

Shape-Invariant Approach to Study Relativistic Symmetries of the Dirac Equation with a New Hyperbolical Potential Combination

Akpan N. Ikot^a, Elham Maghsoudi^b, Saber Zarrinkamar^c, and Hassan Hassanabadi^b

^a Theoretical Physics Group, Department of Physics, University of Uyo, Nigeria

^b Department of Physics, Shahrood University of Technology, P.O.Box 3619995161-316, Shahrood, Iran

^c Department of Basic Sciences, Garmsar Branch, Islamic Azad University, Garmsar, Iran

Reprint requests to A. N. I.; E-mail: ndemikotphysics@gmail.com

Z. Naturforsch. **68a**, 499–509 (2013) / DOI: 10.5560/ZNA.2013-0028

Received December 4, 2012 / revised February 22, 2013 / published online May 22, 2013

Spin and pseudospin symmetries of the Dirac equation are investigated for a novel interaction term, i.e. the combination of Tietz plus a hyperbolical (Schiöberg) potential besides a Coulomb tensor interaction. This choice of interaction yields many of our significant terms in its special cases. After applying a proper hyperbolical term, we find the corresponding superpotential and thereby construct the partner Hamiltonians which satisfy the shape-invariant condition via a translational mapping. We report the spectrum of the system and comment on the impact of various terms engaged.

Key words: Dirac Equation; Tietz Potential; Hyperbolical Potential; Supersymmetry Quantum Mechanics; Shape Invariant.

PACS numbers: 03.65Ge; 03.65Pm; 03.65Db

1. Introduction

The special relation of scalar $S(r)$ and vector $V(r)$ interactions in the Dirac theory were shown to have significant physical meaning. More precisely speaking, the cases $V(r) + S(r) = c_{ps} = \text{const.}$ and $V(r) - S(r) = c_s = \text{const.}$, which are respectively called the pseudospin and spin symmetries in the nomenclature, were shown to have notable consequences in nuclear and hadronic spectroscopies [1]. These symmetries have been successfully used to describe and investigate complicated phenomena such as the deformed nuclei, the super-deformation effect, and the properties shell model [2–6]. Authors of [7] give, in a detailed and systematic manner, the ins and outs of the theory.

As the forthcoming equations reveal, these symmetries yield Schrödinger-like equations. Consequently, our common tools in non-relativistic quantum mechanics have been applied to the theory to investigate the equation under physical interaction terms. The list is rather lengthy and includes asymptotic iteration method (AIM) [8], Nikiforov–Uvarov (NU) technique [9], supersymmetric quantum mechanics (SUSYQM) [10] and the shape-invariance (SI)

condition [11], exact quantization rule [12], etc. The pseudospin symmetry usually refers to as a quasi-degeneracy of single nucleon doublets with non-relativistic quantum numbers $(n, l, j = l + \frac{1}{2})$ and $(n - 1, l + 2, j = l + \frac{3}{2})$, where n , l , and j are single nucleon radial, orbital, and total angular quantum numbers, respectively [4]. The total angular momentum is $j = \tilde{l} + \tilde{s}$, where $\tilde{l} = l + 1$ denotes a pseudo-angular momentum, and \tilde{s} is the pseudospin angular momentum [13]. The tensor interaction was originally introduced into the Dirac formalism with the replacement $\vec{p} \rightarrow \vec{p} - iM\omega\beta \cdot \hat{r}U(r)$ in the Dirac Hamiltonian [14]. The Dirac equation with different phenomenological interaction in the symmetry limits has been considerably investigated in recent years [15–33]. Among the lengthy list, the exponential-type potentials such as the Morse, Mie-type, Yukawa, and Tietz potential, due to their successful predictions in physical sciences, have received a bold attention [33–36].

The main aim of the present paper is to obtain approximate analytical solutions of the Dirac equation with a Tietz plus hyperbolical potential (T-H) including the Coulomb potential under the above men-

tioned symmetry limits. The paper is organized as follows. In Section 2, we give a brief introduction to supersymmetry quantum mechanics (SUSYQM) by which we solve our obtained differential equation. In Section 3, the Dirac equation is written for spin and pseudospin symmetries with a Coulomb tensor interaction. We solve the Dirac equation under these symmetries in Section 4, and few special cases are discussed in Section 5. Finally, the conclusion is presented in Section 6.

2. Supersymmetry

We include this short introduction to SUSYQM to proceed on a more continues manner. In SUSYQM, we normally deal with the partner Hamiltonians [10]

$$H_{\pm} = \frac{p^2}{2m} + V_{\pm}(x), \quad (1)$$

where

$$V_{\pm}(x) = \Phi^2(x) \pm \Phi'(x). \quad (2)$$

In the case of good SUSY, i. e. $E_0 = 0$, the ground state of the system is obtained via

$$\varphi_0^-(x) = Ce^{-U(x)}, \quad (3)$$

where C is a normalization constant and

$$U(x) = \int_{x_0}^x dz \Phi(z). \quad (4)$$

Now, if the shape invariant condition

$$V_+(a_0, x) = V_-(a_1, x) + R(a_1) \quad (5)$$

exists (a_1 is a new set of parameters uniquely determined from the old set a_0 via the mapping $F : a_0 \mapsto a_1 = F(a_0)$, and $R(a_1)$ does not include x), then the higher state solutions are obtained via

$$E_n = \sum_{j=1}^n R(a_j), \quad (6a)$$

$$\varphi_n^-(a_0, x) = \prod_{j=0}^{n-1} \left(\frac{A^\dagger(a_j)}{[E_n - E_j]^{1/2}} \right) \varphi_0^-(a_n, x), \quad (6b)$$

$$\varphi_0^-(a_n, x) = C \exp \left\{ - \int_0^x dz \Phi(a_n, z) \right\}, \quad (6c)$$

where

$$A_j^\dagger = -\frac{\partial}{\partial x} + \Phi(a_j, x). \quad (7)$$

Therefore, this condition determines the spectrum of the bound-states of the Hamiltonian

$$H_j = -\frac{\partial^2}{\partial x^2} + V_-(a_j, x) + E_j, \quad (8)$$

and the energy eigenfunctions of

$$H_j \varphi_{n-j}^-(a_j, x) = E_n \varphi_{n-j}^-(a_j, x), \quad n \geq j, \quad (9)$$

are related via [1–3]:

$$\varphi_{n-j}^-(a_j, x) = \frac{A^\dagger}{[E_n - E_j]^{1/2}} \varphi_{n-(j+1)}^-(a_{j+1}, x). \quad (10)$$

3. Dirac Equation with a Tensor Coupling

The Dirac equation for spin- $\frac{1}{2}$ particles moving in an attractive scalar potential $S(r)$, a repulsive vector potential $V(r)$, and a tensor potential $U(r)$ in the relativistic unit ($\hbar = c = 1$) is [25]

$$\begin{aligned} & [\vec{\alpha} \cdot \vec{p} + \beta(M + S(r)) - i\beta \vec{\alpha} \cdot \hat{r} U(r)] \psi(r) \\ & = [E - V(r)] \psi(r), \end{aligned} \quad (11)$$

where E is the relativistic energy of the system, $\vec{p} = -i\vec{\nabla}$ is the three dimensional momentum operator, r and M is the mass of the fermionic particle. $\vec{\alpha}$, β are the 4×4 Dirac matrices given as

$$\vec{\alpha} = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad (12)$$

where I is the 2×2 unitary matrix, and $\vec{\sigma}_i$ are the Pauli three-vector matrices:

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\ \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \quad (13)$$

