Flow of a Giesekus Fluid in a Planar Channel due to Peristalsis

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An attempt is made to investigate the peristaltic motion of a Giesekus fluid in a planar channel under long wavelength and low Reynolds number approximations. Under these assumptions, the flow problem is modelled as a second-order nonlinear ordinary differential equation. Both approximate and exact solution of this equation are presented. The validity of the approximate solution is examined by comparing it with the exact solution. A parametric study is performed to analyze the effects of non-dimensional parameters associated with the Giesekus fluid model (α and We) on flow velocity, pressure rise per wavelength, and trapping phenomenon. It is found that the behaviour of longitudinal velocity and pattern of streamlines for a Giesekus fluid deviate from their counterparts for a Newtonian fluid by changing the parameters α and We. In fact, the magnitude of the longitudinal velocity at the center of the channel for a Giesekus fluid is less than that for a Newtonian fluid. It is also observed that the pressure rise per wavelength decreases in going form Newtonian to Giesekus fluid. Moreover, the size of trapped bolus is large and it circulates faster for a Newtonian fluid in comparison to a Giesekus fluid.

Key words: Peristalsis; Giesekus Fluid; Channel Flow; Exact Solution; Trapping.

1. Introduction

Peristalsis is a mechanism used by many biological ducts such as esophagus and ureter to convey their fluid contents. The mechanism works when a progressive wave of muscular contraction propogates along the wall of the organ. Industrial roller and finger pumps also operate according to the principle of peristalsis.

Initial studies on peristaltic motion were focused on exploring the fluid mechanics of the problem under the assumptions of low Reynolds number and long peristaltic waves [1 – 3]. The latter developments in the field include the work of Pozrikidis [4] for non-slender geometry, Takabatake et al. [5] for higher Reynolds numbers, Böhme and Friedrich [6], Raju and Devanathan [7, 8], Srivastava and Srivastava [9], Siddiqui and Schwarz [10], Siddiqui et al. [11], Mekheimer et al. [12], Mekheimer [13], Hayat et al. [14, 15], and Haroun et al. [16] for non-Newtonian fluids. However, the survey of literature reveals that very little attention is given to peristaltic flows of viscoelastic fluid models represented by nonlinear differential constitutive equations under widely used assumptions of long wavelength and low Reynolds number. Much of the work is based on the constitutive equations of generalized Newtonian fluid models [17 – 20], retarded motion expansion [21, 22], and polar fluids [23, 24].

The constitutive equations proposed by Oldroyd [25], White and Metzner [26], and Giesekus [27, 28] have been rarely used in the study of peristaltic flows. Few studies have been conducted using Oldroyd constitutive equations. For example, Hayat et al. [29] analyzed the peristaltic motion of an Oldroyd-B fluid in a planar channel by assuming the wave number to be small. They have not adopted long wavelength and low Reynolds number assumptions in their analysis. The problem with the constitutive equation of Oldroyd-B fluid is that it reduces to the constitutive equation of a Newtonian fluid under long wavelength and low Reynolds number assumptions. The simplest model proposed by Oldroyd which captures viscoelastic features under long wavelength assumption is the Oldroyd 4-constant model. Similarly, the Oldroyd 8-constant model can also be used under long wavelength assumption. Ali et al. [30, 31] were the first to discuss the peristaltic motion of Oldroyd 4-constant and Oldroyd 8-constant models under long wavelength approximation. However, no attempt has been made to use the Giesekus constitutive equation for studying peristaltic flow under long wavelength assumption. The Giesekus constitutive equation is proved to be more useful than the Oldroyd 8-constant constitutive equation because it gives material functions that are much more realistic than those obtained from the Oldroyd 8-constant constitutive equation [32]. Some recent studies regarding elementary flows of the Giesekus model can be found in [33–35]. Motivated by the above facts, in the present paper, we study the peristaltic motion of a Giesekus fluid under long wavelength and low Reynolds number assumptions.

In Section 2, we formulate the problem by stating the underlying assumptions and governing equations. The solution of the problem is obtained in Section 3. Section 4 presents a brief discussion of the results. Finally, we conclude the paper in Section 5.

2. Governing Equations

The flow is assumed to be incompressible, therefore the laws of conservation of mass and momentum take the following form:

$$\mathrm{div}\bar{V} = 0\,, \tag{1}$$

$$\rho \frac{\mathrm{d}\bar{V}}{\mathrm{d}\bar{t}} = -\nabla \bar{p} + \mathrm{div}\bar{S}\,, \tag{2}$$

where $\bar{V}$ is the velocity, ρ the density, $\mathrm{d}/\mathrm{d}\bar{t}$ the material derivative, $\bar{p}$ the pressure, and $\bar{S}$ the extra stress tensor. The extra stress tensor for a Giesekus fluid satisfies the expression [27, 28, 32]

$$\bar{S} + \frac{\alpha\bar{\lambda}}{\mu}\left(\bar{S}\cdot\bar{S}\right) + \bar{\lambda}\frac{D\bar{S}}{D\bar{t}} = \mu \bar{A}_1\,, \tag{3}$$

in which μ and $\bar{\lambda}$ are model parameters representing zero-shear viscosity and zero-shear relaxation time, respectively [32]. $\bar{A}_1$ is the first Rivlin–Ericksen tensor, defined by

