable x,t. Substituting the above expansion into (4), we obtain the following bosonic system of 2^N equations:

$$3u_{0,x}u_{0,t} + u_{0,txx} - u_{0,x} - u_{0,t} = 0,$$

$$\hat{L}_o p_{i_1 i_2\cdots i_{2n-1}} = \begin{cases} 0, \ n=1; \\ -\sum_{W_1}(-1)^{\tau(j_1,j_2,\ldots,j_{2n-1})} \\ \cdot(2u_{i_{j_1}i_{j_2}\cdots i_{j_{2l}},t}\,p_{i_{j_{2l+1}}i_{j_{2l+2}}\cdots i_{j_{2n-1}},x} \\ +u_{i_{j_1}i_{j_2}\cdots i_{j_{2l}},x}\,p_{i_{j_{2l+1}}i_{j_{2l+2}}\cdots i_{j_{2n-1}},t}), \\ \quad n=2,3,\ldots,[\frac{N+1}{2}], \end{cases}$$

$$\hat{L}_e u_{i_1 i_2\cdots i_{2n}} = \begin{cases} \sum_{W_2}(-1)^{\tau(j_1,j_2)}(2p_{i_{j_1},x}p_{i_{j_2},xt} \\ +p_{i_{j_1},t}p_{i_{j_2},xx}), \quad n=1; \\ \sum_{W_2}(-1)^{\tau(j_1,j_2,\ldots,j_{2n})}(2p_{i_{j_1}i_{j_2}\cdots i_{j_{2l-1}},x} \\ \cdot p_{i_{j_{2l}}i_{j_{2l+1}}\cdots i_{j_{2n}},xt} + p_{i_{j_1}i_{j_2}\cdots i_{j_{2l-1}},t} \\ \cdot p_{i_{j_{2l}}i_{j_{2l+1}}\cdots i_{j_{2n}},xx}) \\ -3\sum_{W_3}(-1)^{\tau(j_1,j_2,\ldots,j_{2n})}u_{i_{j_1}i_{j_2}\cdots i_{j_{2l}},t} \\ u_{i_{j_{2l+1}}i_{j_{2l+2}}\cdots i_{j_{2n}},x}, \quad n=2,3,\ldots,[\frac{N}{2}], \end{cases}$$

where

$$\hat{L}_o = \partial_{txx} + 2u_{0,t}\partial_x + u_{0,x}\partial_t - \partial_x - \partial_t,$$

$$\hat{L}_e = \partial_{txx} + 3u_{0,t}\partial_x + 3u_{0,x}\partial_t - \partial_x - \partial_t,$$

$$W_1 = \Big\{(j_1,j_2,\ldots,j_{2n-1})|1\le j_1<j_2<\cdots<j_{2l} \le 2n-1, \ 1\le j_{2l+1}<j_{2l+2}<\cdots<j_{2n-1} \le 2n-1, \ 1\le l\le n-1, \ j_{h_1}\ne j_{h_2}\,(h_1\ne h_2)\Big\},$$

$$W_2 = \Big\{(j_1,j_2,\ldots,j_{2n})|1\le j_1<j_2<\cdots<j_{2l-1} \le 2n, \ 1\le j_{2l}<j_{2l+1}<\cdots<j_{2n}\le 2n, \ 1\le l\le n, \ j_{h_1}\ne j_{h_2}\,(h_1\ne h_2)\Big\},$$

$$W_3 = \Big\{(j_1,j_2,\ldots,j_{2n})|1\le j_1<j_2<\cdots<j_{2l}\le 2n, \ 1\le j_{2l+1}<j_{2l+2}<\cdots<j_{2n}\le 2n, \ 1\le l\le n-1, \ j_{h_1}\ne j_{h_2}\,(h_1\ne h_2)\Big\}.$$

Then the component field u with N fermionic parameters can be written as

$$u(x,t) = u_0 + \sum_{n=1}^{[\frac{N+1}{2}]} \cdot \sum_{1\le i_1<\cdots<i_{2n}\le N} u_{i_1 i_2\cdots i_{2n}}\zeta_{i_1}\zeta_{i_2}\cdots\zeta_{i_{2n}}, \tag{25}$$

where

$$u_{i_1 i_2 \cdots i_{2n}} = U_{i_1 i_2 \cdots i_{2n}}(u_0) = \frac{h_{i_1 i_2 \cdots i_{2n}} G(u_0)}{\omega u_0^2 - \omega - k} + g_{i_1 i_2 \cdots i_{2n}}(5\omega u_0^2 - \omega - k) - \frac{15}{16\sqrt{\omega(\omega+k)}(\omega+k)^3((u_0^2-1)\omega - k)} \cdot \left[((-5u_0^2+1)\omega + k)((u_0^2-1)\omega - k) \cdot \int^{u_0} G(y)\widetilde{F}_{i_1 i_2 \cdots i_{2n}}(y)\,\mathrm{d}y + G(u_0) \cdot \int^{u_0} (5\omega y^2 - \omega - k)(\omega(y^2-1) - k)\widetilde{F}_{i_1 i_2 \cdots i_{2n}}(y)\,\mathrm{d}y\right],$$