The eigenvalues of the spin-orbit coupling operator are $\kappa = (j + \frac{1}{2}) \succ 0$, $\kappa = -(j + \frac{1}{2}) \prec 0$ for unaligned $j = l - \frac{1}{2}$ and the aligned spin $j = l + \frac{1}{2}$, respectively. The set (H, K, J^2, J_z) forms a complete set of conserved quantities. Thus, we can write the spinors as [26]

$$\psi_{n\kappa}(r) = \frac{1}{r} \begin{pmatrix} F_{n\kappa}(r) Y_{jm}^l(\theta, \varphi) \\ iG_{n\kappa}(r) Y_{jm}^{\bar{l}}(\theta, \varphi) \end{pmatrix}, \quad (14)$$

where $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ represent the upper and lower components of the Dirac spinors, respectively. $Y_{jm}^l(\theta, \varphi)$, $Y_{jm}^{\tilde{l}}(\theta, \varphi)$ are the spin and pseudospin spherical harmonics, and m is the projection on the z -axis. With other known identities [27]

$$\begin{aligned} (\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B}) &= \vec{A} \cdot \vec{B} + i \vec{\sigma} \cdot (\vec{A} \times \vec{B}), \\ \vec{\sigma} \cdot \vec{p} &= \vec{\sigma} \cdot \hat{r} \left(\hat{r} \cdot \vec{p} + i \frac{\vec{\sigma} \cdot \vec{L}}{r} \right) \end{aligned} \quad (15)$$

as well as

$$\begin{aligned} (\vec{\sigma} \cdot \vec{L}) Y_{jm}^{\tilde{l}}(\theta, \varphi) &= (\kappa - 1) Y_{jm}^l(\theta, \varphi), \\ (\vec{\sigma} \cdot \vec{L}) Y_{jm}^l(\theta, \varphi) &= -(\kappa - 1) Y_{jm}^l(\theta, \varphi), \\ (\vec{\sigma} \cdot \hat{r}) Y_{jm}^l(\theta, \varphi) &= -Y_{jm}^{\tilde{l}}(\theta, \varphi), \\ (\vec{\sigma} \cdot \hat{r}) Y_{jm}^{\tilde{l}}(\theta, \varphi) &= -Y_{jm}^l(\theta, \varphi), \end{aligned} \quad (16)$$

we find the following two coupled first-order Dirac equation [27]:

$$\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r) \right) F_{n\kappa}(r) = (M + E_{n\kappa} - \Delta(r)) \cdot G_{n\kappa}(r), \quad (17)$$

$$\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r) \right) G_{n\kappa}(r) = (M - E_{n\kappa} + \Sigma(r)) \cdot F_{n\kappa}(r), \quad (18)$$

where

$$\Delta(r) = V(r) - S(r), \quad (19)$$

$$\Sigma(r) = V(r) + S(r). \quad (20)$$

Eliminating $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ in (17) and (18), we obtain the second-order Schrödinger-like equation as

$$\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{2\kappa U(r)}{r} - \frac{dU(r)}{dr} - U^2(r) - (M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r)) + \frac{\frac{d\Delta(r)}{dr} \left(\frac{d}{dr} + \frac{\kappa}{r} - U(r) \right)}{(M + E_{n\kappa} - \Delta(r))} \right\} F_{n\kappa}(r) = 0, \quad (21)$$

$$\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} + \frac{2\kappa U(r)}{r} + \frac{dU(r)}{dr} - U^2(r) - (M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r)) + \frac{\frac{d\Sigma(r)}{dr} \left(\frac{d}{dr} - \frac{\kappa}{r} + U(r) \right)}{(M + E_{n\kappa} - \Sigma(r))} \right\} G_{n\kappa}(r) = 0, \quad (22)$$

where $\kappa(\kappa-1) = \tilde{l}(\tilde{l}+1)$, $\kappa(\kappa+1) = l(l+1)$. The radial wave functions are required to satisfy the necessary conditions, i.e. $F_{n\kappa}$ and $G_{n\kappa}$ vanish at the origin and the infinity. At this stage, we take $\Delta(r)$ or $\Sigma(r)$ as the T-H potential. Equations (21) and (22) can be exactly solved for $\kappa = 0, -1$ and $\kappa = 0, 1$, respectively, as the spin-orbit centrifugal term vanishes.

4. Solution of the Dirac Equation

In this section, we are going to solve the Dirac equation with the T-H potential and tensor potential by using the SUSYQM.

4.1. Pseudospin Symmetry Limit

The exact pseudospin symmetry was proved by Meng et al. [28]. It occurs in the Dirac equation when $\frac{d\Sigma(r)}{dr} = 0$ or $\Sigma(r) = C_{ps} = \text{const.}$ [5]. In this limit, we take $\Delta(r)$ as the T-H potentials [27, 36] besides a Coulomb tensor potential [29]:

$$\Delta(r) = V_0 \left(\frac{\sinh \alpha(r - r_0)}{\sinh(\alpha r)} \right)^2 + V_1 (1 - \sigma \coth(\alpha r))^2, \quad (23)$$

$$U(r) = -\frac{H}{r}, \quad H = \frac{z_a z_b e^2}{4\pi \epsilon_0}, \quad r \geq R_e, \quad (24)$$

where V_0 , V_1 , σ , $\alpha = \frac{1}{a}$, and H are constant coefficients, $R_e = 7.78$ fm is the Coulomb radius, z_a and z_b denote the charges of the projectile a and the target nuclei b , respectively [29]. Since the Dirac equation

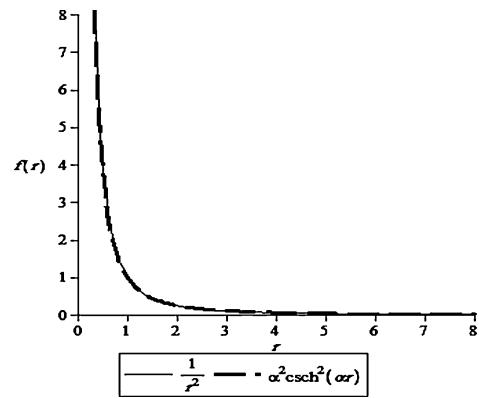


Fig. 1. Centrifugal term ($1/r^2$) and its approximation for $\alpha = 0.01$.

with the T-H potential has no exact solution, we use an approximation for the centrifugal term as shown in Figure 1:

$$\frac{1}{r^2} \approx \frac{\alpha^2}{\sinh^2(\alpha r)} = \alpha^2 \operatorname{csch}^2(\alpha r). \quad (25)$$

Using this term in (22) yields

$$\left\{ -\frac{d^2}{dr^2} + \left[\alpha^2 \Lambda_\kappa (\Lambda_\kappa - 1) - V_0 (M - E_{n\kappa}^{\text{ps}} + C_{\text{ps}}) \right. \right. \\ \cdot \sinh^2(\alpha r_0) - V_1 \sigma^2 (M - E_{n\kappa}^{\text{ps}} + C_{\text{ps}}) \Big] \operatorname{csch}^2(\alpha r) \\ + \left[2V_0 \sinh(\alpha r_0) \cosh(\alpha r_0) (M - E_{n\kappa}^{\text{ps}} + C_{\text{ps}}) \right. \\ \left. \left. + 2V_1 \sigma (M - E_{n\kappa}^{\text{ps}} + C_{\text{ps}}) \right] \coth(\alpha r) \right\} G_{n\kappa}^{\text{ps}}(r) = \quad (26)$$

$$\left\{ -M^2 - MC_{\text{ps}} + (E_{n\kappa}^{\text{ps}})^2 - E_{n\kappa}^{\text{ps}} C_{\text{ps}} + V_0 (M - E_{n\kappa}^{\text{ps}} + C_{\text{ps}}) \right. \\ \left. + C_{\text{ps}} \cosh^2(\alpha r_0) + V_0 (M - E_{n\kappa}^{\text{ps}} + C_{\text{ps}}) \sinh^2(\alpha r_0) \right. \\ \left. + V_1 (M - E_{n\kappa}^{\text{ps}} + C_{\text{ps}}) + V_1 \sigma^2 (M - E_{n\kappa}^{\text{ps}} + C_{\text{ps}}) \right\} G_{n\kappa}^{\text{ps}}(r),$$

where $\kappa = -\tilde{\ell}$ and $\kappa = \tilde{\ell} + 1$ for $\kappa < 0$ and $\kappa > 0$ and $\Lambda_\kappa = \kappa + H$.