$$\bar{A}_1 = \bar{L} + \bar{L}^{\mathrm{T}}\,, \tag{4}$$

where $\bar{L}$ is the velocity gradient and

$$\frac{D\bar{S}}{D\bar{t}} = \frac{\mathrm{d}\bar{S}}{\mathrm{d}\bar{t}} - \bar{L}\bar{S} - \bar{S}\bar{L}^{\mathrm{T}} \tag{5}$$

is the upper convected time derivative. The parameter $\bar{\alpha}$ appearing in (3) is another model parameter, and according to Bird et al. [32], the term containing $\bar{\alpha}$ is due to the anisotropic hydrodynamic drag on the constituent polymer molecules. In view of Giesekus [27, 28] the values of $\bar{\alpha}$ should be such that $0 \le \alpha \le 1$. However, Bird et al. [32] proposed that for realistic properties $0 \le \alpha \le 0.5$. It should be noted that the model (3) includes *the convected* Maxwell model (for $\bar{\alpha} = 0$) and the Newtonian fluid model (for $\bar{\alpha} = \bar{\lambda} = 0$) as limiting cases.

3. Problem Formulation

We consider a channel of width $2a$ filled with a homogenous incompressible Giesekus fluid. The walls of the channel are assumed to be flexible. Further assume two symmetric infinite wave trains travelling with velocity c along the walls. If $\bar{X}$ and $\bar{Y}$ denote the longitudinal and transverse coordinates, respectively, then the wall surface is given by

$$\bar{h}\left(\bar{X},\bar{t}\right) = a + b\cos\left[\frac{2\pi}{\lambda^*}\left(\bar{X} - c\bar{t}\right)\right]. \tag{6}$$

In (6), b is the wave amplitude, λ^* the wavelength, and $\bar{t}$ the time.

Since the flow is two-dimensional, therefore, we define

$$\bar{V} = \left[\bar{U}\left(\bar{X},\bar{Y},\bar{t}\right), \bar{V}\left(\bar{X},\bar{Y},\bar{t}\right), 0\right], \tag{7}$$

in which $\bar{U}$ and $\bar{V}$ are the longitudinal and transverse velocity components, respectively.

With the above definition of velocity field, (1)–(5) give

$$\frac{\partial \bar{U}}{\partial \bar{X}} + \frac{\partial \bar{V}}{\partial \bar{Y}} = 0\,, \tag{8}$$

$$\rho\left(\frac{\partial}{\partial \bar{t}} + \bar{U}\frac{\partial}{\partial \bar{X}} + \bar{V}\frac{\partial}{\partial \bar{Y}}\right)\bar{U} = -\frac{\partial \bar{p}}{\partial \bar{X}} + \frac{\partial \bar{S}_{\bar{X}\bar{X}}}{\partial \bar{X}} + \frac{\partial \bar{S}_{\bar{X}\bar{Y}}}{\partial \bar{Y}}\,, \tag{9}$$

$$\rho\left(\frac{\partial}{\partial \bar{t}} + \bar{U}\frac{\partial}{\partial \bar{X}} + \bar{V}\frac{\partial}{\partial \bar{Y}}\right)\bar{V} = -\frac{\partial \bar{p}}{\partial \bar{Y}} + \frac{\partial \bar{S}_{\bar{X}\bar{Y}}}{\partial \bar{X}} + \frac{\partial \bar{S}_{\bar{Y}\bar{Y}}}{\partial \bar{Y}}\,, \tag{10}$$

$$\bar{S}_{\bar{X}\bar{X}} + \bar{\lambda}\left[\left(\frac{\partial}{\partial \bar{t}} + \bar{U}\frac{\partial}{\partial \bar{X}} + \bar{V}\frac{\partial}{\partial \bar{Y}}\right)\bar{S}_{\bar{X}\bar{X}} - 2\frac{\partial \bar{U}}{\partial \bar{X}}\bar{S}_{\bar{X}\bar{X}} - 2\frac{\partial \bar{U}}{\partial \bar{Y}}\bar{S}_{\bar{X}\bar{Y}}\right] + \frac{\alpha\bar{\lambda}}{\mu}\left(\bar{S}^2_{\bar{X}\bar{X}} + \bar{S}^2_{\bar{X}\bar{Y}}\right) = 2\mu\frac{\partial \bar{U}}{\partial \bar{X}}\,, \tag{11}$$

$$\bar{S}_{\bar{X}\bar{Y}} + \bar{\lambda}\left[\left(\frac{\partial}{\partial \bar{t}} + \bar{U}\frac{\partial}{\partial \bar{X}} + \bar{V}\frac{\partial}{\partial \bar{Y}}\right)\bar{S}_{\bar{X}\bar{Y}} - \frac{\partial \bar{V}}{\partial \bar{X}}\bar{S}_{\bar{X}\bar{X}} - \frac{\partial \bar{U}}{\partial \bar{Y}}\bar{S}_{\bar{Y}\bar{Y}}\right] + \frac{\alpha\bar{\lambda}}{\mu}\left(\bar{S}_{\bar{X}\bar{X}} + \bar{S}_{\bar{Y}\bar{Y}}\right)\bar{S}_{\bar{X}\bar{Y}} = \mu\left(\frac{\partial \bar{U}}{\partial \bar{Y}} + \frac{\partial \bar{V}}{\partial \bar{X}}\right), \tag{12}$$

$$\bar{S}_{\bar{Y}\bar{Y}} + \bar{\lambda}\left[\left(\frac{\partial}{\partial \bar{t}} + \bar{U}\frac{\partial}{\partial \bar{X}} + \bar{V}\frac{\partial}{\partial \bar{Y}}\right)\bar{S}_{\bar{Y}\bar{Y}} - 2\frac{\partial \bar{V}}{\partial \bar{X}}\bar{S}_{\bar{X}\bar{Y}} - 2\frac{\partial \bar{V}}{\partial \bar{Y}}\bar{S}_{\bar{Y}\bar{Y}}\right]\frac{\alpha\bar{\lambda}}{\mu}\left(\bar{S}^2_{\bar{Y}\bar{Y}} + \bar{S}^2_{\bar{X}\bar{Y}}\right) = 2\mu\frac{\partial \bar{V}}{\partial \bar{Y}}. \tag{13}$$

