

Trace for Differential Pencils on a Star-Type Graph

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In this work, we consider the spectral problem for differential pencils on a star-type graph with a Kirchhoff-type condition in the internal vertex. The regularized trace formula of this operator is established with the contour integration method in complex analysis.

Key words: Differential Pencils; Star-Type Graph; Trace Formula.

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1. Introduction

We consider the differential equations on a graph

$$-y_i''(x) + [2\rho p_i(x) + q_i(x)]y_i(x) = \rho^2 y_i(x), \quad (1)$$
$$i = \overline{1, r}; \quad r \geq 2, \quad r \in \mathbb{N},$$

where ρ is the spectral parameter, while the functions $y_i(x)$ and $y'_i(x)$ for $i = \overline{1, r}$ are absolutely continuous on $[0, \pi]$ and satisfy the matching conditions

$$y_i(\pi) = y_j(\pi), \quad i, j = \overline{1, r};$$
$$\sum_{i=1}^r y'_i(\pi) + \beta y_1(\pi) = 0 \quad (2)$$

at the central vertex v_0 . We assume that real-valued functions $p_i(x) \in W_2^2[0, \pi]$ with $\int_0^\pi p_i(t)dt = 0$, $i = \overline{1, r}$, $q_i(x) \in W_2^1[0, \pi]$; β is a real constant given by (2). The real-valued functions $p(x) = \{p_i(x)\}_{i=\overline{1,r}}$ and $q(x) = \{q_i(x)\}_{i=\overline{1,r}}$ on a graph are called potentials. The matching conditions (2) are called the continuity condition together with a Kirchhoff-type condition.

We consider on a graph the boundary value problem for (1) with the matching conditions (2) and the following boundary conditions at the boundary vertices v_1, \dots, v_r :

$$y'_i(0) - h_i y_i(0) = 0, \quad i = \overline{1, r}, \quad (3)$$

where h_i are real numbers. Denote the problem (1), (2), and (3) by $L = L(p, q, h, \beta)$, where $h = \{h_i\}_{i=\overline{1,r}}$. Then (1) describes the wave propagation on a graph, where ρ is the wave number (momentum) and ρ^2 the energy;

$p(x)$ and $q(x)$ describe the joint effect of absorption and generation of energy and the regeneration of the force density, respectively. Because of important applications in quantum mechanics, it is interesting to investigate the spectral characteristics of the operator pencil L , and if at least one of the colliding particles is a fermion, it is relevant to solve the inverse problem in the presence of central and spin-orbital potentials.

The theory of regularized traces of ordinary differential operators has a long history. First, the trace formulas for the Sturm–Liouville operator with the Dirichlet boundary conditions and sufficiently smooth potential were established in [1]. Afterwards, these investigations were continued in many directions (see, [2–7], etc.). The trace formulas can be used for approximate calculation of the first eigenvalues of an operator, and in order to establish necessary and sufficient conditions for a set of complex numbers to be the spectrum of an operator.

Differential operators on graphs (networks and trees) often arise in natural sciences and technology. The spectral problems of quantum graphs have been studied in [8–18], etc. Some results on trace formulas for Laplacians on metric graphs have appeared in [10, 14] and other articles.

Recently, the author considered the spectral problem for the Sturm–Liouville differential operator on r -star-type graph [16], and later found that by using a refined estimate for a fundamental pair of solutions to equation (1) in [19], we can establish the trace formula for the operator L , which extends the result of the previous work of the author [16].

2. Result

In [15], the author considered the spectral problem for the differential pencil L and gave asymptotics of the eigenvalues for L . The eigenvalue set $\bigcup_{j=1}^r \{\rho_n^{(j)}\}_{-\infty}^\infty$ for the differential pencil L behaves asymptotically as follows (see Theorem 2.3 in [15]):

$$\rho_n^{(1)} = n + \frac{\omega + \beta}{rn} + O\left(\frac{1}{n^2}\right)$$

and

$$\rho_n^{(j)} = n + \frac{1}{2} + \frac{c_j}{n + \frac{1}{2}} + O\left(\frac{1}{n^2}\right), \quad j = \overline{2, r},$$

where c_j , $j = \overline{2, r}$, are the solutions (all real but not necessarily different) of the equation for x

$$P(x) \stackrel{\text{def}}{=} \sum_{i=1}^r \prod_{l \neq i} (x - \omega_l) = 0,$$

where

$$\omega_i(x) = \frac{1}{2} \int_0^x [p_i^2(t) + q_i(t)] dt + h_i, \quad i = \overline{1, r},$$

and

$$\omega_0 = \sum_{i=1}^r \frac{(2-r)p_i(\pi) + rp_i(0)}{2}, \quad \omega = \sum_{i=1}^r \omega_i(\pi).$$

The trace formula is interesting and of significance for the inverse spectral problem. How is it possible to obtain the second-order trace formula for eigenvalues of the problem L ? It was noted that the formula of the second-order trace comprises lots of information on the operator spectrum, the boundary condition parameters, and the potential of a graph.