4.2. Solution Pseudospin Symmetry Limit

In the previous section, we obtained a Schrödinger-like equation of the form

$$-\frac{d^2 G_{n\kappa}^{\text{ps}}(r)}{dr^2} + V_{\text{eff}}(r) G_{n\kappa}^{\text{ps}}(r) = \tilde{E}_{n\kappa}^{\text{ps}} G_{n\kappa}^{\text{ps}}(r) \quad (27)$$

with the effective potential being

$$V_{\text{eff}} = \tilde{V}_{1\text{ps}} \operatorname{csch}^2(\alpha r) + \tilde{V}_{2\text{ps}} \coth(\alpha r), \quad (28)$$

where

$$\tilde{V}_{1\text{ps}} = \alpha^2 \Lambda_\kappa (\Lambda_\kappa - 1) - MV_1 \sigma^2 + V_1 \sigma^2 E_{n\kappa}^{\text{ps}} \\ - V_1 \sigma^2 C_{\text{ps}} - MV_0 \sinh^2(\alpha r_0) \\ + V_0 E_{n\kappa} \sinh^2(\alpha r_0) - V_0 C_{\text{ps}} \sinh^2(\alpha r_0), \quad (29)$$

$$\tilde{V}_{2\text{ps}} = 2MV_0 \sinh(\alpha r_0) \cosh(\alpha r_0) - 2V_0 E_{n\kappa}^{\text{ps}} \sinh(\alpha r_0) \\ \cdot \cosh(\alpha r_0) + 2V_0 C_{\text{ps}} \sinh(\alpha r_0) \cosh(\alpha r_0) \\ + 2MV_1 \sigma - 2V_1 \sigma E_{n\kappa}^{\text{ps}} + 2V_1 \sigma C_{\text{ps}}.$$

The corresponding effective energy is given by

$$\begin{aligned} \tilde{E}_{n\kappa}^{\text{ps}} = & -M^2 - MC_{\text{ps}} + (E_{n\kappa}^{\text{ps}})^2 - E_{n\kappa}^{\text{ps}} C_{\text{ps}} \\ & + MV_0 \cosh^2(\alpha r_0) - V_0 E_{n\kappa}^{\text{ps}} \cosh^2(\alpha r_0) \\ & + V_0 C_{\text{ps}} \cosh^2(\alpha r_0) + MV_0 \sinh^2(\alpha r_0) \\ & - V_0 E_{n\kappa}^{\text{ps}} \sinh^2(\alpha r_0) + V_0 C_{\text{ps}} \sinh^2(\alpha r_0) + MV_1 \\ & - V_1 E_{n\kappa}^{\text{ps}} + V_1 C_{\text{ps}} + MV_1 \sigma^2 - V_1 \sigma^2 E_{n\kappa}^{\text{ps}} + V_1 \sigma^2 C_{\text{ps}}. \end{aligned} \quad (30)$$

In the SUSYQM formalism, the ground-state wave function for the lower component is given as

$$G_{0\kappa}^{\text{ps}}(r) = \exp \left(- \int \varphi(r) dr \right). \quad (31)$$

Thus, we are dealing with the Riccati equation

$$\varphi^2 - \varphi' = V_{\text{eff}} - \tilde{E}_{0\kappa}^{\text{ps}}, \quad (32)$$

for which we take a superpotential of the form

$$\varphi(r) = C_p + D_p \coth(\alpha r). \quad (33)$$

Therefore, the exact parameters of our study are obtained via

$$\begin{aligned} C_p^2 + D_p^2 + 2C_p D_p \coth(\alpha r) + (D_p^2 + \alpha D_p) \operatorname{csch}^2(\alpha r) \\ = \tilde{V}_{1\text{ps}} \operatorname{csch}^2(\alpha r) + \tilde{V}_{2\text{ps}} \coth(\alpha r) - \tilde{E}_{0,\kappa}^{\text{ps}}. \end{aligned} \quad (34)$$

Solving (34) yields

$$C_p = \frac{\tilde{V}_{2\text{ps}}}{2D_p}, \quad (35)$$

$$D_p = \frac{-\alpha \pm \sqrt{\alpha^2 + 4\tilde{V}_{1\text{ps}}}}{2}, \quad (36)$$

$$\tilde{E}_{0,\kappa}^{\text{ps}} = -\frac{\tilde{V}_{2\text{ps}}^2}{4D_p^2} - D_p^2. \quad (37)$$

After constructing the partner Hamiltonians

$$\begin{aligned} V_{\text{eff}+}(r) = & \varphi^2 + \frac{d\varphi}{dr} \\ = & \tilde{V}_{2\text{ps}} \coth(\alpha r) + (D_p^2 - \alpha D_p) \\ & \cdot \operatorname{csch}^2(\alpha r) + \frac{\tilde{V}_{2\text{ps}}^2}{4D_p^2} + D_p^2, \end{aligned} \quad (38a)$$

$$\begin{aligned} V_{\text{eff}-}(r) = & \varphi^2 - \frac{d\varphi}{dr} \\ = & \tilde{V}_{2\text{ps}} \coth(\alpha r) + (D_p^2 + \alpha D_p) \\ & \cdot \operatorname{csch}^2(\alpha r) + \frac{\tilde{V}_{2\text{ps}}^2}{4D_p^2} + D_p^2, \end{aligned} \quad (38b)$$

where $a_0 = D_p$ and a_i is a function of a_0 , i.e., $a_1 = f(a_0) = a_0 - \alpha$. Consequently, $a_n = f(a_0) = a_0 - n\alpha$. One can see that the shape-invariance holds via a mapping of the form $D_p \rightarrow D_p - \alpha$. Thus, from (5), we have [29, 30]

$$\begin{aligned} R(a_1) &= \left(\frac{\tilde{V}_{2ps}^2}{4a_0^2} + a_0^2 \right) - \left(\frac{\tilde{V}_{2ps}^2}{4a_1^2} + a_1^2 \right), \\ R(a_2) &= \left(\frac{\tilde{V}_{2ps}^2}{4a_1^2} + a_1^2 \right) - \left(\frac{\tilde{V}_{2ps}^2}{4a_2^2} + a_2^2 \right), \\ R(a_3) &= \left(\frac{\tilde{V}_{2ps}^2}{4a_2^2} + a_2^2 \right) - \left(\frac{\tilde{V}_{2ps}^2}{4a_3^2} + a_3^2 \right), \\ &\vdots \end{aligned} \quad (39)$$

$$\begin{aligned} R(a_n) &= \left(\frac{\tilde{V}_{2ps}^2}{4a_{n-1}^2} + a_{n-1}^2 \right) - \left(\frac{\tilde{V}_{2ps}^2}{4a_n^2} + a_n^2 \right), \\ \tilde{E}_{0\kappa}^- &= 0. \end{aligned} \quad (40)$$

Therefore, from (6a), the eigenvalues can be found as

$$\begin{aligned} \tilde{E}_{n\kappa}^{ps-} &= \sum_{k=1}^n R(a_\kappa) \\ &= \left(\frac{\tilde{V}_{2ps}^2}{4a_0^2} + a_0^2 \right) - \left(\frac{\tilde{V}_{2ps}^2}{4a_n^2} + a_n^2 \right), \end{aligned} \quad (41a)$$

$$\tilde{E}_{n\kappa}^{ps} = \tilde{E}_{n\kappa}^{ps-} + \tilde{E}_{0\kappa}^+ = - \left(\frac{\tilde{V}_{2ps}^2}{4a_n^2} + a_n^2 \right). \quad (41b)$$