Now, for subsequent analysis, we switch from laboratory frame $(\bar{X}, \bar{Y})$ to wave frame $(\bar{x}, \bar{y})$ which is moving with the wave speed c. In the wave frame, the flow becomes steady. The coordinates and velocities in the two frames are related through

$$\bar{x} = \bar{X} - c\bar{t}, \quad \bar{y} = \bar{Y}, \qquad \bar{u} = \bar{U} - c, \quad \bar{v} = \bar{V}, \tag{14}$$

where $\bar{u}$ and $\bar{v}$ are respective dimensional velocity components parallel to $\bar{x}$ and $\bar{y}$ in the wave frame. The governing equation can be made dimensionless by introducing the following dimensionless variables:

$$x = \frac{2\pi\bar{x}}{\lambda^*}, \quad y = \frac{\bar{y}}{a}, \quad u = \frac{\bar{u}}{c}, \quad v = \frac{\bar{v}}{c}, \qquad h = \frac{\bar{h}}{a}, \quad S = \frac{a\bar{S}}{\mu c}, \quad p = \frac{2\pi a^2}{\lambda^*\mu c}\bar{p}. \tag{15}$$

Finally, the governing equation in terms of stream function $\psi(x,y)$ defined by the relations

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\delta\frac{\partial \psi}{\partial x}, \tag{16}$$

can be written as

$$\delta \mathrm{Re}\left[\left(\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial \psi}{\partial y}\right] = -\frac{\partial p}{\partial x} + \delta\frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{xy}}{\partial y}, \tag{17}$$

$$-\delta^3 \mathrm{Re}\left[\left(\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial \psi}{\partial x}\right] = -\frac{\partial p}{\partial y} + \delta^2\frac{\partial S_{xy}}{\partial x} + \delta\frac{\partial S_{yy}}{\partial y}, \tag{18}$$

$$\delta \mathrm{Re}\left[\left(\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\right)\left(\frac{\partial^2 \psi}{\partial y^2} + \delta^2\frac{\partial^2 \psi}{\partial x^2}\right)\right] = \delta\frac{\partial^2\left(S_{xx} - S_{yy}\right)}{\partial x \partial y} + \left(\frac{\partial^2}{\partial y^2} - \delta^2\frac{\partial^2}{\partial x^2}\right)S_{xy}, \tag{19}$$

$$S_{xx} + \mathrm{We}\left[\delta\left(\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\right)S_{xx} - 2\delta\frac{\partial^2 \psi}{\partial x \partial y}S_{xx} - 2\frac{\partial^2 \psi}{\partial y^2}S_{xy}\right] + \alpha \mathrm{We}\left(S^2_{xx} + S^2_{xy}\right) = 2\delta\frac{\partial^2 \psi}{\partial x \partial y}, \tag{20}$$

$$S_{xy} + \mathrm{We}\left[\delta\left(\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\right)S_{xy} + \delta^2\frac{\partial^2 \psi}{\partial x^2}S_{xx} - \frac{\partial^2 \psi}{\partial y^2}S_{yy}\right] + \alpha \mathrm{We} S_{xy}\left(S_{xx} + S_{yy}\right) = \left(\frac{\partial^2 \psi}{\partial y^2} - \delta^2\frac{\partial^2 \psi}{\partial x^2}\right), \tag{21}$$

$$S_{yy} + \mathrm{We}\left[\delta\left(\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\right)S_{yy} + 2\delta^2\frac{\partial^2 \psi}{\partial x^2}S_{xy} + 2\delta\frac{\partial^2 \psi}{\partial x \partial y}S_{yy}\right] + \alpha \mathrm{We}\left(S^2_{xy} + S^2_{yy}\right) = -2\delta\frac{\partial^2 \psi}{\partial x \partial y}, \tag{22}$$

where δ $(= 2\pi a/\lambda^*)$ is the dimensionless wave number, Re $(= \rho c a/\mu)$ is the Reynolds number, and We $(= \bar{\lambda}c/a)$ is the Weissenberg number. It should be noted that by defining the stream function, the continuity equation (8) is automatically satisfied, and the compatibility equation (19) is obtained by eliminating p between (17) and (18) (it represents the vorticity transport equation). Now, in view of long wavelength and low Reynolds number approximations [2, 10, 12 – 19], (17) – (22) reduce to

$$\frac{\partial S_{xy}}{\partial y} = \frac{\mathrm{d}p}{\mathrm{d}x}, \tag{23}$$

$$\frac{\partial^2 S_{xy}}{\partial y^2} = 0, \tag{24}$$

$$S_{xx} - 2\mathrm{We}\frac{\partial^2 \psi}{\partial y^2}S_{xy} + \alpha \mathrm{We}\left(S^2_{xx} + S^2_{xy}\right) = 0, \tag{25}$$

$$S_{xy} - \mathrm{We}\frac{\partial^2 \psi}{\partial y^2}S_{yy} + \alpha \mathrm{We} S_{xy}\left(S_{xx} + S_{yy}\right) = \frac{\partial^2 \psi}{\partial y^2}, \tag{26}$$

$$S_{yy} + \alpha \mathrm{We}\left(S^2_{xy} + S^2_{yy}\right) = 0. \tag{27}$$

The dimensionless pressure rise over one wavelength can be calculated via the expression

$$\Delta p = \int_0^{2\pi} \frac{\mathrm{d}p}{\mathrm{d}x}\,\mathrm{d}x. \tag{28}$$

