Exact Group Invariant Solutions and Conservation Laws of the Complex Modified Korteweg–de Vries Equation

Andrew G. Johnpillai[a], Abdul H. Kara[b], and Anjan Biswas[c,d]

[a] Department of Mathematics, Eastern University, Sri Lanka
[b] School of Mathematics and Centre for Differential Equations, Continuum Mechanics and Applications, University of the Witwatersrand, Wits 2050, Johannesburg, South Africa
[c] Department of Mathematical Sciences, Delaware State University, Dover, DE 19901-2277, USA
[d] Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia

Reprint requests to A. B.; E-mail: biswas.anjan@gmail.com

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We study the scalar complex modified Korteweg–de Vries (cmKdV) equation by analyzing a system of partial differential equations (PDEs) from the Lie symmetry point of view. These systems of PDEs are obtained by decomposing the underlying cmKdV equation into real and imaginary components. We derive the Lie point symmetry generators of the system of PDEs and classify them to get the optimal system of one-dimensional subalgebras of the Lie symmetry algebra of the system of PDEs. These subalgebras are then used to construct a number of symmetry reductions and exact group invariant solutions to the system of PDEs. Finally, using the Lie symmetry approach, a couple of new conservation laws are constructed. Subsequently, respective conserved quantities from their respective conserved densities are computed.

Key words: Complex Modified KdV Equation; Solitons; Lie Symmetries; Optimal System; Symmetry Reduction; Group Invariant Solutions; Conservation Laws.
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1. Introduction

In this paper, we study the exact solutions and conservation laws of the dimensionless form of the complex modified Korteweg–de Vries (cmKdV) equation

$$q_t + f_0|q|^2 q_x + g_0 q_{xxx} = 0\,, \tag{1}$$

where q is the complex valued dependent variable, x,t are the independent variables, and f_0 and g_0 are arbitrary real valued non-zero constants. Equation (1) arises in many areas of physics and mathematics, particularly in nonlinear optics and in the area of plasma physics (see for e. g., [1 – 13]). Let us denote $q(x,t) = u(x,t) + \mathrm{i}v(x,t)$. The transformation

$$\tilde{t} = g_0 t\,,\ \ \tilde{x} = x\,,\ \ \tilde{q} = q \tag{2}$$

maps (1) to

$$\tilde{q}_t + \tilde{a}|\tilde{q}|^2 \tilde{q}_x + \tilde{q}_{xxx} = 0\,, \tag{3}$$

where $\tilde{a} = f_0/g_0$. Therefore, without loss of generality, we can consider the equations of the general form

$$q_t + a|q|^2 q_x + q_{xxx} = 0\,, \tag{4}$$

where a is an arbitrary non-zero constant. By decomposing (4) into real and imaginary parts, we obtain the following system of partial differential equations (PDEs):

$$\begin{aligned} &u_t + a(u^2+v^2)u_x + u_{xxx} = 0\,,\\ &v_t + a(u^2+v^2)v_x + v_{xxx} = 0\,. \end{aligned} \tag{5}$$

Therefore, in the sequel, we will consider in our analysis the system of PDEs (5) as all the results of the system of equations (5) are equivalent to the class of equations (4).

During the past four decades, the Lie symmetry analysis has proved to be a powerful tool for solving nonlinear problems characterized by the differential equations arising in mathematics, physics, and in many other scientific fields of study. For the theory and application of the Lie symmetry methods, see e. g., [14–17].

Our aim in the present work is to obtain symmetry reductions and exact solutions for the system of PDEs (5) using the similarity transformations. These similarity transformations are constructed by utilizing the Lie point symmetry generators admitted by the system of PDEs (5).

The outline of the paper is as follows. In Section 2, we present the Lie point symmetries of the system of PDEs (5), and in Section 3, we construct the optimal system of one-dimensional subalgebras of the Lie symmetry algebra of the system of PDEs (5). Moreover, using the optimal system of subalgebras, symmetry reductions and exact group-invariant solutions of the system of PDEs (5) are obtained. In Section 4, the method of multipliers is used to obtain conserved quantities for the cmKdV equation (1). Finally, in Section 5, concluding remarks are made.