This completely determines the energy of the pseudospin symmetry limit. With the aid of (29) and (35)–(37), we obtain the energy eigenvalues as

$$\begin{aligned} &-M^2 - MC_{ps} + (E_{n\kappa}^{ps})^2 - E_{n\kappa}^{ps}C_{ps} + MV_0 \cosh^2(\alpha r_0) - V_0 E_{n\kappa}^{ps} \cosh^2(\alpha r_0) + V_0 C_{ps} \cosh^2(\alpha r_0) + MV_0 \sinh^2(\alpha r_0) \\ &- V_0 E_{n\kappa}^{ps} \sinh^2(\alpha r_0) + V_0 C_{ps} \sinh^2(\alpha r_0) + MV_1 - V_1 E_{n\kappa}^{ps} + V_1 C_{ps} + MV_1 \sigma^2 - V_1 \sigma^2 E_{n\kappa}^{ps} + V_1 \sigma^2 C_{ps} \\ &+ \left\{ \frac{1}{4 \left(\frac{-\alpha \pm \sqrt{\alpha^2 + 4 [\alpha^2 \Lambda_\kappa (\Lambda_\kappa - 1) - MV_1 \sigma^2 + V_1 \sigma^2 E_{n\kappa}^{ps} - V_1 \sigma^2 C_{ps} - MV_0 \sinh^2(\alpha r_0) + V_0 E_{n\kappa} \sinh^2(\alpha r_0) - V_0 C_{ps} \sinh^2(\alpha r_0)]}}{2} - n\alpha \right)^2} \right. \\ &\cdot \left[2MV_0 \sinh(\alpha r_0) \cosh(\alpha r_0) - 2V_0 E_{n\kappa}^{ps} \sinh(\alpha r_0) \cosh(\alpha r_0) \right. \\ &+ 2V_0 C_{ps} \sinh(\alpha r_0) \cosh(\alpha r_0) + 2MV_1 \sigma - 2V_1 \sigma E_{n\kappa}^{ps} + 2V_1 \sigma C_{ps} \left. \right]^2 \\ &+ \left. \left(\frac{-\alpha \pm \sqrt{\alpha^2 + 4 [\alpha^2 \Lambda_\kappa (\Lambda_\kappa - 1) - MV_1 \sigma^2 + V_1 \sigma^2 E_{n\kappa}^{ps} - V_1 \sigma^2 C_{ps} - MV_0 \sinh^2(\alpha r_0) + V_0 E_{n\kappa} \sinh^2(\alpha r_0) - V_0 C_{ps} \sinh^2(\alpha r_0)]}}{2} - n\alpha \right)^2 \right\} = 0. \end{aligned} \quad (42)$$

Thus, the lower component of the wave function is

$$\begin{aligned} G_{n\kappa}^{ps}(r) &= N_{n\kappa} \left(\frac{1}{2} - \frac{\coth(\alpha r)}{2} \right) \sqrt{\frac{\tilde{V}_{2ps}}{4\alpha^2} - \frac{\tilde{E}_{n\kappa}^{ps}}{4\alpha^2}} \\ &\cdot \left(\frac{1}{2} + \frac{\coth(\alpha r)}{2} \right) \sqrt{-\frac{\tilde{V}_{2ps}}{4\alpha^2} - \frac{\tilde{E}_{n\kappa}^{ps}}{4\alpha^2}} \\ &\times P_n \left(2\sqrt{\frac{\tilde{V}_{2ps}}{4\alpha^2} - \frac{\tilde{E}_{n\kappa}^{ps}}{4\alpha^2}}, 2\sqrt{-\frac{\tilde{V}_{2ps}}{4\alpha^2} - \frac{\tilde{E}_{n\kappa}^{ps}}{4\alpha^2}} \right) (\coth(\alpha r)), \end{aligned} \quad (43)$$

and the other component can be simply found as

$$F_{n\kappa}^{ps}(r) = \frac{1}{M - E_{n\kappa}^{ps} + C_{ps}} \left(\frac{d}{dr} - \frac{\kappa}{r} + U(r) \right) G_{n\kappa}^{ps}(r), \quad (44)$$

where $N_{n\kappa}$ is the normalization constant.

4.3. Spin Symmetry Limit

In the spin symmetry limit $\frac{d\Delta(r)}{dr} = 0$ or $\Delta(r) = C_s = \text{const.}$ [17, 18]. As the previous section, we consider [27, 36]

$$\Sigma(r) = V_0 \left(\frac{\sinh \alpha(r - r_0)}{\sinh(\alpha r)} \right)^2 + V_1 (1 - \sigma \coth(\alpha r))^2. \quad (45)$$

Substitution of the latter in (9) gives

$$\begin{aligned} & \left\{ -\frac{d^2}{dr^2} + \left[\alpha^2 \eta_\kappa (\eta_\kappa - 1) + V_1 \sigma^2 (M + E_{n\kappa}^s - C_s) \right. \right. \\ & + V_0 (M + E_{n\kappa}^s - C_s) \sinh^2(\alpha r_0) \left. \right] \operatorname{csch}^2(\alpha r) \\ & - \left[2V_0 (M + E_{n\kappa}^s - C_s) \sinh(\alpha r_0) \cosh(\alpha r_0) \right. \\ & \left. + 2\sigma V_1 (M + E_{n\kappa}^s - C_s) \right] \coth(\alpha r) \left. \right\} F_{n\kappa}^s(r) \quad (46) \end{aligned}$$

$$= \left\{ -M^2 + MC_s + (E_{n\kappa}^s)^2 - E_{n\kappa}^s C_s - V_0 (M + E_{n\kappa}^s \right. \\ \left. - C_s) \cosh^2(\alpha r_0) - V_0 (M + E_{n\kappa}^s - C_s) \sinh^2(\alpha r_0) \right. \\ \left. - V_1 (M + E_{n\kappa}^s - C_s) - V_1 \sigma^2 (M + E_{n\kappa}^s - C_s) \right\} F_{n\kappa}^s(r),$$

where $\kappa = \ell$ and $\kappa = -\ell - 1$ for $\kappa < 0$ and $\kappa > 0$, respectively, and $\eta_\kappa = \kappa + H + 1$.

4.4. Solution of the Spin Symmetry Limit

In this case,

$$-\frac{d^2 F_{n\kappa}^s(r)}{dr^2} + V_{\text{eff}}(r) F_{n\kappa}^s(r) = \tilde{E}_{n\kappa}^s F_{n\kappa}^s(r) \quad (47)$$

with

$$V_{\text{eff}}(r) = \tilde{V}_{1s} \operatorname{csch}^2(\alpha r) + \tilde{V}_{2s} \coth(\alpha r), \quad (48)$$

where

$$\begin{aligned} \tilde{V}_{1s} &= \alpha^2 \eta_\kappa (\eta_\kappa - 1) + MV_1 \sigma^2 + V_1 \sigma^2 E_{n\kappa}^s - V_1 \sigma^2 C_s \\ &+ MV_0 \sinh^2(\alpha r_0) + V_0 E_{n\kappa}^s \sinh^2(\alpha r_0) \\ &- V_0 C_s \sinh^2(\alpha r_0), \quad (49) \end{aligned}$$

$$\begin{aligned} \tilde{V}_{2s} &= -2MV_0 \sinh(\alpha r_0) \cosh(\alpha r_0) - 2V_0 E_{n\kappa}^s \sinh(\alpha r_0) \\ &\cdot \cosh(\alpha r_0) + 2V_0 C_s \sinh(\alpha r_0) \cosh(\alpha r_0) \\ &- 2MV_1 \sigma - 2V_1 E_{n\kappa}^s \sigma + 2V_1 \sigma C_s, \end{aligned}$$

and

$$\begin{aligned} \tilde{E}_{n\kappa}^s &= -M^2 + MC_s + (E_{n\kappa}^s)^2 - E_{n\kappa}^s C_s - MV_0 \cosh^2(\alpha r_0) \\ &- V_0 E_{n\kappa}^s \cosh^2(\alpha r_0) + V_0 C_s \cosh^2(\alpha r_0) - MV_0 \\ &\cdot \sinh^2(\alpha r_0) - V_0 E_{n\kappa}^s \sinh^2(\alpha r_0) + V_0 C_s \sinh^2(\alpha r_0) \\ &- MV_1 - V_1 E_{n\kappa}^s + V_1 C_s - MV_1 \sigma^2 - V_1 \sigma^2 E_{n\kappa}^s + V_1 \sigma^2 C_s. \end{aligned} \quad (50)$$