Exploiting the flow symmetry about the x-axis, we shall solve the flow problem only in the half flow domain $y \in [0, h]$.

The appropriate boundary conditions in the wave frame are [16 – 18]

$$\psi = 0\,, \quad \frac{\partial^2 \psi}{\partial y^2} = 0 \ \text{ at } \ y = 0\,, \tag{29}$$

$$\psi = F\,, \quad \frac{\partial \psi}{\partial y} = -1\,, \ \text{ at } \ y = h = 1 + \varphi \cos x\,, \tag{30}$$

where $\varphi = b/a$ is the amplitude ratio. The dimensionless mean flows Θ, in laboratory frame, and F in the wave frame are related according to the following expression [18]:

$$\Theta = F + 1\,. \tag{31}$$

4. Solution of the Problem

Equation (24) can be integrated to give

$$S_{xy} = Ay\,, \tag{32}$$

where we have used the second boundary condition in (2). The value of the integration constant A physically represents the value pressure gradient $\mathrm{d}p/\mathrm{d}x$. Now, from (26) and (27), we can write

$$S_{xx} = \frac{(1 + \mathrm{We} S_{yy})\,\partial^2 \psi/\partial y^2}{\alpha \mathrm{We} S_{xy}} - \frac{1 + \alpha \mathrm{We} S_{yy}}{\alpha \mathrm{We}}\,, \tag{33}$$

$$S_{yy} = \frac{-1 \pm \sqrt{1 - 4\alpha^2 \mathrm{We}^2 S_{xy}^2}}{2\alpha \mathrm{We}}\,. \tag{34}$$

The appropriate sign in (34) must be positive as discussed by Schleiniger and Weinacht [33]. Insertion of (33) into (25) yields the following determining equation for ψ:

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{1 + (2\alpha - 1)\mathrm{We} S_{yy}}{(1 + \mathrm{We} S_{yy})^2} S_{xy}\,. \tag{35}$$

With the help of (32) and (34), (35) can be put in the form

$$\begin{aligned}\frac{\partial^2 \psi}{\partial y^2} = &\left[1 + (2\alpha - 1)\left\{\left(-1 + \sqrt{1 - 4\alpha^2 \mathrm{We}^2 A^2 y^2}\right)\right.\right. \\ &\left.\left./2\alpha\right\}\right]\left[\left(1 + \left\{\left(-1 + \sqrt{1 - 4\alpha^2 \mathrm{We}^2 A^2 y^2}\right)\right.\right.\right. \\ &\left.\left.\left./2\alpha\right\}\right)^2\right]^{-1} Ay\,. \end{aligned}\tag{36}$$

The above equation is subject to the boundary conditions $\psi(0) = 0$, $\psi(h) = F$, and $\partial \psi/\partial y|_{y=h} = -1$.

Integration of above equation twice and utilization of boundary conditions $\psi(0) = 0$ and $\partial \psi/\partial y|_{y=h} = -1$ yields the following expression of ψ:

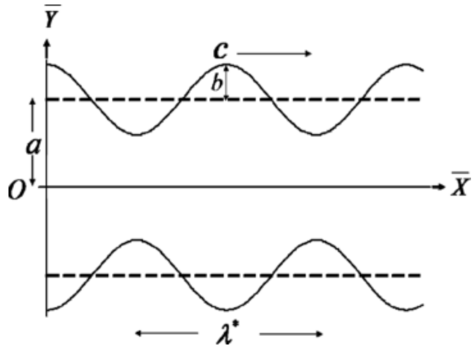

Fig. 1. Schematic diagram of the flow geometry for $a = 1$, $b = 0.4$, $\lambda^* = 2$, and $t = 0$.

