Analytical Solutions of Unsteady Blood Flow of Jeffery Fluid Through Stenosed Arteries with Permeable Walls

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Z. Naturforsch. **68a,** 489 – 498 (2013) / DOI: 10.5560/ZNA.2013-0032
Received November 12, 2012 / revised February 27, 2013 / published online May 22, 2013

This problem deals with the theoretical study of unsteady blood flow of a Jeffery fluid. Two types of arteries, namely (i) composite stenosed artery and (ii) anistropically tapered stenosed artery with permeable walls are considered. The highly nonlinear momentum equations of the Jeffery fluid model are simplified by considering the case of mild stenosis, and finally the exact solutions are found. The expressions for velocity, flow impedance, pressure rise, and stream function are computed and discussed through graphs for different physical quantities of interest.

Key words: Unsteady Flow; Jeffery Fluid; Blood Flow; Permeable Walls; Stenosed Arteries; Exact Solutions.

1. Introduction

A stenosis is an abnormal narrowing in a blood vessel or other tabular organ or structure. It is also some times called a stricture. Most of the times these stenosis cause death when the degree of narrowing becomes significant enough to impede the flow of blood. Due to stenosis in the human artery, the flow of blood is disturbed and resistance to flow becomes higher than that of normal one. The main cause of formation of such stenosis is not yet known clearly but their consequences can be recognized easily. The important contribution of recent years to the topic are referred in [1 – 6]. Many researchers in the field of arteriosclerotic development indicate that the studies are mainly concerned with single symmetric and non-symmetric stenosis. Some stenosis may develop in series or may be of irregular shapes or are overlapping or of composite in nature. Ismail et al. [7] studied the power law model of blood flow through an overlapping stenosed artery where an improved shape of the time variant stenosed in the tapered arterial lumen is given.

In a number of papers, Mekheimer and El Kot [8 – 11] have discussed the different aspects of blood flow analysis in stenosed arteries. The blood flow analysis for different non-Newtonian fluid models have been examined by Akbar and Nadeem [12 – 16]. Very recently, Mishra and Siddiqui [17] have studied the blood flow through a composite stenosis in an artery with permeable wall. Sinha and Misra discussed the influence of slip velocity on the blood flow through an artery with permeable wall [18]. However, the blood flow of a Jeffery fluid through stenosed arteries with permeable walls is not explored sofar.

The aim of the present paper is to see the effects of permeable walls along with slip on the blood flow of the Jeffery fluid model through stenosed arteries for two types of arterial shapes called composite stenosed artery and anistropically tapered stenosed artery. The flow in the permeable boundary is described by the Darcy law which states that the rate at which fluids flow through a permeable substance per unit area is equal to the permeability times the pressure drop per unit length of flow, divided by the viscosity of the flow [17]. Because of the permeability at the wall, slip effects are also taken into account. The governing equations of a Jeffery fluid are presented. The highly nonlinear partial differential equations are simplified by considering the observations of mild stenosis. The exact solutions are carried out subject to the boundary conditions of blood flow for two types of geometries with a permeable wall. The physical features of the major parameters have been discussed through the graphs.

2. Geometry of the Problem

2.1. Composite Stenosed Artery

The geometry of the composite stenosis (see Fig. 1) is assumed to be manifested in the arterial segment and is described as

$$R(z)/R_0 = \begin{cases} 1-\dfrac{2\delta}{R_0L_0}(z-d)\,, \ \ d<z\le d+L_0/2\,, \\ 1-\dfrac{\delta}{2R_0}\left(1+\cos\dfrac{2\pi}{L_0}(z-d-L_0/2)\right), \\ \qquad d+L_0/2<z\le d+L_0\,, \\ 1 \ \ \text{otherwise}\,. \end{cases} \quad (1)$$

2.2. Anisotropically Tapered Stenosed Artery

The geometry of the anistropically tapered arteries (see Fig. 2) with time dependant stenosis is defined by

$$R(z,t)/R_0 = \begin{cases} \Bigg(mz+R_0-\dfrac{\delta\cos\phi}{L_0}(z-d) \\ \quad \cdot\left(11-\dfrac{94}{3L_0}(z-d)+\dfrac{32}{L_0^2}(z-d)^2\right. \\ \quad \left.\left.-\dfrac{32}{3L_0^3}(z-d)^3\right)\right), \\ \quad d\le z\le d+3L_0/2\,, \\ (mz+1) \ \ \text{otherwise}\,, \end{cases} \quad (2)$$

where $R(z)$ and $R(z,t)$ denote the steady and unsteady radii of the arteries, R_0 is the constant radius of the normal artery in the non-stenotic region, ϕ is the angle of tapering, and $m=(\tan\phi)$ represents the slope of the tapered vessel such as $\phi<0$ is for converging tapering, $\phi=0$ is the non-tapered artery, and diverging artery is for $\phi>0$.