In the SUSYQM theory, the ground-state wave function for the lower component is given as

$$F_{0,\kappa}^s(r) = \exp \left(- \int \varphi(r) dr \right). \quad (51)$$

Thus, we are dealing with the Riccati equation of the form

$$\varphi^2 - \varphi' = V_{\text{eff}} - \tilde{E}_{0,\kappa}^{\text{ps}}, \quad (52)$$

which corresponds to the superpotential

$$\varphi(r) = A_s + B_s \coth(\alpha r). \quad (53)$$

Thus, the exact parameters of our study are obtained from

$$\begin{aligned} A_s^2 + B_s^2 + 2A_s B_s \coth(\alpha r) + (B_s^2 + \alpha B_s) \operatorname{csch}^2(\alpha r) \\ = \tilde{V}_{1s} \operatorname{csch}^2(\alpha r) + \tilde{V}_{2s} \coth(\alpha r) - \tilde{E}_{0,\kappa}^s. \end{aligned} \quad (54)$$

In this case, the coefficients are explicitly given as

$$\tilde{E}_{0,\kappa}^s = -\frac{\tilde{V}_{2s}^2}{4B_s^2} - B_s^2, \quad (55a)$$

$$A_s = \frac{\tilde{V}_{2s}}{2B_s}, \quad (55b)$$

$$B_s = \frac{-\alpha \pm \sqrt{\alpha^2 + 4\tilde{V}_{1s}}}{2}. \quad (55c)$$

We construct the partner Hamiltonians as

$$\begin{aligned} V_{\text{eff}+}(r) &= \varphi^2 + \frac{d\varphi}{dr} \\ &= B_s^2 + \frac{\tilde{V}_{2s}^2}{4B_s^2} + \tilde{V}_{2s} \coth(\alpha r) \\ &\quad + (B_s^2 - \alpha B_s) \operatorname{csch}^2(\alpha r), \end{aligned} \quad (56a)$$

$$\begin{aligned} V_{\text{eff}-}(r) &= \varphi^2 - \frac{d\varphi}{dr} \\ &= B_s^2 + \frac{\tilde{V}_{2s}^2}{4B_s^2} + \tilde{V}_{2s} \coth(\alpha r) \\ &\quad + (B_s^2 + \alpha B_s) \operatorname{csch}^2(\alpha r), \end{aligned} \quad (56b)$$

where $a_0 = B_s$ and a_i is a function of a_0 , i.e. $a_i = f(a_0) = a_0 - \alpha$. Therefore, $a_n = f(a_0) = a_0 - n\alpha$. It

is not difficult to see that the shape invariance holds of the form $B_s \rightarrow B_s - \alpha$. Like the previous symmetry, we find

$$\begin{aligned} R(a_1) &= V_+(a_0, r) - V_-(a_1, r) \\ &= \left(\frac{\tilde{V}_{2s}^2}{4a_0^2} + a_0^2 \right) - \left(\frac{\tilde{V}_{2s}^2}{4a_1^2} + a_1^2 \right), \\ R(a_2) &= \left(\frac{\tilde{V}_{2s}^2}{4a_1^2} + a_1^2 \right) - \left(\frac{\tilde{V}_{2s}^2}{4a_2^2} + a_2^2 \right), \\ &\vdots \\ R(a_n) &= \left(\frac{\tilde{V}_{2s}^2}{4a_{n-1}^2} + a_{n-1}^2 \right) - \left(\frac{\tilde{V}_{2s}^2}{4a_n^2} + a_n^2 \right). \end{aligned} \quad (57)$$

From the shape-invariance condition, we can determine the energy spectra of the $V_{\text{eff}-}(r)$ potential by using

$$\tilde{E}_{n\kappa}^s = \tilde{E}_{n\kappa}^{s-} + \tilde{E}_{0,\kappa}^s, \quad (58)$$

$$\tilde{E}_{0,\kappa}^s = 0, \quad (59)$$

$$\tilde{E}_{n\kappa}^{s-} = \sum_{i=1}^n R(a_i). \quad (60)$$

By virtue of (60), we find

$$\tilde{E}_{n\kappa}^{s-} = - \left(\frac{\tilde{V}_{2s}^2}{4a_n^2} + a_n^2 \right). \quad (61)$$

By considering the same way of solving (46), the energy equation for the T-H potential in the presence of a tensor interaction is obtained as follows:

$$\begin{aligned} &-M^2 + MC_s + (E_{n\kappa}^s)^2 - E_{n\kappa}^s C_s - MV_0 \cosh^2(\alpha r_0) - V_0 E_{n\kappa}^s \cosh^2(\alpha r_0) + V_0 C_s \cosh^2(\alpha r_0) - MV_0 \sinh^2(\alpha r_0) \\ &- V_0 E_{n\kappa}^s \sinh^2(\alpha r_0) + V_0 C_s \sinh^2(\alpha r_0) - MV_1 - V_1 E_{n\kappa}^s + V_1 C_s - MV_1 \sigma^2 - V_1 \sigma^2 E_{n\kappa}^s \\ &+ V_1 \sigma^2 C_s \left\{ \frac{1}{4 \left(\frac{-\alpha \pm \sqrt{\alpha^2 + 4 \left[\alpha^2 \eta_\kappa (\eta_\kappa - 1) + MV_1 \sigma^2 + V_1 \sigma^2 E_{n\kappa}^s - V_1 \sigma^2 C_s + MV_0 \sinh^2(\alpha r_0) + V_0 E_{n\kappa}^s \sinh^2(\alpha r_0) - V_0 C_s \sinh^2(\alpha r_0) \right]}}{2} - n\alpha \right)^2} \right. \\ &\cdot \left[-2MV_0 \sinh(\alpha r_0) \cosh(\alpha r_0) - 2V_0 E_{n\kappa}^s \sinh(\alpha r_0) \cosh(\alpha r_0) \right. \\ &+ 2V_0 C_s \sinh(\alpha r_0) \cosh(\alpha r_0) - 2MV_1 \sigma - 2V_1 E_{n\kappa}^s \sigma + 2V_1 \sigma C_s \left. \right]^2 \\ &+ \left. \left(\frac{-\alpha \pm \sqrt{\alpha^2 + 4 \left[\alpha^2 \eta_\kappa (\eta_\kappa - 1) + MV_1 \sigma^2 + V_1 \sigma^2 E_{n\kappa}^s - V_1 \sigma^2 C_s + MV_0 \sinh^2(\alpha r_0) + V_0 E_{n\kappa}^s \sinh^2(\alpha r_0) - V_0 C_s \sinh^2(\alpha r_0) \right]}}{2} - n\alpha \right)^2 \right\} = 0 \end{aligned} \quad (62)$$

Therefore, the upper component is obtained as

$$\begin{aligned} F_{n\kappa}^s(r) &= D_{n\kappa} \left(\frac{1}{2} - \frac{\coth(\alpha r)}{2} \right) \sqrt{\frac{\tilde{V}_{2s}}{4\alpha^2} - \frac{\tilde{E}_{n\kappa}^s}{4\alpha^2}} \\ &\cdot \left(\frac{1}{2} + \frac{\coth(\alpha r)}{2} \right) \sqrt{-\frac{\tilde{V}_{2s}}{4\alpha^2} - \frac{\tilde{E}_{n\kappa}^s}{4\alpha^2}} \\ &\times P_n \left(2\sqrt{\frac{\tilde{V}_{2s}}{4\alpha^2} - \frac{\tilde{E}_{n\kappa}^s}{4\alpha^2}}, 2\sqrt{-\frac{\tilde{V}_{2s}}{4\alpha^2} - \frac{\tilde{E}_{n\kappa}^s}{4\alpha^2}} \right) (\coth(\alpha r)), \end{aligned} \quad (63)$$

and the other component can be simply found from

$$G_{n\kappa}^s(r) = \frac{1}{M + E_{n\kappa}^s - C_s} \left(\frac{d}{dr} + \frac{\kappa}{r} - U(r) \right) F_{n\kappa}^s(r), \quad (64)$$

where $D_{n\kappa}$ is the normalization constant.

