A Note on the Perturbation of Mixed Percolation on the Hierarchical Group

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We study mixed percolation on the hierarchical group of order N where each node is open with probability $1-\gamma$, $0 \le \gamma \le 1$, and the probability of connection between two open nodes separated by a distance k is of the form $1-\exp(-\alpha\beta^{-k})$, $\alpha \ge 0$, and $\beta > 0$. The parameters α and γ are the percolation parameters, while β describes the long-range nature of the model. In terms of parameters α, β, and γ, we show some perturbation results for the percolation function $\theta(\alpha,\beta,\gamma)$, which is the probability of existing an infinite component containing a prescribed node.

Key words: Mixed Percolation; Hierarchical Group; Percolation Function.
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1. Introduction and the Model

The percolation theory of the Euclidean lattice $\mathbb{Z}^d$ started with the work of Broadbent and Hammersley in 1957. The infinity of the space of sites (or nodes) and its geometry are premier features of this model (see e. g. [1, 2] for background). Some questions of percolation in other non-Euclidean infinite systems have been formulated in [3]. The study of long-range percolation on $\mathbb{Z}^d$ stretches back to [4] and leads to a range of interesting results in mathematical physics [5 – 10]. On the other hand, hierarchical structures have been widely used in applications in the physical, biological, and social sciences due to the multi-scale organization of many natural objects [11 – 13].

Recently, long-range percolation is studied on the hierarchical group Ω_N of order N (to be defined below), where classical methods for the usual lattice break down. The asymptotic long-range percolation on Ω_N is addressed in [14] for $N \to \infty$. A series of works [15 – 17] analyze the phase transition of long-range percolation on Ω_N for finite N using different connection probabilities and methodologies. The contact process on Ω_N for fixed N has been investigated in [18]. In this note, we take a closer look at the mixed percolation (i. e. site-bond percolation) on Ω_N with finite N, noting that only bond percolation is considered in all the above cited works. We present some interesting perturbation results for the percolation function in terms of the model parameters. The node and edge deletion processes performing on $\mathbb{Z}^d$ and complete graphs are usually described as mixed percolation [2], and the resulting graph is sometimes called the faulty graph, which has many applications in communication networks [19, 20]. From this point of view, the model considered here can be regarded as an extension of mixed percolation from homogeneous structures to a hierarchical one.

In the sequel, we will introduce the model. For an integer $N \ge 2$, we define the set

$$\Omega_N := \Big\{ x = (x_1, x_2, \ldots) : x_i \in \{0, 1, \ldots, N-1\}, \quad i = 1, 2, \ldots, \ x_i \ne 0 \text{ only for finitely many } i \Big\}, \tag{1}$$

and define a metric d on it:

$$d(x,y) = \begin{cases} 0, & x = y, \\ \max\{i : x_i \ne y_i\}, & x \ne y, \end{cases} \tag{2}$$

where $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots) \in \Omega_N$. The pair (Ω_N, d) is called the hierarchical group (or lattice) of order N, which may be thought of as the set of leaves at the bottom of an infinite regular tree without a root, where the distance between two nodes is the number of levels (generations) from the bottom to their most recent common ancestor; see Figure 1 for an illustration.

Such a distance d satisfies the strong triangle inequality

$$d(x,y) \leq \max\{d(x,z), d(z,y)\} \tag{3}$$

for any triple $x,y,z \in \Omega_N$. Hence, (Ω_N, d) is an ultrametric (or non-Archimedean) space [21]. From its ultrametricity, it is clear that for every $x \in \Omega_N$ there are $(N-1)N^{k-1}$ nodes at distance k from it.

Now consider a long-range mixed percolation on Ω_N. For $0 \leq \gamma \leq 1$, a node $x \in \Omega_N$ is open with probability $1-\gamma$ and closed with probability γ, independently of the states of other nodes in Ω_N. For each $k \geq 1$, the probability of connection between two open nodes x and y such that $d(x,y) = k$ is given by

$$p_k = 1 - \exp\left(-\frac{\alpha}{\beta^k}\right), \tag{4}$$

where $0 \leq \alpha < \infty$ and $0 < \beta < \infty$, all connections being independent. No edge is incident to a closed node. Two nodes $x, y \in \Omega_N$ are in the same component if there exists a finite sequence $x = x_0, x_1, \ldots, x_n = y$ of nodes such that each pair (x_{i-1}, x_i), $i = 1, \ldots, n$, of nodes presents an edge.