2. Lie Point Symmetries

In this section, we will derive the Lie point symmetries of the system of PDEs (5).

A vector field

$$X = \tau(t,x,u,v)\partial_t + \xi(t,x,u,v)\partial_x + \eta^1(t,x,u,v)\partial_u + \eta^2(t,x,u,v)\partial_v \tag{6}$$

is a generator of point symmetry of (5) if

$$\begin{aligned} X^{[3]}\left[u_t + a(u^2+v^2)u_x + u_{xxx} = 0\right] &= 0\,,\\ X^{[3]}\left[v_t + a(u^2+v^2)v_x + v_{xxx} = 0\right] &= 0 \end{aligned} \tag{7}$$

whenever the system of PDEs (5) is satisfied. Here the operator $X^{[3]}$ is the third prolongation of the operator X defined by

$$X^{[3]} = X + \zeta_1^1\partial_{u_t} + \zeta_2^1\partial_{u_x} + \zeta_1^2\partial_{v_t} + \zeta_2^2\partial_{v_x} + \zeta_{222}^1\partial_{u_{xxx}} + \zeta_{222}^2\partial_{v_{xxx}}$$

and the coefficients ζ_j^i are given by the prolongation formulae

$$\begin{aligned} \zeta_1^1 &= D_t(\eta^1) - u_t D_t(\tau) - u_x D_t(\xi)\,,\\ \zeta_2^1 &= D_x(\eta^1) - u_t D_x(\tau) - u_x D_x(\xi)\,,\\ \zeta_1^2 &= D_t(\eta^2) - v_t D_t(\tau) - v_x D_t(\xi)\,,\\ \zeta_2^2 &= D_x(\eta^2) - v_t D_x(\tau) - v_x D_x(\xi)\,,\\ \zeta_{222}^1 &= D_x(\zeta_{22}^1) - u_{xxt} D_x(\tau) - u_{xxx} D_x(\xi)\,,\\ \zeta_{222}^2 &= D_x(\zeta_{22}^2) - v_{xxt} D_x(\tau) - v_{xxx} D_x(\xi)\,. \end{aligned}$$

Here D_t and D_x are the total derivative operators defined by

$$\begin{aligned} D_t &= \partial_t + u_t\partial_u + v_t\partial_v, \ldots\,,\\ D_x &= \partial_x + u_x\partial_u + v_x\partial_v, \ldots\,. \end{aligned} \tag{8}$$

The coefficient functions τ, ξ, η^1, and η^2 are calculated by solving the determining equation (7). Since τ, ξ, η^1, and η^2 are independent of the derivatives of u and v, the coefficients of like derivatives of u and v in (7) can be equated to yield an over determined system of linear PDEs. Solving the determining equation for the infinitesimal coefficients τ, ξ, η^1, and η^2 in this case is cumbersome, and after the lengthy calculations, we obtain the following Lie point symmetries admitted by the system of PDEs (5):

$$\begin{aligned} &X_1 = \partial_t\,,\ X_2 = \partial_x\,,\ X_3 = -v\partial_u + u\partial_v\,,\\ &X_4 = 3t\partial_t + x\partial_x - u\partial_u - v\partial_v\,. \end{aligned} \tag{9}$$

3. Symmetry Reductions and Exact Group-Invariant Solutions of the System (5)

Here we first construct the optimal system of one-dimensional subalgebras of the Lie algebra admitted by the system of PDEs (5). The classification of the one-dimensional subalgebras are then used to obtain symmetry reductions and exact group invariant solutions for the system of PDEs (5).

The results on the classification of the Lie point symmetries (9) of the system of the PDEs (5) are summarized in Tables 1, 2, and 3. The commutator table of the Lie point symmetries of (5) and the adjoint representations of the symmetry group of (5) on its Lie algebra are given in Table 1 and Table 2, respectively. The Table 1 and Table 2 are used to construct the optimal system of one-dimensional subalgebras for the system of PDEs (5) which is given in Table 3 (for more details of the approach see [16, and the references therein]).