3. Formulation of the Problem

Consider an incompressible Jeffrey fluid flowing through a stenosed circular artery with permeable walls. We are considering cylindrical coordinates (r,θ,z) in such a way that the z-axis is taken along the axis of the artery and r, θ are the radial and circumferential directions, respectively. Let $r=0$ be considered as the axis of the symmetry of the tube. Then the governing equations for the flow problem are defined as

$$\frac{\partial v}{\partial r}+\frac{v}{r}+\frac{\partial u}{\partial z}=0\,, \quad (3)$$

$$\rho\left(\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial r}+u\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}(rS_{rz}) + \frac{\partial}{\partial z}(S_{zz})\,, \quad (4)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial z}+v\frac{\partial v}{\partial r}\right)=-\frac{\partial p}{\partial r}+\frac{1}{r}\frac{\partial}{\partial r}(rS_{rr}) + \frac{\partial}{\partial z}(S_{rz})\,, \quad (5)$$

where

$$S_{rr}=\frac{2\mu}{1+\lambda_1}\left(1+\lambda_2\left(v\frac{\partial}{\partial r}+u\frac{\partial}{\partial z}\right)\right)\frac{\partial v}{\partial r}\,, \quad (6)$$

$$S_{rz}=\frac{\mu}{1+\lambda_1}\left(1+\lambda_2\left(v\frac{\partial}{\partial r}+u\frac{\partial}{\partial z}\right)\right) \cdot\left(\frac{\partial v}{\partial z}+\frac{\partial u}{\partial r}\right), \quad (7)$$

$$S_{zz}=\frac{2\mu}{1+\lambda_1}\left(1+\lambda_2\left(v\frac{\partial}{\partial r}+u\frac{\partial}{\partial z}\right)\right)\frac{\partial u}{\partial z}\,. \quad (8)$$

Introducing non-dimensional variables

$$\begin{gathered} \widetilde{v}=\frac{1}{\varepsilon U_0}v\,,\ \widetilde{u}=\frac{1}{U_0}u\,,\ \widetilde{t}=\frac{\varepsilon U_0}{R_0}t\,, \\ \widetilde{r}=\frac{r}{R_0}\,,\ \widetilde{z}=\frac{1}{R_0}z\,,\ \widetilde{R}(z)=\frac{R(z)}{R_0}\,, \\ \widetilde{p}=\frac{\varepsilon R_0}{\mu U_0}p\,,\ \widetilde{R}(z,t)=\frac{R(z,t)}{R_0}\,,\ \widetilde{\delta}=\frac{\delta}{R_0}\,, \\ \widetilde{d}=\frac{d}{L_0}\,,\ \widetilde{L}=\frac{L}{L_0}\,,\ \widetilde{m}=\frac{L_0 m}{R_0}\,. \end{gathered} \quad (9)$$

Using the above non-dimensional variables, then

$$\begin{aligned} \mathrm{Re}\,\varepsilon\left(\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial r}+u\frac{\partial u}{\partial z}\right) &= -\frac{\partial p}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}\Bigg(\frac{r}{1+\lambda_1} \\ &\cdot\left(1+\frac{\varepsilon U_0\lambda_2}{R_0}\left(v\frac{\partial}{\partial r}+u\frac{\partial}{\partial z}\right)\right)\left(\varepsilon\frac{\partial v}{\partial z}+\frac{\partial u}{\partial r}\right)\Bigg) \\ &+\frac{\partial}{\partial z}\left(\frac{2\varepsilon}{1+\lambda_1}\left(1+\frac{U_0\lambda_2}{R_0}\left(v\frac{\partial}{\partial r}+u\frac{\partial}{\partial z}\right)\right)\frac{\partial u}{\partial z}\right), \end{aligned} \quad (10)$$

$$\begin{aligned} \mathrm{Re}\,\varepsilon^2\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial z}+v\frac{\partial v}{\partial r}\right) &= -\frac{\partial p}{\partial r}+\frac{1}{r}\frac{\partial}{\partial r}\Bigg(\frac{2\varepsilon r}{1+\lambda_1} \\ &\cdot\left(1+\frac{U_0\lambda_2}{R_0}\left(\varepsilon v\frac{\partial}{\partial r}+u\frac{\partial}{\partial z}\right)\right)\frac{\partial v}{\partial r}\Bigg)+\frac{\partial}{\partial z} \\ &\cdot\Bigg(\frac{2\varepsilon^2}{1+\lambda_1}\left(1+\frac{U_0\lambda_2}{R_0}\left(v\frac{\partial}{\partial r}+u\frac{\partial}{\partial z}\right)\right)\left(\frac{\partial v}{\partial z}+\frac{\partial u}{\partial r}\right)\Bigg) \end{aligned} \quad (11)$$

where $\mathrm{Re} = \rho U_0 R_0/\mu$ is the Reynolds number. Under the assumption of low Reynolds number for mild stenosis, with additional condition $\varepsilon = \frac{R_0}{L_0} = o(1)$, the non-dimensional problem is given by

$$\frac{\partial p}{\partial r} = 0\,, \tag{12}$$

$$\frac{\partial p}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{1+\lambda_1}\left(\frac{\partial u}{\partial r}\right)\right). \tag{13}$$

4. Solution of the Problem

4.1. Composite Stenosed Artery

The non-dimensional expression of (1) reads

$$R(z) = \begin{cases} 1-2\delta(z-d) \ \text{ for } d<z\le d+1/2\,, \\ 1-\dfrac{\delta}{2}(1+\cos 2\pi(z-d-1/2)) \\ \qquad \text{for } d+1/2<z\le d+1\,, \\ 1 \ \text{ otherwise}\,. \end{cases} \tag{14}$$

The integration of (13) yields

$$u = \left(\frac{\partial p}{\partial z}\right)\frac{(1+\lambda_1)}{2}r^2 - (1+\lambda_1)C_1\log_{\mathrm{e}} r + C_2\,. \tag{15}$$