$$\begin{aligned}\psi = &\frac{1}{4} y\left[-4 - \left(2Ah^2(2\alpha - 1)\left(1 - 2\alpha\right.\right.\right. \\ &\left.\left.+ \sqrt{1 - 4\alpha^2 A^2 h^2 \mathrm{We}^2}\right)\right)\Big/(\alpha - 1 + \alpha A^2 h^2 \mathrm{We}^2) \\ &+ \left\{24\alpha(1 - \alpha) - 4 + (2\alpha - 1)\left(4\sqrt{1 - 4\alpha^2 A^2 h^2 \mathrm{We}^2}\right.\right. \\ &\left.\left. - \sqrt{1 - 4\alpha^2 A^2 y^2 \mathrm{We}^2}\right)\right\}\Big/A\alpha \mathrm{We}^2\Big] + \frac{1}{8\alpha^2 A^2 \mathrm{We}^3} \\ &\cdot\Big[(2\alpha - 1)(1 - 24\alpha(1 - \alpha))\sin^{-1}(2Ay\alpha \mathrm{We}) + 8\alpha^{3/2} \\ &\cdot\sqrt{(\alpha - 1)/\alpha^2}(1 - 6\alpha(1 - \alpha))\tan^{-1}\left(\frac{Ay\mathrm{We}}{\sqrt{(\alpha - 1)/\alpha}}\right) \\ &- \tan^{-1}\left(Ay\mathrm{We}(2\alpha - 1)\Big/\sqrt{\frac{(1 - 4\alpha^2 A^2 y^2 \mathrm{We}^2)(\alpha - 1)}{\alpha}}\right) \\ &\cdot 8\sqrt{\alpha(\alpha - 1)}(1 - 6\alpha(1 - \alpha)) + \ln\left\{\left(2\alpha - 1\right.\right. \\ &\left. + \sqrt{1 - 4\alpha^2 A^2 y^2 \mathrm{We}^2}\right)\Big/\left(2\alpha - 1\right. \\ &\left.\left. + \sqrt{1 - 4\alpha^2 A^2 h^2 \mathrm{We}^2}\right)\right\}\cdot 4\alpha Ay\mathrm{We}(1 - 8\alpha(1 - \alpha))\Big]\,. \end{aligned}\tag{37}$$

To calculate the remaining unknown constant A, we make use of the boundary condition $\psi(h) = F$. This gives

$$\begin{aligned}&\frac{1}{4} h\left[-4 - \left(2Ah^2(2\alpha - 1)\left(1 - 2\alpha\right.\right.\right. \\ &\left.\left.+ \sqrt{1 - 4\alpha^2 A^2 h^2 \mathrm{We}^2}\right)\right)\Big/(\alpha - 1 + \alpha A^2 h^2 \mathrm{We}^2)\end{aligned}\tag{38}$$

$$+\left\{24\alpha(1-\alpha)-4+3(2\alpha-1)\sqrt{1-4\alpha^2A^2h^2\mathrm{We}^2}\right\}\Big/A\alpha\mathrm{We}^2\Bigg]+\frac{1}{8\alpha^2A^2\mathrm{We}^3}\Bigg[(2\alpha-1)(1-24\alpha(1-\alpha))\cdot\sin^{-1}(2Ah\alpha\mathrm{We})+8\alpha^{3/2}\sqrt{(\alpha-1)/\alpha^2}(1-6\alpha(1-\alpha))\cdot\tan^{-1}\left(Ah\mathrm{We}\Big/\sqrt{(\alpha-1)/\alpha}\right)-\tan^{-1}\left(Ah\mathrm{We}(2\alpha-1)\Big/\sqrt{(1-4\alpha^2A^2h^2\mathrm{We}^2)(\alpha-1)/\alpha}\right)\cdot 8\sqrt{\alpha(\alpha-1)}(1-6\alpha(1-\alpha))\Bigg]=F\,,$$

which is a strongly nonlinear algebraic equation. This equation is solved using symbolic software Mathematica 6 and at each cross-section x. Having the value of A, the solution given by (37) is completely known at each cross-section x.

An approximate solution of (36) can be obtained by expanding the radical term in power series using binomial expansion and retaining the first two terms. This yields

$$\frac{\partial^2\psi}{\partial y^2}=\frac{1-\alpha(2\alpha-1)\mathrm{We}^2A^2y^2}{(1-\alpha\mathrm{We}^2A^2y^2)^2}Ay\,. \tag{39}$$

Integration of above equation gives

$$\psi=\frac{1}{2\alpha^{3/2}\mathrm{We}^3A^2}\Bigg[(4-6\alpha)\tanh^{-1}\left(\sqrt{\alpha}\mathrm{We}Ay\right)-\left(\sqrt{\alpha}\mathrm{We}Ay\Big/(\alpha\mathrm{We}^2A^2h^2-1)\right)\Big\{-2+2\alpha-2\alpha\mathrm{We}^2A+2\alpha^2\mathrm{We}^4A^3h^2+\big(-1-2\alpha^2\mathrm{We}^2A^2h^2$$