5. Few Special Cases

In this section, we will consider two special cases of the T-H potentials as follows. The relativistic symmetries of the Dirac equation with Tietz potential were studied in [36]. If we set $V_1 = 0$, the T-H potential reduces to the Tietz potential and we recover the result

Table 1. Energies in the pseudospin symmetry limit for $\alpha = 0.01$, $M = 1 \text{ fm}^{-1}$, $V_0 = -1$, $V_1 = -0.8$, $C_{\text{ps}} = -3$, $\sigma = 0.1$, and $r_0 = 0.2$.

ℓ	$n, \kappa < 0$	(ℓ, j)	$E_{n\kappa}^{\text{ps}}(\text{fm}^{-1})$ ($H = 0$)	$E_{n\kappa}^{\text{ps}}(\text{fm}^{-1})$ ($H = 0.5$)	$n - 1, \kappa > 0$	$(\ell + 2, j + 1)$	$E_{n\kappa}^{\text{ps}}(\text{fm}^{-1})$ ($H = 0$)	$E_{n\kappa}^{\text{ps}}(\text{fm}^{-1})$ ($H = 0.5$)
1	1, -1	$1S_{\frac{1}{2}}$	-2.323832627	-2.311939149	0, 2	$0d_{\frac{3}{2}}$	-2.323832627	-2.338610527
2	1, -2	$1P_{\frac{3}{2}}$	-2.355189421	-2.338610527	0, 3	$0f_{\frac{5}{2}}$	-2.355189421	-2.372718826
3	1, -3	$1d_{\frac{5}{2}}$	-2.390581334	-2.372718826	0, 4	$0g_{\frac{7}{2}}$	-2.390581334	-2.408347992
4	1, -4	$1f_{\frac{7}{2}}$	-2.425729246	-2.408347992	0, 5	$0h_{\frac{9}{2}}$	-2.425729246	-2.44253477
1	2, -1	$2S_{\frac{1}{2}}$	-2.403911016	-2.396525934	1, 2	$1d_{\frac{3}{2}}$	-2.403911016	-2.413436227
2	2, -2	$2P_{\frac{3}{2}}$	-2.424525886	-2.413436227	1, 3	$1f_{\frac{5}{2}}$	-2.424525886	-2.436651812
3	2, -3	$2d_{\frac{5}{2}}$	-2.449370879	-2.436651812	1, 4	$1g_{\frac{7}{2}}$	-2.449370879	-2.462332558
4	2, -4	$2f_{\frac{7}{2}}$	-2.475270629	-2.462332558	1, 5	$1h_{\frac{9}{2}}$	-2.475270629	-2.487989087
1	3, -1	$3S_{\frac{1}{2}}$	-2.461587101	-2.456499112	2, 2	$2d_{\frac{3}{2}}$	-2.461587101	-2.468263772
2	3, -2	$3P_{\frac{3}{2}}$	-2.476184142	-2.468263772	2, 3	$2f_{\frac{5}{2}}$	-2.476184142	-2.485006403
3	3, -3	$3d_{\frac{5}{2}}$	-2.49442071	-2.485006403	2, 4	$2g_{\frac{7}{2}}$	-2.49442071	-2.504163106
4	3, -4	$3f_{\frac{7}{2}}$	-2.514018629	-2.504163106	2, 5	$2h_{\frac{9}{2}}$	-2.514018629	-2.523818158

Table 2. Energies in the spin symmetry limit for $\alpha = 0.01$, $M = 1 \text{ fm}^{-1}$, $V_0 = 1$, $V_1 = 0.8$, $C_s = -3$, $\sigma = 0.1$, and $r_0 = 0.2$.

ℓ	$n, \kappa < 0$	(ℓ, j)	$E_{n\kappa}^s(\text{fm}^{-1})$ ($H = 0$)	$E_{n\kappa}^s(\text{fm}^{-1})$ ($H = 0.5$)	$n - 1, \kappa > 0$	$(\ell + 2, j + 1)$	$E_{n\kappa}^s(\text{fm}^{-1})$ ($H = 0$)	$E_{n\kappa}^s(\text{fm}^{-1})$ ($H = 0.5$)
1	0, -2	$0P_{\frac{3}{2}}$	2.00023829	1.998236113	0, 1	$0P_{\frac{1}{2}}$	2.00023829	2.003021964
2	0, -3	$0d_{\frac{5}{2}}$	2.006568204	2.003021964	0, 2	$0d_{\frac{3}{2}}$	2.006568204	2.010853282
3	0, -4	$0f_{\frac{7}{2}}$	2.015849098	2.010853282	0, 3	$0f_{\frac{5}{2}}$	2.015849098	2.021523673
4	0, -5	$0g_{\frac{9}{2}}$	2.027841679	2.021523673	0, 4	$0g_{\frac{7}{2}}$	2.027841679	2.034765004
1	1, -2	$1P_{\frac{3}{2}}$	2.065615749	2.063887412	1, 1	$1P_{\frac{1}{2}}$	2.065615749	2.068019667
2	1, -3	$1d_{\frac{5}{2}}$	2.071083749	2.068019667	1, 2	$1d_{\frac{3}{2}}$	2.071083749	2.074788636
3	1, -4	$1f_{\frac{7}{2}}$	2.079111358	2.074788636	1, 3	$1f_{\frac{5}{2}}$	2.079111358	2.084025711
4	1, -5	$1g_{\frac{9}{2}}$	2.089502665	2.084025711	1, 4	$1g_{\frac{7}{2}}$	2.089502665	2.095510804
1	2, -2	$2P_{\frac{3}{2}}$	2.122599868	2.121094528	2, 1	$2P_{\frac{1}{2}}$	2.122599868	2.124694325
2	2, -3	$2d_{\frac{5}{2}}$	2.12736514	2.124694325	2, 2	$2d_{\frac{3}{2}}$	2.12736514	2.130596274
3	2, -4	$2f_{\frac{7}{2}}$	2.134368662	2.130596274	2, 3	$2f_{\frac{5}{2}}$	2.134368662	2.13866051
4	2, -5	$2g_{\frac{9}{2}}$	2.14344762	2.13866051	2, 4	$2g_{\frac{7}{2}}$	2.14344762	2.148703739

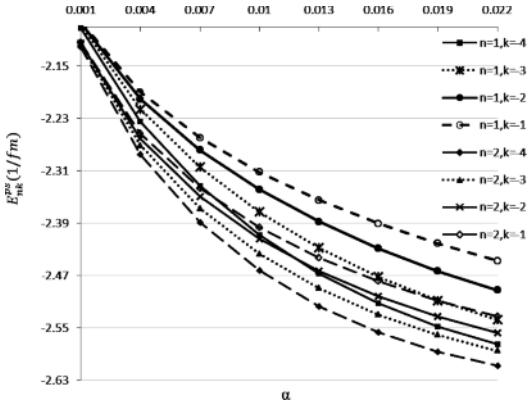


Fig. 2. Energy vs. α for pseudospin symmetry limit for $H = 0.5$, $M = 1 \text{ fm}^{-1}$, $V_0 = -1$, $V_1 = -0.8$, $C_{\text{ps}} = -3$, $\sigma = 0.1$, and $r_0 = 0.2$.