Incorporating the non-dimensional boundary conditions

$$\frac{\partial u}{\partial r} = 0 \ \text{ at } \ r=0\,, \tag{16}$$

$$u = u_\mathrm{B} \ \text{ and } \ \frac{\partial u}{\partial r} = \frac{\alpha}{\sqrt{\mathrm{Da}}}(u_\mathrm{B} - u_\mathrm{porous}) \quad \text{at } \ r = R(z)\,, \tag{17}$$

we arrive at

$$u = \frac{\partial p}{\partial z}\frac{(1+\lambda_1)}{4}(r^2 - R^2(z)) + u_\mathrm{B}\,, \tag{18}$$

where u_B is the slip velocity and is given by

$$u_\mathrm{B} = -\frac{\partial p}{\partial z}\left(\mathrm{Da} + \frac{\sqrt{\mathrm{Da}}}{2\alpha}(1+\lambda_1)R(z)\right). \tag{19}$$

The solution in terms of the stream function is given by

$$\Psi = \frac{1}{16}r^2\left(8u_\mathrm{B} + \frac{\partial p}{\partial z}(r^2 - 2u^2)(1+\lambda_1)\right). \tag{20}$$

The volumetric flow flux Q is thus calculated as

$$Q = 2\int_0^{R(z)} ru\,\mathrm{d}r \tag{21}$$

or

$$Q = -\frac{\partial p}{\partial z}F(z)\,, \tag{22}$$

where

$$F(z) = \Big(R^2\big(8\mathrm{Da}\alpha + (1+\lambda_1)\big(4\sqrt{\mathrm{Da}}R(z) + R^2(z)\alpha\big)\big)\Big)(16\alpha)^{-1}\,, \tag{23}$$

$$\nabla p = \int_0^L \left(-\frac{\partial p}{\partial z}\right), \tag{24}$$

$$\lambda = \frac{1}{Q}\int_0^L \left(-\frac{\partial p}{\partial z}\right), \tag{25}$$

$$\lambda = \int_0^L G(z)\,\mathrm{d}z\,, \tag{26}$$

$$G(z) = \frac{1}{F(z)}\,. \tag{27}$$

Finally, we get

$$\lambda = \int_0^d \frac{1}{F(z)_{R=1}}\mathrm{d}z + \int_d^{d+1/2}\frac{1}{F(z)_{R \text{ from } (12)}}\mathrm{d}z + \int_{d+1/2}^{d+1}\frac{1}{F(z)_{R \text{ from } (13)}}\mathrm{d}z + \int_{d+1}^{L}\frac{1}{F(z)_{R=1}}\mathrm{d}z\,. \tag{28}$$

4.2. Anisotropically Tapered Stenosed Artery

The non-dimensional expression of (2) reads

$$R(z,t) = \begin{cases} \Big((mz+R_0) - \delta\cos\phi(z-d)\big(11 - \dfrac{94}{3}(z \\ \quad -d) + 32(z-d)^2 - \dfrac{32}{3}(z-d)^3\big)\Big) \\ \quad \cdot\,\Omega(t) \ \text{ for } d\le z\le d+3/2\,, \\ (mz+1)\Omega(t) \ \text{ otherwise}\,. \end{cases} \tag{29}$$

Corresponding boundary conditions are

$$\frac{\partial u}{\partial r} = 0 \ \text{ at } \ r=0\,, \tag{30}$$

$$u = u_\mathrm{B} \ \text{ and } \ \frac{\partial u}{\partial r} = \frac{\alpha}{\sqrt{\mathrm{Da}}}(u_\mathrm{B} - u_\mathrm{porous}) \quad \text{at } \ r = R(z,t)\,, \tag{31}$$

where $u_\mathrm{porous} = -\mathrm{Da}\frac{\partial p}{\partial z}$, u_porous is the velocity in the permeable boundary, Da is the Darcy number, and α

(called the slip parameter) is a dimensionless quantity depending on the material parameters which characterize the structure of the permeable material within the boundary region. We arrive at

$$u = \frac{\partial p}{\partial z}\frac{(1+\lambda_1)}{4}\left(r^2 - R^2(z,t)\right) + u_{\mathrm{B}}\,, \tag{32}$$

where u_{B} is slip velocity and is given by

$$u_{\mathrm{B}} = -\frac{\partial p}{\partial z}\left(\mathrm{Da} + \frac{\sqrt{\mathrm{Da}}}{2\alpha}(1+\lambda_1)R(z,t)\right). \tag{33}$$

The volumetric flow flux Q is thus calculated as

$$Q = 2\int_0^{R(z,t)} ru\,\mathrm{d}r \tag{34}$$

or

$$Q = -\frac{\partial p}{\partial z}F(z,t)\,, \tag{35}$$

$$\lambda = \int_0^L G(z,t)\,\mathrm{d}z\,. \tag{36}$$

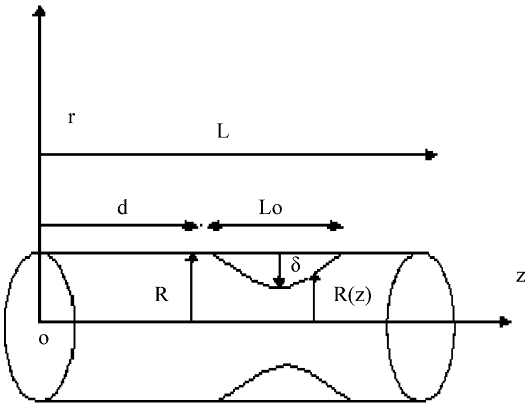

Fig. 1. Geometry of the problem for composite stenosed artery.