$$v_{i_1 i_2 \cdots i_{2n-1}} = P_{i_1 i_2 \cdots i_{2n-1}}(u_0) = \begin{cases} \frac{r_{i_1 i_2 \cdots i_{2n-1}}}{\omega(\omega+k)(\omega u_0^2 - \omega - k)} - \frac{3 r_{i_1 i_2 \cdots i_{2n-1}} u_0 \tanh^{-1}\left(\frac{\omega u_0}{\sqrt{\omega(\omega+k)}}\right)}{(\omega+k)^2\sqrt{\omega(\omega+k)}} + \frac{3 r_{i_1 i_2 \cdots i_{2n-1}} \ln(-(\omega+k))}{2(\omega+k)^2 \omega} + s_{i_1 i_2 \cdots i_{2n-1}} u_0 + \alpha_{i_1 i_2 \cdots i_{2n-1}}, \quad n = 1; \\ c_{i_1 i_2 \cdots i_{2n-1}} u_0 + \frac{c_{i_1 i_2 \cdots i_{2n-1}} H(u_0)}{\omega u_0^2 - \omega - k} + \frac{1}{2\sqrt{\omega(\omega+k)}((u_0^2-1)\omega - k)(\omega+k)^2} \Big[-3((u_0^2-1)\omega - k)u_0 \int^{u_0} H(y) E_{i_1 i_2 \cdots i_{2n-1}}(y)\,\mathrm{d}y + 3H(u_0) \cdot \int^{u_0} y E_{i_1 i_2 \cdots i_{2n-1}}(y)(\omega(y^2-1) - k)\,\mathrm{d}y\Big], \\ \quad n = 2, 3, \ldots, \left[\frac{N+1}{2}\right], \end{cases}$$

with $\widetilde{F}_{i_1 i_2 \cdots i_{2n}}(u_0) = \int^{u_0} F_{i_1 i_2 \cdots i_{2n}}(y)\,\mathrm{d}y + b_{i_1, i_2, \cdots, i_{2n}}$ and

$$F_{i_1 i_2 \cdots i_{2n}}(u_0) = \begin{cases} 3\sum_{W_2}\left[(-\omega u_0^2 + \omega + k)(P_{i_{j_1}})_{u_0} \cdot (P_{i_{j_2}})_{u_0 u_0}\right], \quad n = 1; \\ 3\sum_{W_2}\left[(\omega u_0^2 - \omega - k)(P_{i_{j_1} i_{j_2} \cdots i_{2l-1}})_{u_0} \cdot (P_{i_{j_{2l}} i_{j_{2l+1}} \cdots i_{j_{2n}}})_{u_0 u_0}\right] + 6\omega u_0 \cdot \sum_{W_3}(U_{i_{j_1} i_{j_2} \cdots i_{j_{2l}}})_{u_0} \cdot (U_{i_{j_{2l+1}} i_{j_{2l+2}} \cdots i_{j_{2n}}})_{u_0}, \\ \quad n = 2, 3, \cdots, \left[\frac{N}{2}\right], \end{cases}$$

$$E_{i_1 i_2 \cdots i_{2n-1}}(u_0) = -6\omega \sum_{W_1} (U_{i_{j_1} i_{j_2} \cdots i_{j_{2l}}})_{u_0} \cdot (P_{i_{j_1} i_{j_2} \cdots i_{j_{2n-1}}})_{u_0}, \quad n = 2, 3, \ldots, \left[\frac{N+1}{2}\right],$$

where u_0 represents the solution of the usual Hirota–Satsuma equation and $b_{i_1, i_2, \cdots, i_{2n}}$ are arbitrary integral constants.

5. Conclusion

In summary, with a simple bosonization procedure, the sHS system is transformed into a coupled linear system without the intractable fermionic parameters. The travelling wave solutions of the bosonization systems have been obtained simply by the mapping and deformation method for the two and three fermionic parameter cases, respectively, then the general travelling wave solutions for arbitrary $N \geq 2$ fermionic parameters also have been achieved. Especially, some special types of exact solution of sHS equation can be obtained straightforwardly through the exact solutions of the Hirota–Satsuma equation and the related symmetries.

It should be noted that any kind of solutions of the usual Hirota–Satsuma equation such as the solitary solutions can be extended to those of the sHS equation, and these solutions are completely different from those obtained via other methods such as the bilinear approach [32]. This fact shows us that for the sHS equation besides the super solitons existing in the super space-time there exist various kinds of localized excitations in the usual space-time. So, it is important to further develop the bosonization procedure such that the known solutions obtained in other approaches can also be included.

In this pape,r we have dealt with an integrable supersymmetric equation but from the procedure of bosonization, we conclude that for the nonintegrable ones this method also work well. So this method provides an efficient way to solve supersymmetric systems and further work on this aspect needs to be enlarged.

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