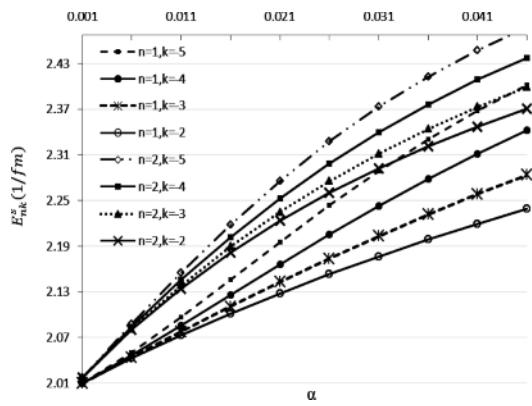


Fig. 3. Energy vs. α for spin symmetry limit for $H = 0.5$, $M = 1 \text{ fm}^{-1}$, $V_0 = 1$, $V_1 = 0.8$, $C_s = -3$, $\sigma = 0.1$, and $r_0 = 0.2$.

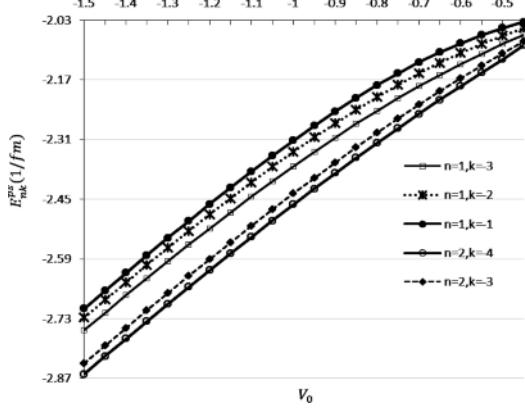


Fig. 4. Energy vs. V_0 for pseudospin symmetry limit for $\alpha = 0.01$, $M = 1 \text{ fm}^{-1}$, $H = 0.5$, $V_1 = -0.8$, $C_{\text{ps}} = -3$, $\sigma = 0.1$, and $r_0 = 0.2$.

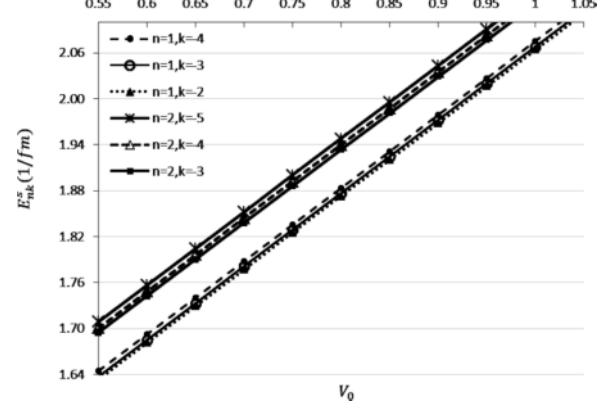


Fig. 5. Energy vs. V_0 for spin symmetry limit for $\alpha = 0.01$, $M = 1 \text{ fm}^{-1}$, $H = 0.5$, $V_1 = 0.8$, $C_s = -3$, $\sigma = 0.1$, and $r_0 = 0.2$.

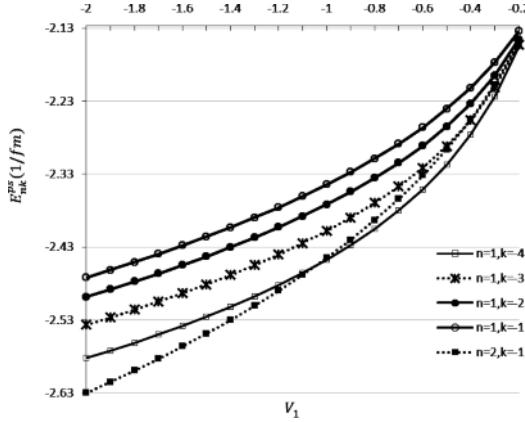


Fig. 6. Energy vs. V_1 for pseudospin symmetry limit for $\alpha = 0.01$, $M = 1 \text{ fm}^{-1}$, $V_0 = -1$, $H = 0.5$, $C_{\text{ps}} = -3$, $\sigma = 0.1$, and $r_0 = 0.2$.

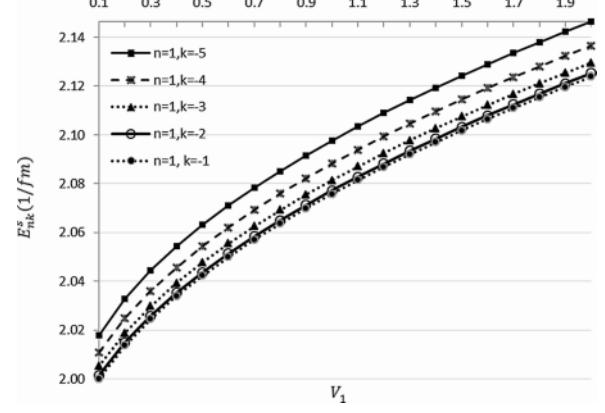


Fig. 7. Energy vs. V_1 for spin symmetry limit for $\alpha = 0.01$, $M = 1 \text{ fm}^{-1}$, $V_0 = 1$, $H = 0.5$, $C_s = -3$, $\sigma = 0.1$, and $r_0 = 0.2$.

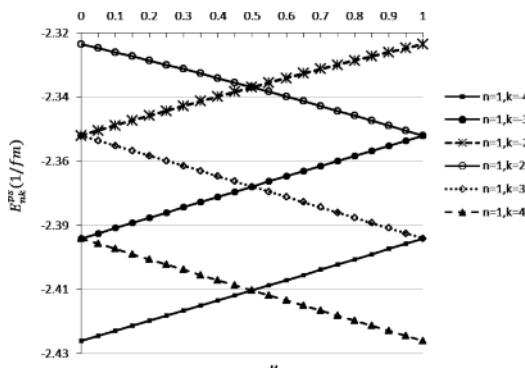


Fig. 8. PSS: Energy vs. H for pseudospin symmetry limit for $\alpha = 0.01$, $M = 1 \text{ fm}^{-1}$, $V_0 = -1$, $V_1 = -0.8$, $C_{\text{ps}} = -3$, $\sigma = 0.1$, and $r_0 = 0.2$.

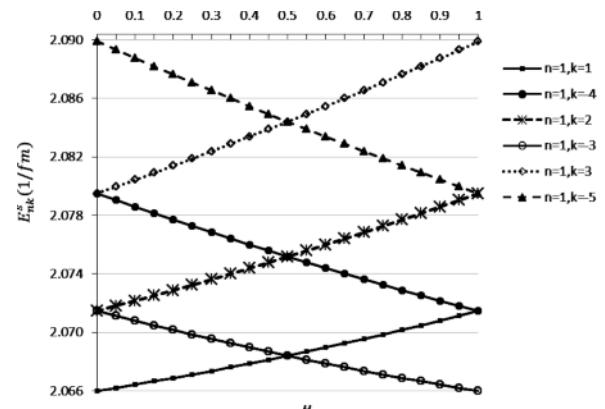


Fig. 9. Energy vs. H for spin symmetry limit for first choice $\alpha = 0.01$, $M = 1 \text{ fm}^{-1}$, $V_0 = 1$, $V_1 = 0.8$, $C_s = -3$, $\sigma = 0.1$, and $r_0 = 0.2$.