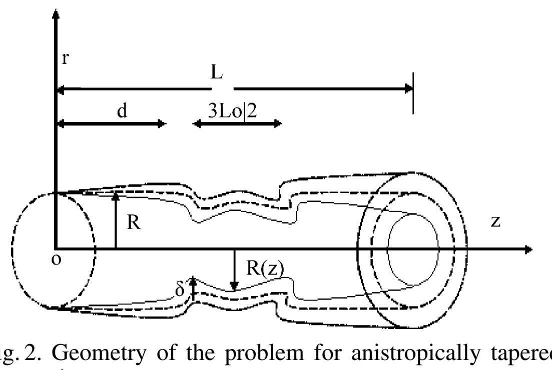

Fig. 2. Geometry of the problem for anistropically tapered stenosed artery.

Over the tapered arterial domain, the expression for the impedance will finally be

$$\lambda = \int_0^d G(z,t)\,\mathrm{d}z + \int_d^{d+3/2} G(z,t)\,\mathrm{d}z + \int_{d+3/2}^L G(z,t)\,\mathrm{d}z\,. \tag{37}$$

5. Discussion

5.1. Composite Stenosed Artery

To observe the quantitative effects of the Jeffrey parameter λ_1 and other various parameters on flow impedance λ, we have sketched a number of graphs. In Figure 3, the flow impedance λ is plotted against the slip parameter α for composite stenosed arteries. It is observed that by increasing Jeffrey parameter λ_1 and Darcy number $\sqrt{\mathrm{Da}}$, the impedance λ decreases while an increase in stenosis height δ results in an increase

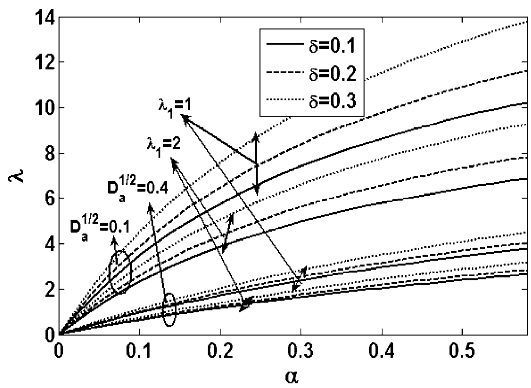

Fig. 3. Variation of resistance to flow λ with α.

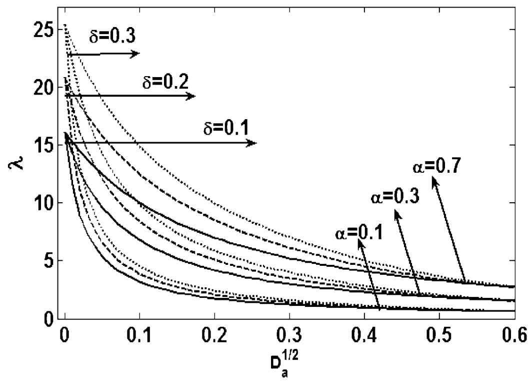

Fig. 4. Variation of resistance to flow λ with $\sqrt{\mathrm{Da}}$.

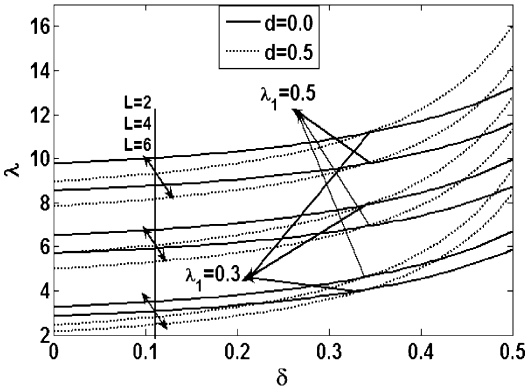

Fig. 5. Variation of resistance to flow λ with time δ.

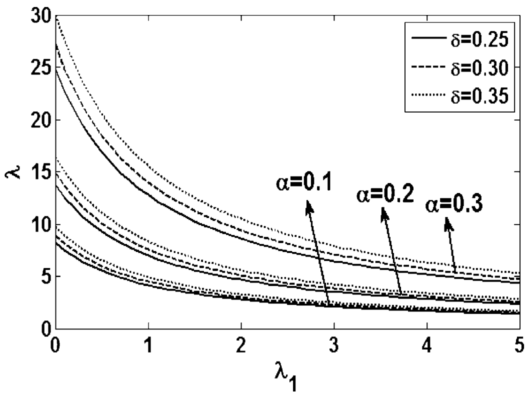

Fig. 6. Variation of resistance to flow λ with λ_1.

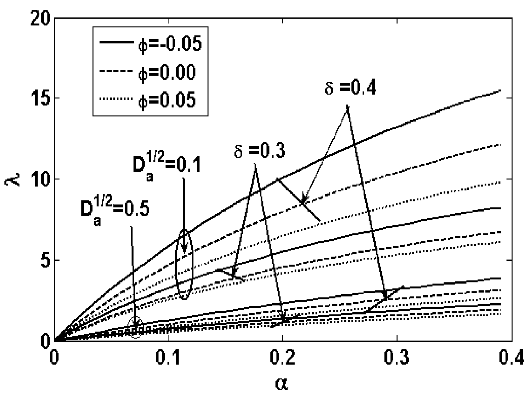

Fig. 7. Variation of resistance to flow λ with α.