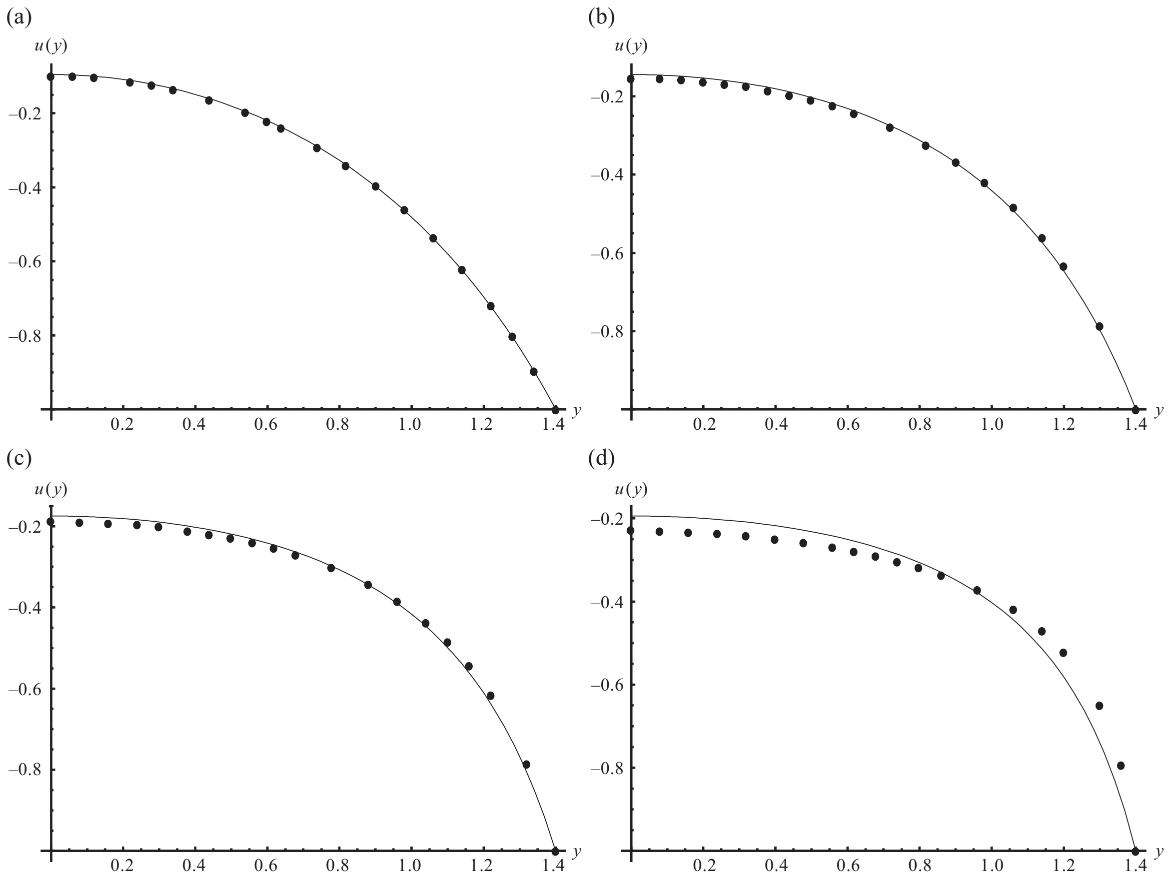

Fig. 2. Plots showing the comparison of approximate (solid line) and exact solution (dots) for $F=-0.5$, $\varphi=0.4$, and $x=0$. (a) $\alpha=0.3$, We $=1$; (b) $\alpha=0.3$, We $=2$; (c) $\alpha=0.3$, We $=3$; (d) $\alpha=0.5$, We $=3$.

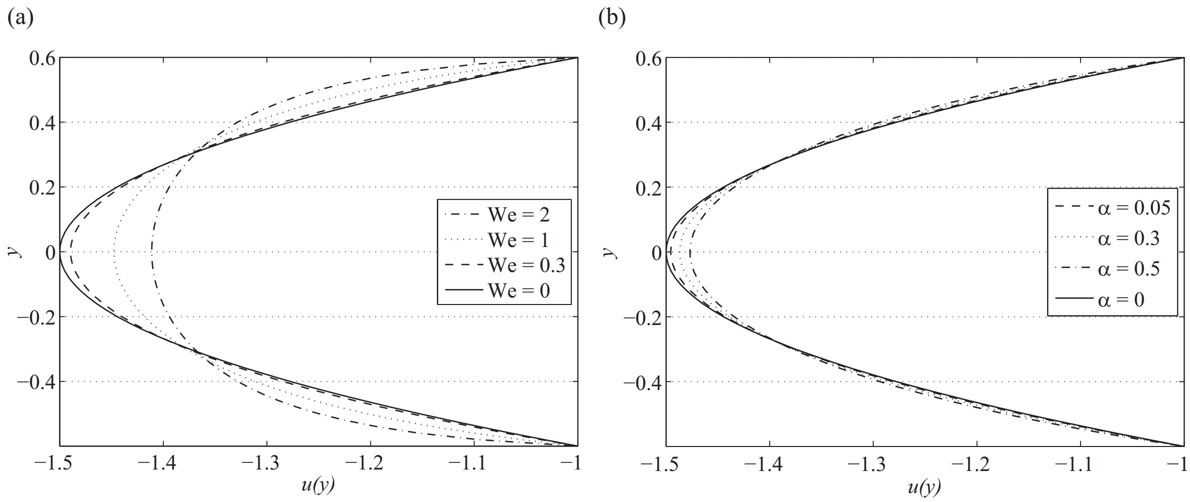

Fig. 3. Plots showing $u(y)$ at cross-section $x=\pi$ for different values of (a) We $(\alpha=0.5)$ and (b) α $(\mathrm{We}=0.5)$. The other parameters are $F=-0.8$ and $\varphi=0.4$.