The rest of the note is organized as follows. In Section 2, we present our perturbation results. Section 3 is devoted to the proofs. Finally, a conclusion is drawn in Section 4.

2. Main Results

Let $|S|$ denote the size of a set S. The connected component containing the node $x \in \Omega_N$ is denoted by $C(x)$. Since, for every $x \in \Omega_N$, $|C(x)|$ has the same distribution, it suffices to consider only $|C(0)|$.

Denote by $P_{\alpha,\beta,\gamma}$ the probability measure governing the above mixed percolation process on the appropriate probability space and sigma algebra. The percolation function is defined as

$$\theta(\alpha,\beta,\gamma) := P_{\alpha,\beta,\gamma}(|C(0)| = \infty). \tag{5}$$

It follows from a standard coupling argument that $\theta(\alpha,\beta,\gamma)$ is increasing in α and decreasing in β and γ. It is also known that, when $\gamma = 0$, i. e., in the case of pure bond percolation, the percolation function $\theta(\alpha,\beta,0) = 1$ as long as $\alpha > 0$ and $\beta \leq N$, while $\theta(\alpha,\beta,0) < 1$ for $\beta > N$ [16]. Our main perturbation result is established as follows.

Theorem 1. *For any $0 \leq \delta < 1$, $\varepsilon_1 \geq 0$, and $0 < \varepsilon_2 \leq 1$ satisfying $\beta\varepsilon_2 > N$ and $\varepsilon_1 - \varepsilon_2 > -1$, there exists a $\gamma > \delta$, such that*

$$\theta(\alpha,\beta,\delta) \leq \theta(\alpha(1+\varepsilon_1), \beta\varepsilon_2, \gamma). \tag{6}$$

We give some remarks here. Firstly, it follows from the above comments that the inequality (6) holds if $\gamma \leq \delta$. Secondly, the assumption $\varepsilon_1 - \varepsilon_2 > -1$ aims to exclude the situation where $\varepsilon_1 = 0$ and $\varepsilon_2 = 1$ hold simultaneously, since we have $\theta(\alpha,\beta,\delta) > \theta(\alpha,\beta,\gamma)$ as long as $\gamma > \delta$.

On the other direction of perturbation, the following result holds.

Theorem 2. *Let $\beta > N$. For any $0 < \delta \leq 1$, $0 < \varepsilon_1 \leq 1$, and $\varepsilon_2 \geq 0$ satisfying $\varepsilon_2 - \varepsilon_1 > -1$, there exists a $\gamma < \delta$, such that*

$$\theta(\alpha,\beta,\delta) \geq \theta(\alpha\varepsilon_1, \beta(1+\varepsilon_2), \gamma). \tag{7}$$

3. Proof

In this section, we only prove Theorem 1 and leave the similar proof of Theorem 2 for the interested reader.

Proof of Theorem 1. We first define a directed version of the mixed percolation on Ω_N. For more information for directed percolation see e. g. [22]. Given $0 \leq \gamma \leq 1$, a node $x \in \Omega_N$ is open with probability $1-\gamma$ and closed with probability γ, independently of the states of other nodes in Ω_N. If node x is open, then a directed edge from x to y is present with probability $1 - \exp\left(-\alpha\beta^{d(x,y)}\right)$, where $0 \leq \alpha < \infty$ and $0 < \beta < \infty$. Conditioned on the states of nodes, all connections are independent. No directed edge starts at a closed node. The corresponding probability measure of this directed version from now on is denoted by $\hat{P}_{\alpha,\beta,\gamma}$. The set of nodes which can be reached by a directed path starting from node x is denoted by $\hat{C}(x)$. A standard argument (see e. g. [23, 24]) can be applied to show that

$$\begin{aligned} P_{\alpha,\beta,\gamma}(|C(0)| = \infty) &= \theta(\alpha,\beta,\gamma) \\ &= \hat{P}_{\alpha,\beta,\gamma}\left(|\hat{C}(0)| = \infty\right). \end{aligned} \tag{8}$$