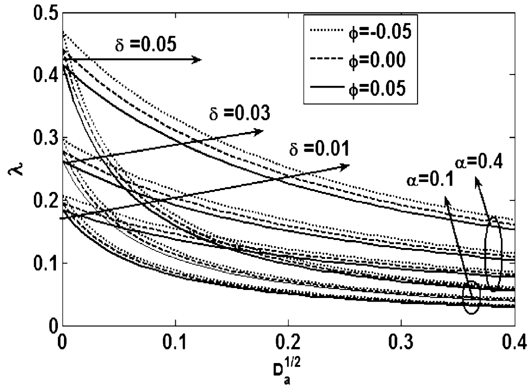

Fig. 8. Variation of resistance to flow λ with $\sqrt{\mathrm{Da}}$.

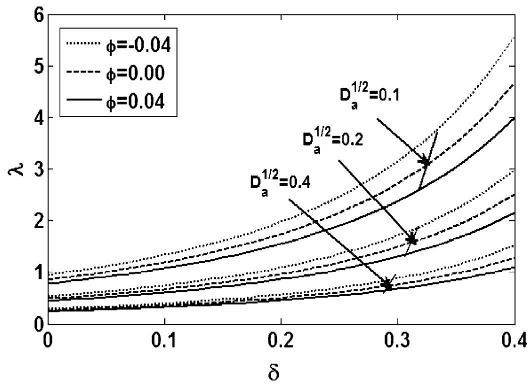

Fig. 9. Variation of resistance to flow λ with time δ.

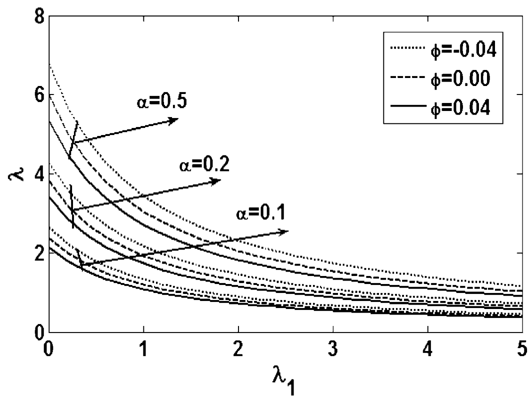

Fig. 10. Variation of resistance to flow λ with λ_1.

of the impedance λ. It is also noticed that $\sqrt{\mathrm{Da}}$ plays an inverse role against impedance λ. Figure 4 is the profile of λ against $\sqrt{\mathrm{Da}}$ for various values of stenosis height δ and slip parameter α. It shows that the impedance λ increases by increasing slip parameter α.

The effects of the length of the artery L with the position of stenosis d and Jeffrey parameter λ_1 on the flow impedance λ are discussed in Figure 5. It is observed that by increasing L, the impedance λ decreases. The effects of λ_1 are same as observed before. Figure 6