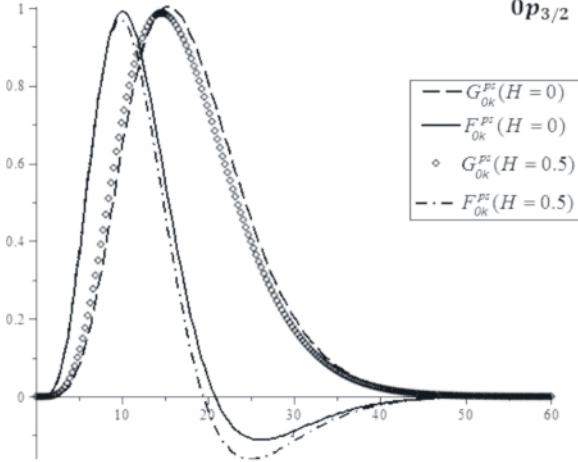


Fig. 10. Lower and upper radial wave functions in view of the pseudospin symmetry for $\alpha = 0.01$, $M = 1 \text{ fm}^{-1}$, $V_0 = -1$, $V_1 = -0.8$, $C_{\text{ps}} = -3$, $\sigma = 0.1$, and $r_0 = 0.2$.

of [36]. Similarly, if we set $V_0 = 0$, the T-H potential reduces to the hyperbolical potential reported in [27]. We depict in Figures 2–9 the energy vs. α , V_0 , V_1 , and H for pseudospin and spin symmetry limits, respectively. Also, we have shown in Figures 10 and 11 the wave function for spin and pseudospin symmetry limits with and without a tensor interaction, respectively. Finally, we have portrayed the energy values for various values of H for pseudospin and spin symmetry limits in Tables 1–2.

- [1] J. N. Ginocchio, Phys. Rep. **414**, 165 (2005).
- [2] M. Hamzavi, A. A. Rajabi, and H. Hassanabadi, Few-Body Syst. **52**, 19 (2012).
- [3] D. Troltenier, C. Bahri, and J. P. Draayer, Nucl. Phys. A **586**, 53 (1995).
- [4] J. N. Ginocchio, Phys. Rev. Lett. **78**, 436 (1997).
- [5] J. N. Ginocchio, A. Leviatan, J. Meng, and S. G. Zhou, Phys. Rev. C **69**, 34303 (2004).
- [6] J. N. Ginocchio and A. Leviatan, Phys. Lett. B **425**, 1 (1998).
- [7] H. Hassanabadi, E. Maghsoudi, and S. Zarrinkamar, Euro. Phys. J. Plus **127**, 31 (2012).
- [8] H. Hamzavi, A. A. Rajabi, and H. Hassanabadi, Phys. Lett. A **374**, 4303 (2010).
- [9] A. F. Nikforov and V. B. Uvarov, Special Functions of Mathematical Physics, Birkhauser, Basel 1988.
- [10] E. Maghsoudi, H. Hassanabadi, S. Zarrinkamar, and H. Rahimov, Phys. Scr. **85**, 55007 (2012).
- [11] M. R. Setare and Z. Nazari, Acta Phys. Pol. B **40**, 2809 (2009).
- [12] W. C. Qiang, Y. Gao and R. S. Zhou, Cent. Euro. J. Phys. **6**, 356 (2008).
- [13] S. M. Ikhdair and R. Sever, Appl. Math. Comput. **216**, 911 (2010).
- [14] M. Moshinsky and A. Szczepanika, J. Phys. A: Math. Gen. **22**, 1817 (1989).
- [15] H. Hassanabadi, E. Maghsoudi, S. Zarrinkamar, and H. Rahimov, J. Math. Phys. **53**, 22104 (2012).
- [16] S. M. Ikhdair and M. Hamzavi, Few-Body Syst. **53**, 487 (2012).
- [17] H. Akcay, Phys. Lett. A **373**, 616 (2009).
- [18] H. Akcay, J. Phys. A: Math. Theor. **40**, 6427 (2007).
- [19] O. Aydogdu and R. Sever, Few-Body Syst. **47**, 193 (2010).
- [20] H. Hassanabadi, S. Zarrinkamar, and M. Hamzavi, Few-Body Syst. **37**, 209 (2012).
- [21] M. Hamzavi and S. M. Ikhdair, arxiv: quant-ph/1207.0681.
- [22] H. Hassanabadi, S. Zarrinkamar, and A. A. Rajabi, Commun. Theor. Phys. **55**, 541 (2011).

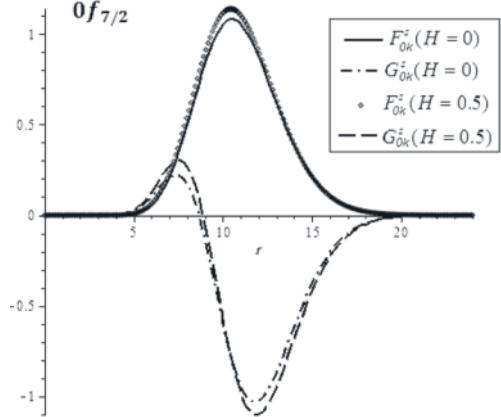


Fig. 11. Lower and upper radial wave functions in view of the spin symmetry for $\alpha = 0.01$, $M = 1 \text{ fm}^{-1}$, $V_0 = 1$, $V_1 = 0.8$, $C_s = -3$, $\sigma = 0.1$, and $r_0 = 0.2$.

6. Conclusions

In this paper, we have obtained the approximate analytical solutions of the Dirac equation for the T-H potential including a tensor Coulomb interaction term within the framework of pseudospin and spin symmetry limits using the SUSYQM. Thereby, we reported the arbitrary-state solutions for the energy eigenvalues and the components of the wave function in terms of the Jacobi polynomials. As we expected, the results are in exact agreement with the special cases previously published in the literature.

- [23] C. Tezcan and R. Sever, *Int. J. Theor. Phys.* **48**, 337 (2009).
- [24] B. H. Yazarloo, H. Hassanabadi, and S. Zarrinkamar, *Euro. Phys. J. Plus* **127**, 51 (2012).
- [25] R. Lisboa, M. Malheiro, A. S. Castro, P. Alberto, and M. Fiolhais, *Phys. Rev. C* **69**, 24319 (2004).
- [26] P. Alberto, R. Lisboa, M. Malheiro, and A. S. Castro, *Phys. Rev. C* **71**, 34313 (2005).
- [27] H. Hassanabadi, E. Maghsoodi, S. Zarrinkamar, and H. Rahimov, *Mod. Phys. Lett. A* **26**, 2703 (2011).
- [28] J. Meng, K. Sugawara-Tanabe, S. Yamaji, P. Ring, and A. Arima, *Phys. Rev. C* **58**, 628 (1998).
- [29] A. N. Ikot, *Few-Body Syst.* **53**, 549 (2012).
- [30] R. L. Greene and C. Aldrich, *Phys. Rev. A* **14**, 2363 (1976).
- [31] E. Maghsoodi, H. Hassanabadi, and S. Zarrinkamar, *Few-Body Syst.* **53**, 525 (2012).
- [32] S. M. Ikhdair, *J. Math. Phys.* **52**, 52303 (2011).
- [33] H. Hamzavi, M. Eshghi, and S. M. Ikhdair, *J. Math. Phys.* **53**, 082101 (2012).
- [34] H. Hamzavi, A. A. Rajabi, and H. Hassanabadi, *Few-Body Syst.* **52**, 19 (2012).
- [35] E. Maghsoodi, H. Hassanabadi, and O. Aydogdu, *Phys. Scr.* **86**, 15005 (2012).
- [36] H. Hassanabadi, E. Maghsoodi, and S. Zarrinkamar, *Eur. Phys. J. Plus* **127**, 31 (2012).