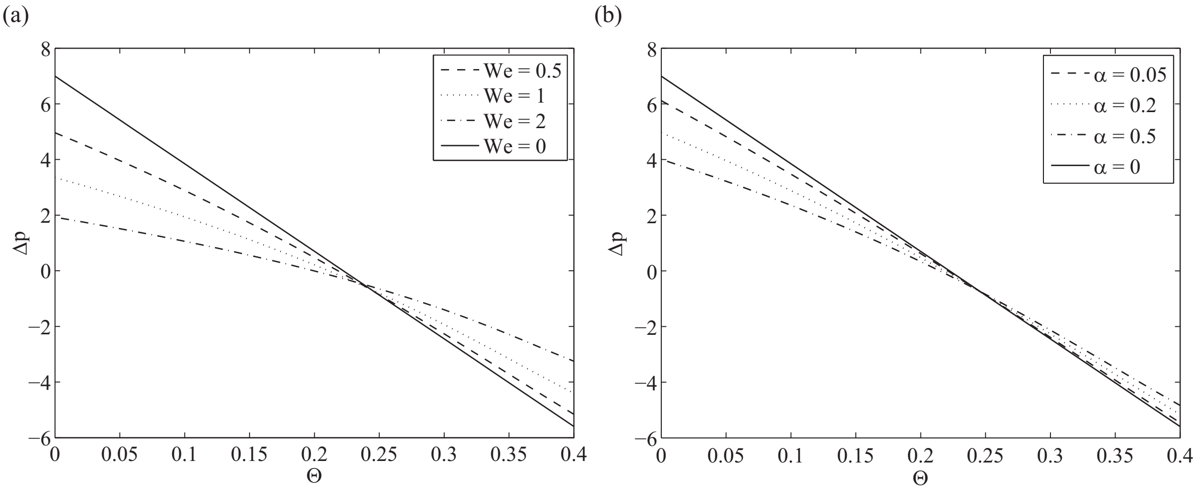

Fig. 4. Plots showing Δp versus Θ for different values of (a) We $(\alpha=0.2)$ and (b) α $(\mathrm{We}=0.5)$ with $\varphi=0.4$.

$$+2\alpha+\alpha\mathrm{We}^2A^2h^2)\ln(1-\alpha\mathrm{We}^2A^2h^2)\Big\} \tag{40}$$

$$-\sqrt{\alpha}(2\alpha-1)\mathrm{We}Ay\Big\{-2+\ln\Big(1-\alpha\mathrm{We}^2A^2y^2\Big)\Big\}\Big],$$

where the unknown constant A at each cross-section can be obtained by solving the following transcendental equation:

$$\frac{1}{2\alpha^{3/2}\mathrm{We}^3A^2}\Big[(4-6\alpha)\tanh^{-1}\left(\sqrt{\alpha}\mathrm{We}Ah\right)$$

$$-\Big(\sqrt{\alpha}\mathrm{We}Ah\Big/\Big(\alpha\mathrm{We}^2A^2h^2-1\Big)\Big)\Big\{-2+2\alpha$$

$$-2\alpha\mathrm{We}^2A+2\alpha^2\mathrm{We}^4A^3h^2+\Big(-1-2\alpha^2\mathrm{We}^2A^2h^2+2\alpha$$

$$+\alpha\mathrm{We}^2A^2h^2\Big)\ln\left(1-\alpha\mathrm{We}^2A^2h^2\right)\Big\}-\sqrt{\alpha}(2\alpha-1)$$

$$\cdot\mathrm{We}Ah\Big\{-2+\ln\left(1-\alpha\mathrm{We}^2A^2h^2\right)\Big\}\Big]=F. \tag{41}$$

It is worth mentioning that in the limiting case when $\alpha\to 0$ or $\mathrm{We}\to 0$ both solutions of (36) reduce to the solution of the corresponding equation for a Newtonian fluid [2]. This also demonstrates the validity of our scheme for finding the solution of (36). In the next section, the validity of the approximate solution is demon-