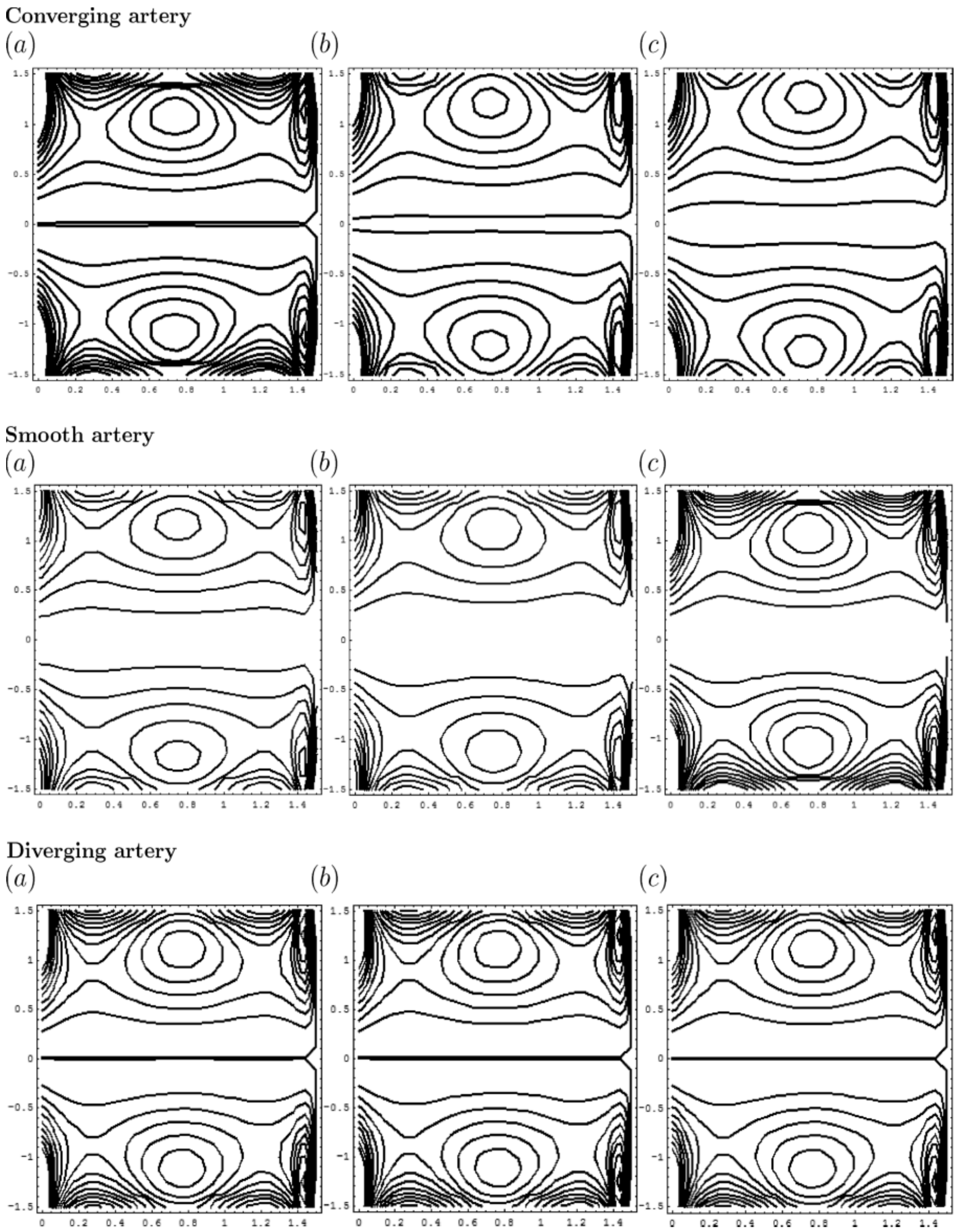

Fig. 11. Streamlines for various values of δ: (a) $\delta = 0.1$, (b) $\delta = 0.2$, (c) $\delta = 0.3$.

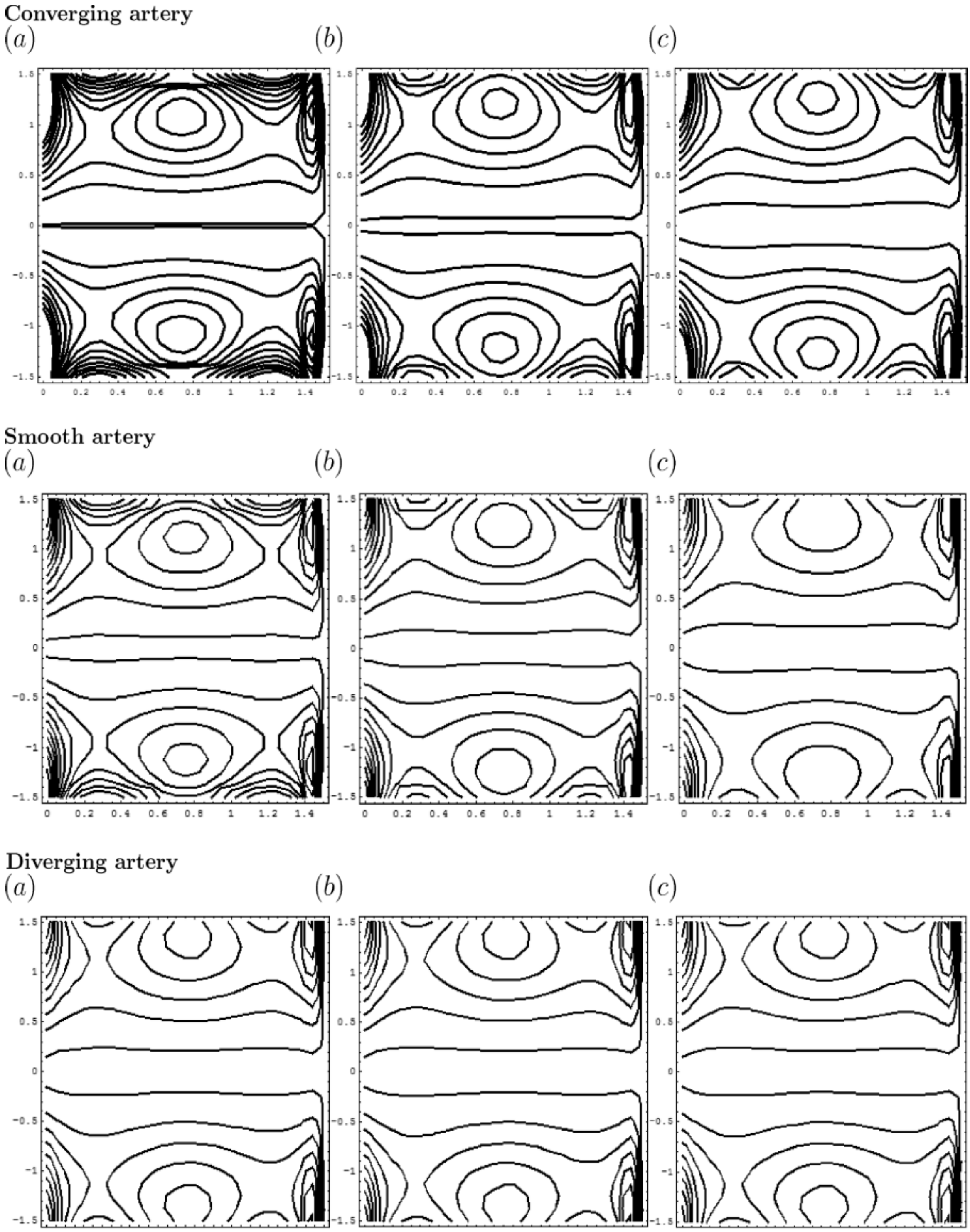

Fig. 12. Streamlines for various values of $\sqrt{\mathrm{Da}}$: (a) $\sqrt{\mathrm{Da}} = 0.1$, (b) $\sqrt{\mathrm{Da}} = 0.2$, (c) $\sqrt{\mathrm{Da}} = 0.3$.