Table 1. Commutator table of the Lie algebra of (5).

	X_1	X_2	X_3	X_4
X_1	0	0	0	$3X_1$
X_2	0	0	0	X_2
X_3	0	0	0	0
X_4	$-3X_1$	$-X_2$	0	0

Table 2. Adjoint table of the Lie algebra of (5).

Ad	X_1	X_2	X_3	X_4
X_1	X_1	X_2	X_3	$X_4-3\varepsilon X_1$
X_2	X_1	X_2	X_3	$X_4-\varepsilon X_2$
X_3	X_1	X_2	X_3	X_4
X_4	$\mathrm{e}^{3\varepsilon}X_1$	$\mathrm{e}^{\varepsilon}X_2$	X_3	X_4

Case 1. In this case, the group-invariant solution corresponding to the symmetry generator $X_4+\lambda X_3$ reduces the system of PDEs (5) to the system of nonlinear third-order ordinary differential equations (ODEs)

$$\begin{aligned} &3A'''+3a(A^2+B^2)A'-\gamma A'-A+\lambda B=0\,,\\ &3B'''+3a(A^2+B^2)B'-\gamma B'-B-\lambda A=0\,. \end{aligned} \quad (10)$$

Here 'prime' denotes differentiation with respect to γ.

Case 2. The group invariant solution arising from $X_1+\varepsilon_1 X_2$ reduces the system of PDEs (5) to the system of nonlinear third-order ordinary differential equations (ODEs)

$$\begin{aligned} &A'''+aA^2A'+aB^2A'-\varepsilon_1 A'=0\,,\\ &B'''+aB^2B'+aA^2B'-\varepsilon_1 B'=0\,. \end{aligned} \quad (11)$$

Here 'prime' denotes differentiation with respect to γ.

The system of the ODEs (11) is highly nonlinear, however, if we set $B=\sqrt{a_0}$, then solving the ODE (11) by setting the constants of integration to zero, we obtain the following solutions for A:

$$A=\sqrt{\frac{6\varepsilon_1}{a}-6a_0}\,\mathrm{sech}\left[\sqrt{\varepsilon_1-aa_0}(x-\varepsilon_1 t)+\delta\right],$$

$$A=\sqrt{\frac{3\varepsilon_1}{a}-3a_0}\,\tanh\left[\sqrt{\frac{aa_0-\varepsilon_1}{2}}(x-\varepsilon_1 t)+\delta\right].$$

Hence we have the following solitary wave group invariant solutions for (4):

$$q=\sqrt{\frac{6\varepsilon_1}{a}-6a_0}\,\mathrm{sech}\left[\sqrt{\varepsilon_1-aa_0}(x-\varepsilon_1 t)+\delta\right]+\mathrm{i}a_0\,,$$

$$q(x,t)=\sqrt{\frac{3\varepsilon_1}{a}-3a_0}\,\tanh\left[\sqrt{\frac{aa_0-\varepsilon_1}{2}}(x-\varepsilon_1 t)+\delta\right]+\mathrm{i}a_0\,.$$

Case 3. The group invariant solution that corresponds to $X_2+\varepsilon_2 X_3$ reduces the system of PDEs (5) to the system of nonlinear first-order ODEs

$$\begin{aligned} &A'+(aA^2+aB^2)B-B=0\,,\\ &B'-(aA^2+aB^2)A+A=0\,. \end{aligned} \quad (12)$$

Here 'prime' means differentiation with respect to γ. The system of ODEs (12) has the particular solutions $A=\mathrm{e}^{\mathrm{i}\gamma}$ and $B=\mathrm{i}\,\mathrm{e}^{\mathrm{i}\gamma}$. Thus we have the following group invariant solutions for (5):

$$\begin{aligned} u(x,t)&=\pm\left[-\sin(t+x)+\cos(t+x)\right],\\ v(x,t)&=\left[\cos(t+x)+\sin(t+x)\right]. \end{aligned}$$

Hence the group invariant solution of (4) is

$$q(x,t)=\pm\left[-\sin(t+x)+\cos(t+x)\right]+\mathrm{i}\left[\cos(t+x)+\sin(t+x)\right].$$

If $\varepsilon_2=0$, then the symmetry generator X_2 gives rise to the trivial constant solutions of the system of PDEs (5), that is, $u(x,t)=u_0$ and $v(x,t)=v_0$.