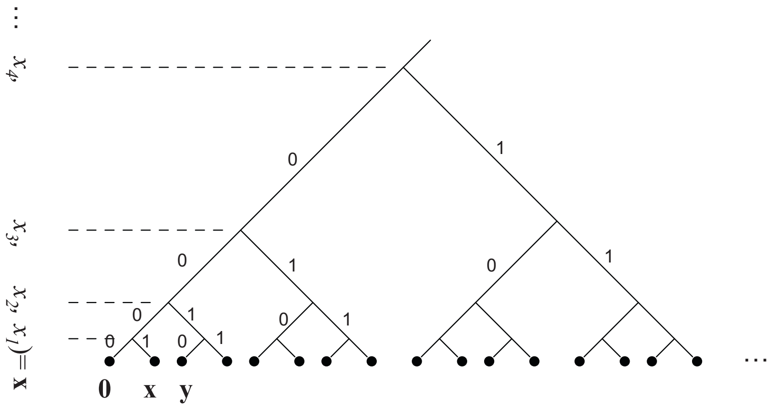

Fig. 1. Illustration of hierarchical group Ω_2 of order 2. The distances between three nodes $0=(0,0,0,\ldots)$, $x=(1,0,0,\ldots)$, and $y=(0,1,0,\ldots)$ are $d(0,x)=1$ and $d(0,y)=d(x,y)=2$.

The resulting graph of the directed mixed percolation with parameters α, β, and γ can be obtain as follows. We assign independent and identically distributed random variables X_x to nodes $x\in\Omega_N$, all being Poisson distributed with parameter $\alpha(N-1)/(\beta-N)$. We then construct a directed multi-graph as follows. Each node is open with probability $1-\gamma$ and closed with probability γ, independently of other nodes. If x is open, then there are X_x directed edges starting from x. The endpoints of these edges are independently chosen from $\Omega_N\backslash\{x\}$, and a node at distance r of x is chosen with probability $(\beta-N)(N-1)^{-1}\beta^{-r}$. If x is closed, no edges start from x. Recall that there are $(N-1)N^{r-1}$ nodes at distance r from x, and

$$\sum_{r=1}^{\infty}\frac{\beta-N}{(N-1)\beta^r}\cdot(N-1)N^{r-1}=1\,. \tag{9}$$

Since

$$\frac{\alpha(N-1)}{\beta-N}\cdot\frac{\beta-N}{(N-1)\beta^r}=\frac{\alpha}{\beta^r}\,, \tag{10}$$

it is easy to see that we arrive at the original directed graph by replacing the collection of all edges from x to y (if there is at least one) by a single edge from x to y, for all $x,y\in\Omega_N$.

Let $Z_1=W_1W_2$, where W_1 is equal to 1 with probability $1-\delta$ and equal to 0 with probability δ, and W_2 is Poisson distributed with parameter $\alpha(N-1)/(\beta-N)$ independent of W_1. In addition, let $Z_2=Y_1Y_2$, where Y_1 is equal to 1 with probability $1-\gamma$ and equal to 0 with probability γ, and Y_2 is Poisson distributed with parameter $\alpha(1+\varepsilon_1)(N-1)/(\beta\varepsilon_2-N)$ independent of Y_1. For the mixed percolation with parameters α,β,δ the number of edges starting from x in the multi-graph is distributed as Z_1, while for the mixed percolation with parameters $\alpha(1+\varepsilon_1),\beta\varepsilon_2,\gamma$ the number of edges starting from x in the multi-graph is distributed as Z_2. Now, it is easy to check that there exists a $\gamma>\delta$, such that $P(Z_1=0)=P(Z_2=0)$ and for this γ and all $k>0$, we obtain

$$\begin{aligned}P\left(Z_2>k|Z_2>0\right)&=P\left(Y_2>k|Y_1=1,Y_2>0\right)\\&=P\left(Y_2>k|Y_2>0\right)\\&>P\left(W_2>k|W_2>0\right)\\&=P\left(W_2>k|W_1=1,W_2>0\right)\\&=P\left(Z_2>k|Z_1>0\right).\end{aligned} \tag{11}$$

The statement of Theorem 1 then follows from a direct coupling argument. □

4. Conclusion

The use of percolation theory in mathematical physics has long been recognized. In this note, we introduced the mixed percolation on the hierarchical group, which differs significantly from Euclidean lattice, and studied some perturbation results for the corresponding percolation functions. One of the important issues left open is the phase diagram of the mixed percolation. In addition, the uniqueness of an infinite component and the continuity of percolation function are worth investigating. A detailed analysis thus appears as a promising target for future work.

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