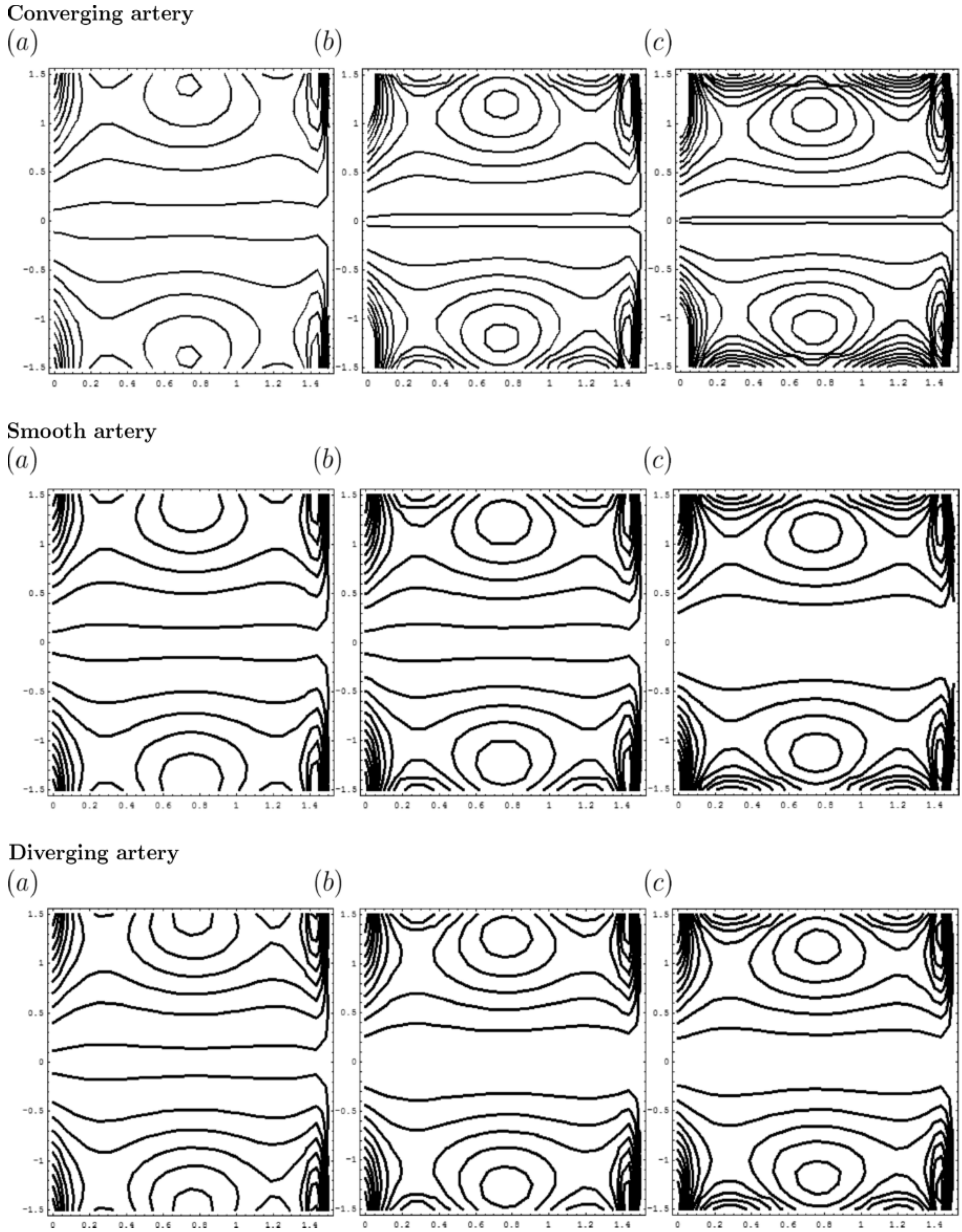

Fig. 13. Streamlines for various values of α: (a) $\alpha = 0.1$, (b) $\alpha = 0.2$, (c) $\alpha = 0.3$.