Case 4. The $X_1+\delta X_3+\varepsilon_3 X_1$-invariant solution reduces the system of PDEs (5) to the system of nonlinear third-order ordinary differential equations (ODEs)

$$\begin{aligned} &A'''+aA^2A'+aB^2A'-\varepsilon_3 A'+B=0\,,\\ &B'''+aB^2B'+aA^2B'-\varepsilon_3 B'-A=0\,. \end{aligned} \quad (13)$$

Here 'prime' denotes differentiation with respect to γ.

Case 5. The symmetry generator X_3 does not give a group invariant solution.

Table 3. Subalgebra, group invariants, group invariant solutions of (5).

N	X	γ	Group-invariant solution
1	$X_4+\lambda X_3$	$xt^{-1/3}$	$u=t^{-1/3}\left[A(\gamma)\cos(\lambda/3\ln t)+B(\gamma)\sin(\lambda/3\ln t)\right]$ $v=t^{-1/3}\left[A(\gamma)\sin(\lambda/3\ln t)-B(\gamma)\cos(\lambda/3\ln t)\right]$
2	$X_1+\varepsilon_1 X_2$	$x-\varepsilon_1 t$	$u=A(\gamma)$, $v=B(\gamma)$
3	$X_2+\varepsilon_2 X_3$	t	$u=\pm\left[-A(\gamma)\sin x+B(\gamma)\cos x\right]$, $v=\left[A(\gamma)\cos x+B(\gamma)\sin x\right]$
4	$X_1+\delta X_3+\varepsilon_2 X_2$	$x-\varepsilon_3 t$	$u=\delta\left[-A(\gamma)\sin t+B(\gamma)\cos t\right]$, $v=\left[A(\gamma)\cos t+B(\gamma)\sin t\right]$
5	X_3	N/A	N/A

Here $\varepsilon_i=0,\pm1$, $i=1,\dots,3$, $\delta=\pm1$, and λ is an arbitrary real constant.

4. Conservation Laws

In this section, the method of multipliers is going to be also used to obtain a few conserved densities of the cmKdV equation (1) in which we use $a = f_0$ and $b = g_0$ for simplicity.

4.1. Method of Multipliers

In order to evaluate conserved quantities, we resort to the invariance and multiplier approach based on the well-known result that the Euler–Lagrange operator annihilates a total divergence. Firstly, if (T^t, T^x) is a conserved vector corresponding to a conservation law, then

$$D_t T^t + D_x T^x = 0$$

along the solutions of the differential equation $(\mathrm{de} = 0)$.

Moreover, if there exists a non-trivial differential function Q, called a ‘multiplier’, such that

$$E_q[Q \cdot (\mathrm{de})] = 0,$$

then

$$Q \cdot (\mathrm{de}) = D_t T^t + D_x T^x,$$

where E_q is the Euler–Lagrange operator for some (conserved) vector (T^t, T^x). Thus, a knowledge of each multiplier Q leads to a conserved vector determined by, inter alia, a homotopy operator. See details and references in [18].

For a system $\mathrm{de1} = 0$, $\mathrm{de2} = 0$, $Q = (f, g)$, say, so that

$$f \cdot (\mathrm{de1}) + g \cdot (de2) = D_t T^t + D_x T^x$$

and

$$E_{(u,v)}\left[D_t T^t + D_x T^x\right] = 0.$$

Here, either T^t or T^x is the *conserved density*.