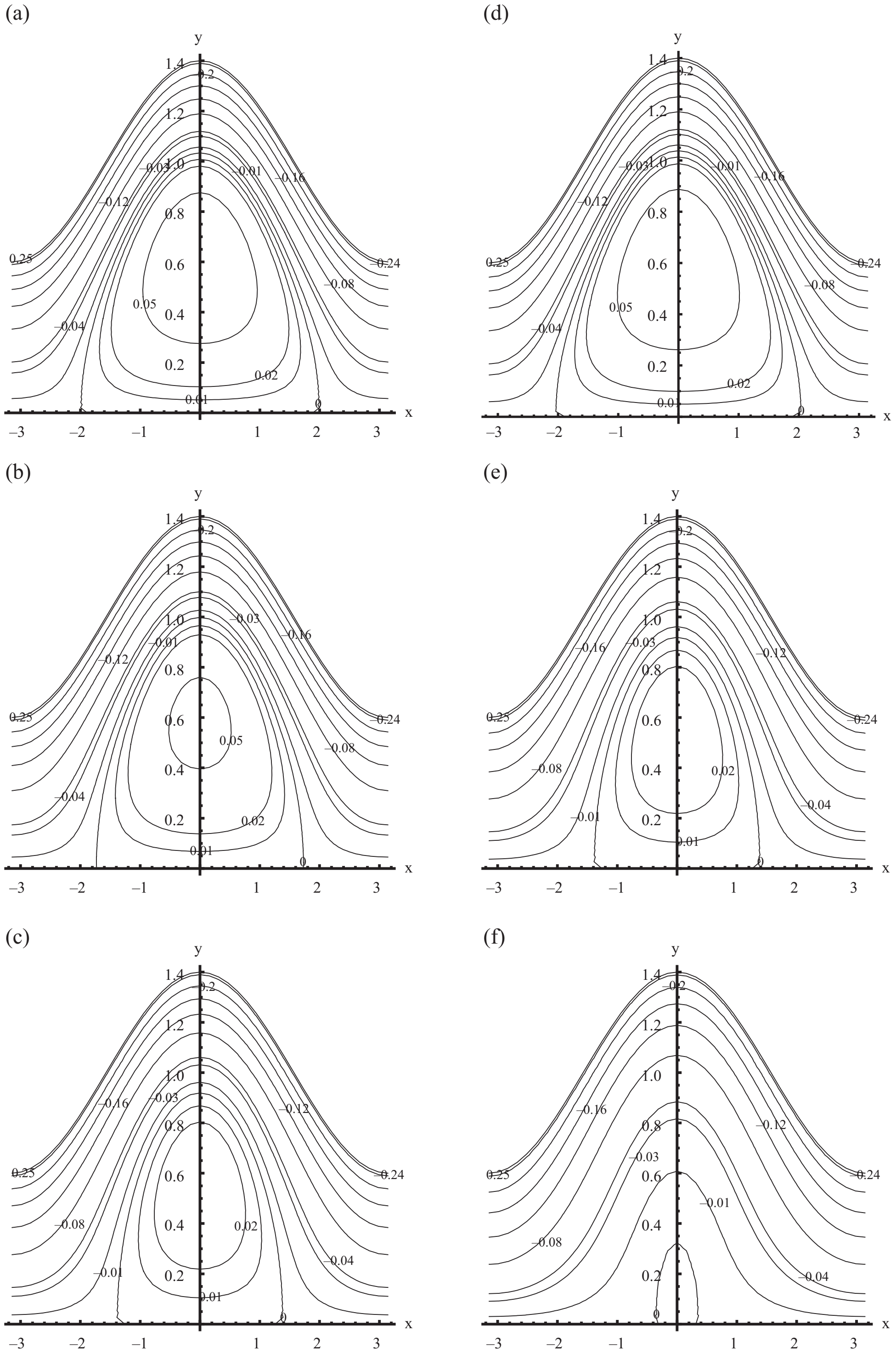

Fig. 5. Streamlines for different values of α $(= 0.05, 0.2, 0.5)$, (left) We $= 1$ and (right) We $(= 0.3, 1, 2)$, $\alpha = 0.5$. The other parameters are $F = -0.25$ and $\varphi = 0.4$.

strated by comparing it with the exact solution. Further, the observations regarding the effects of dimensionless parameters α and We on various features of peristaltic motion such velocity, pressure rise per wavelength, and trapping are also reported using the exact solution in the next section.

5. Results and Discussion

A comparison of exact and approximate solution of (36) is presented through Figure 2. One can observe from panel (a) that the approximate solution is in excellent agreement with the exact solution for $\mathrm{We} = 1$ and $\alpha = 0.3$. However, as evident from panel (b), it deviates slightly from the exact solution by increasing We, i. e. $\mathrm{We} = 2$. The deviation becomes prominent for $\mathrm{We} = 3$ (panel (c)) but still is in acceptable range. Then, for $\alpha = 0.5$ and $\mathrm{We} = 3$, the approximate solution is in total disagreement with the exact solution (panel (d)). Thus it can be concluded that for $0 \le \alpha \le 0.5$, We should be less than 1 for acceptable results. Test computations also confirm this conclusion.

The variation of velocity $u(y)$ at cross-section $x = \pi$ for different values of We and α is shown in Figure 3. The curve for $\alpha = 0$ or $\mathrm{We} = 0$ in each panel of Figures 3 and 4 corresponds to a Newtonian fluid. Figure 3 reveals that We and α has same effects on the velocity profile, i. e. it decreases for their large values. Moreover, one can see that the magnitude of velocity for the Newtonian fluid is greater than for the Giesekus fluid.

One of the important features of peristaltic motion is that it pumps a fluid against the pressure rise per wavelength. To observe this feature, the pressure rise per wavelength Δp is plotted against the flow rate Θ in Figure 4 for different values of We and α. One can see that the maximum pressure p_0, i. e. the value of Δp for $\Theta = 0$, decreases in going from Newtonian to Giesekus fluid. Thus peristalsis has to work against a smaller pressure rise for a Giesekus fluid in comparison to a Newtonian fluid. However, the value of free pumping flux Θ_0, i. e. the value of Θ for $\Delta p = 0$, is greater for a Newtonian fluid as compared to a Giesekus fluid. It is further noted from Figure 4 that an increase in the value of We and α causes a decrease in p_0 and Θ_0. Thus viscoelastic materials with larger relaxation time can be propelled easily by the peristaltic mechanism. Moreover, a rapid decrease in p_0 and Θ_0 is observed by increasing We. On the other side with an increase in α the decrease in p_0 and Θ_0 is slow.

The influence of We and α on the streamlines of the flow is shown in Figure 5. The value of Θ is chosen so that the center streamline $\psi = 0$ splits to enclose a bolus of fluid which moves with the wave. An inspection of panels (a) – (c) reveals that size and circulation of the trapped bolus decrease with an increase in α. Similar observation can be made from panels (d) – (f). But a rapid decrease in size and circulation of the trapped bolus is observed by increasing We, and the bolus disappears for large values of We as evident from panel (f).

6. Concluding Remarks

The flow of a Giesekus fluid in a channel induced by peristaltic waves is analyzed under long wavelength and low Reynolds number assumptions. An exact as well as approximate solution of the governing equation is constructed. Both the solutions are compared, and a range of validity for the approximate solution is provided. Effects of Giesekus fluid parameters We and α on various features of the peristaltic motion are analyzed. In nutshell, the velocity profile, pressure rise per wavelength, and size and circulation of the trapped bolus decrease by increasing We and α. We end up with the remark that the present work may find application in processes where the peristaltic transport of polymeric fluids is involved.

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