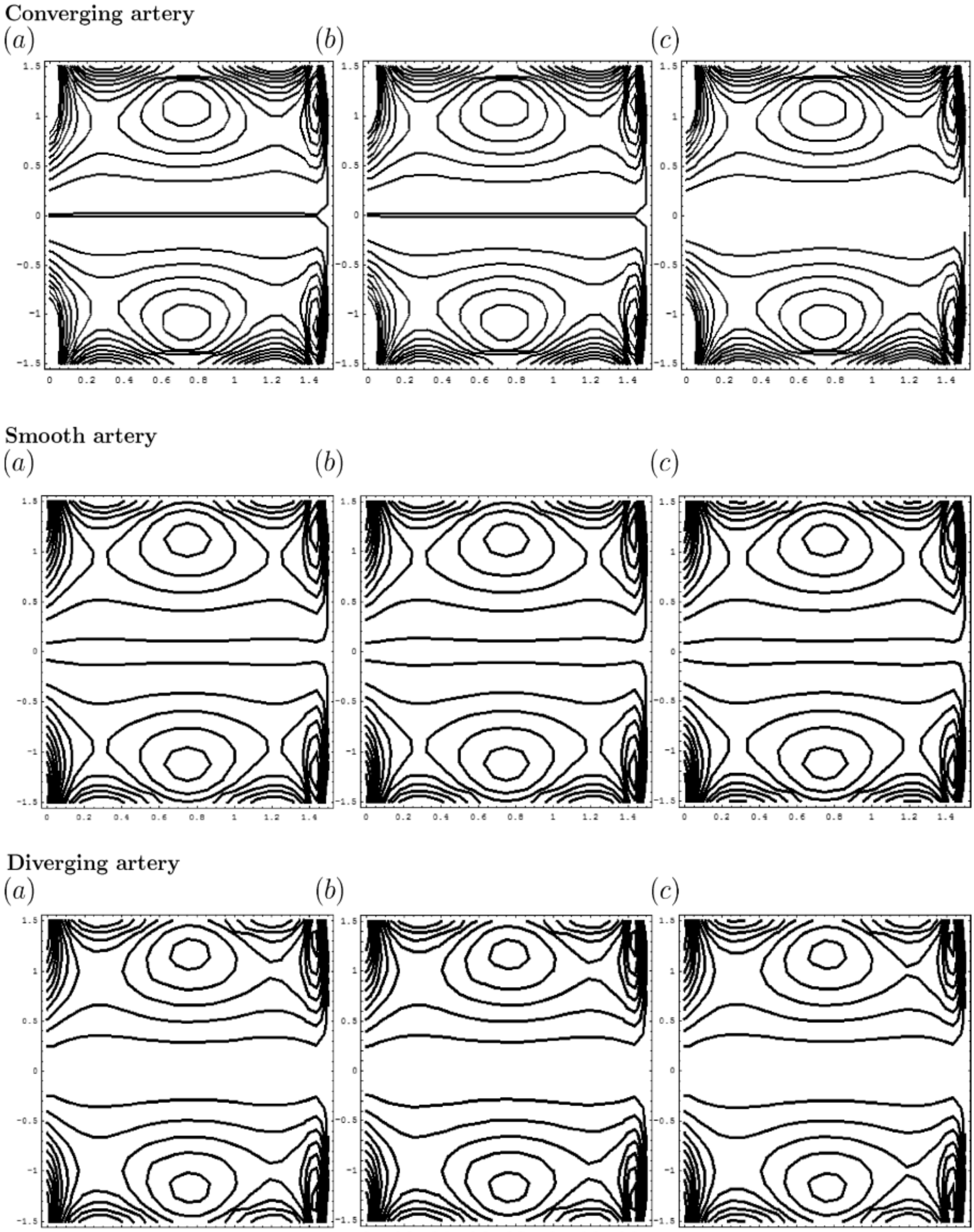

Fig. 14. Streamlines for various values of λ_1: (a) $\lambda_1 = 0.01$, (b) $\lambda_1 = 0.1$, (c) $\lambda_1 = 10$.

describes the impedance profile of λ against Jeffrey parameter λ_1. Again it is clear that an increase in the slip parameter α results an increase in impedance λ.

5.2. Anisotropically Tapered Stenosed Artery

In order to observe the effects of the Jeffrey parameter λ_1 along with all other parameters on flow impedance λ to the blood flow through anistropically tapered stenosed arteries, we have drawn graphs 7 – 12. The effects of tapering angle ϕ is discussed in all profiles with the respective parameters. In Figure 7, the flow impedance λ is plotted against the slip parameter α for tapered stenosed arteries. It is observed that by increasing Darcy number $\sqrt{\mathrm{Da}}$, the impedance λ decreases while an increase in stenosis height δ results in an increase of the impedance λ. Figure 8 describes how λ is related to $\sqrt{\mathrm{Da}}$ under the effects of δ and α. One can easily observe that the impedance λ is directly proportional to the stenosis height δ and the slip parameter α. The effects of Darcy number $\sqrt{\mathrm{Da}}$ on the flow impedance λ are discussed in Figure 9, and one notices that $\sqrt{\mathrm{Da}}$ has the same role as we have observed in previous cases. Figure 10 describes the profiles of impedance λ against Jeffrey parameter λ_1. It is clear that an increase of the slip parameter α results in an increase of impedance λ. The streamlines for converging, diverging, and non-tapered arteries are also plotted against these parameters. It is depicted that the maximum number of trapping bolus are offered by the diverging artery while a converging artery shows the least number of trapping bolus. By increasing the stenosis height δ and the slip parameter α, the number of bolus decreases (see Figs. 8 and 10). An increase in Darcy number $\sqrt{\mathrm{Da}}$ and Jeffrey parameter λ_1 results in a decrease of the number of bolus (see Figs. 9 and 11).

6. Conclusion

In the present study, the mathematical and graphical results of blood flow of a Jeffery fluid in a stenosed artery are discussed. Two types of arteries, namely (i) composite stenosed artery and (ii) anistropically tapered stenosed artery with permeable walls are considered. The governing equations are simplified by employing mild stenosis and low Reynolds number approximations. The exact solutions of the resulting equations are found out. The following main results are observed:

- By increasing δ and α, the impedance λ increases.
- Impedance λ decreases by increasing λ_1 and $\sqrt{\mathrm{Da}}$.
- Maximum impedance is offered by converging arteries while the diverging arteries offer a minimum.
- Maximum number of trapping bolus are offered by the diverging arteries while the converging arteries show the least number of trapping bolus.

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