For the system of PDEs which is derived by decomposing (1) into real and imaginary parts, that is,

$$\begin{aligned} &u_t + a(u^2+v^2)u_x + bu_{xxx} = 0, \\ &v_t + a(u^2+v^2)v_x + bv_{xxx} = 0, \end{aligned} \tag{14}$$

we obtained, inter alia, the higher-order multipliers

(i) $(f,g) = \left(\frac{1}{b}\left(bu_{xx} + \frac{1}{3}au(u^2+v^2)\right),\ \frac{1}{b}\left(-\frac{a}{3}v(u^2+v^2) - bv_{xx}\right)\right)$ and

(ii) $(f,g) = \left(\frac{1}{b}\left(bv_{xxx} + av_x(u^2+v^2)\right),\ \frac{1}{b}\left(-au_x(u^2+v^2) - bu_{xxx}\right)\right)$

which, for the system

$$q_t + a|q|^2 q_x + bq_{xxx} = 0, \tag{15}$$

lead to, respectively,

$$Q_1 = \frac{a}{12}|q|^4 + \frac{b}{4}\left(qq^*_{xx} + q^* q_{xx}\right) \quad \text{and}$$

$$Q_2 = -\frac{\mathrm{i}}{4}\left[\frac{a}{2}|q|^2\left(q^* q_x - qq^*_x\right) + b\left(q^* q_{xxx} - qq^*_{xxx}\right)\right].$$

4.2. Conserved Quantities

In this subsection, the one-soliton solution that was obtained in [19, 20] will be used to compute the conserved quantities. To recall, the one-soliton solution to (15) is given by

$$q(x,t) = A\,\mathrm{sech}\left[B(x - vt)\right] \mathrm{e}^{\mathrm{i}(-\kappa x + \omega t + \theta)}, \tag{16}$$

where the amplitude(A)–width(B) relation is given by

$$B = A\sqrt{\frac{a}{6b}}, \tag{17}$$

and the wave number (ω) is

$$\omega = b\kappa\left(3B^2 - \kappa^2\right), \tag{18}$$

while the relation between the soliton velicity (v) and the soliton frequency (κ) is

$$v = b\left(B^2 - 3\kappa^2\right). \tag{19}$$

Using the one-soliton solution, the conserved quantities, from Q_1 and Q_2 above, are [20]

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} Q_1\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \left\{\frac{a}{12}|q|^4 + \frac{b}{4}\left(qq^*_{xx} + q^* q_{xx}\right)\right\}\mathrm{d}x \\ &= \frac{A^2}{9B}\left\{aA^2 - 3b\left(3\kappa^2 + 5B^2\right)\right\} \end{aligned} \tag{20}$$

and

$$
\begin{aligned}
I_2 &= \int_{-\infty}^{\infty} Q_2\,\mathrm{d}x \\
&= -\frac{\mathrm{i}}{4}\int_{-\infty}^{\infty}\Big\{\frac{a}{2}|q|^2\big(q^* q_x - q q_x^*\big) \\
&\quad + b\big(q^* q_{xxx} - q q_{xxx}^*\big)\Big\}\,\mathrm{d}x \\
&= -\frac{\kappa A^2}{3B}\Big\{aA^2 - 3B\big(\kappa + B^2\big)\Big\}.
\end{aligned} \tag{21}
$$

5. Conclusions

In this paper, we have studied the scalar complex modified Korteweg–de Vries (cmKdV) equation by investigating a system of PDEs using the Lie symmetry group method. The system of PDEs is obtained by decomposing the underlying cmKdV equation into real and imaginary components. We derived the Lie point symmetry generators of the system of PDEs. By classifying these Lie point symmetry generators, we obtained the optimal system of one-dimensional subalgebras of the Lie symmetry algebra of the system of PDEs. We then used these optimal system of subalgebras to construct a number of symmetry reductions and exact group invariant solutions to the system of PDEs. The Lie symmetry method is also employed to extract a couple of conserved densities, and the corresponding conserved quantities are also computed.

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