Before giving the main result, we need the following notations. For $i = \overline{1, r}$, denote

$$\begin{aligned} b_{1,i} &= \frac{1}{2} \int_0^\pi [p_i^2(t) + q_i(t)] dt, \\ a_{1,i} &= \frac{1}{2} [p_i(\pi) + p_i(0)], \\ c_{1,i} &= \frac{1}{2} [p_i(0) - p_i(\pi)], \\ d_{2,i} &= \frac{1}{8} [5p_i^2(\pi) - 2p_i(0)p_i(\pi) - 3p_i^2(0) \\ &\quad + 2q_i(\pi) - 2q_i(0)] - \frac{1}{2} b_{1,i}^2, \\ d_{1,i} &= -\frac{1}{4} [p_i'(\pi) + p_i'(0)] + \frac{1}{2} b_{1,i} [p_i(\pi) \\ &\quad - p_i(0)] + \frac{1}{2} \int_0^\pi p_i(t) [p_i^2(t) + q_i(t)] dt, \end{aligned} \tag{4}$$

$$e_{1,i} = \frac{1}{4} [p_i'(\pi) - p_i'(0)] - \frac{1}{2} b_{1,i} [p_i(\pi) + p_i(0)]$$

$$+ \frac{1}{2} \int_0^\pi p_i(t) [p_i^2(t) + q_i(t)] dt,$$

$$e_{2,i} = \frac{1}{8} [3p_i^2(\pi) - 2p_i(0)p_i(\pi) + 3p_i^2(0)$$

$$+ 2q_i(\pi) + 2q_i(0)] + \frac{1}{2} b_{1,i}^2,$$

$$A_1 = \{\pm 0, \pm 1, \pm 2, \dots\}, \quad A_2 = \{0, \pm 1, \pm 2, \dots\},$$

and

$$\begin{aligned} \rho_{-0}^{(01)} &= \rho_0^{(01)} = 0; \quad \rho_n^{(01)} = n, \quad n \in \mathbb{Z} \setminus \{0\}; \\ \rho_n^{(0j)} &= n + \frac{1}{2}, \quad n \in \mathbb{Z}, \quad j = \overline{2, r}. \end{aligned} \tag{5}$$

The main result of this work reads as follows.

Theorem 1. *For the eigenvalue set $\bigcup_{j=1}^r \{\rho_n^{(j)}\}_{-\infty}^\infty$ for the differential pencil L , we have the trace formula*

$$\begin{aligned} &\sum_{n \in A_1} \left[(\rho_n^{(1)})^2 - (\rho_n^{(01)})^2 - \frac{2}{\pi r} \sum_{i=1}^r (b_{1,i} + h_i) - \frac{2\beta}{\pi r} \right] \\ &+ \sum_{n \in A_2} \left\{ \sum_{i=2}^r \left[(\rho_n^{(i)})^2 - (\rho_n^{(0i)})^2 \right] - \frac{2(r-1)}{\pi r} \sum_{i=1}^r (b_{1,i} + h_i) \right\} \\ &= -\frac{2}{\pi r} \sum_{i=1}^r (b_{1,i} + h_i) - \frac{2\beta}{\pi r} - \sum_{i=1}^r (2d_{2,i} + b_{1,i}^2) \\ &+ \frac{2}{r} \sum_{i=1}^r (d_{2,i} + e_{2,i}) - \sum_{i=1}^r h_i^2 - \frac{\beta^2}{r^2} + \omega_3, \end{aligned} \tag{6}$$

where

$$\begin{aligned} \omega_3 &= \frac{1}{r^2} \left[\sum_{i=1}^r (a_{1,i} - c_{1,i}) \right]^2 \\ &+ \frac{2}{r} \sum_{i=1}^r a_{1,i} c_{1,i} + \frac{r-2}{r} \sum_{i=1}^r c_{1,i}^2. \end{aligned} \tag{7}$$

3. Proof of Theorem 1

Denote by $\varphi_i(\rho, x)$, $i = \overline{1, r}$, the solutions of (1) satisfying the conditions

$$\varphi_i(\rho, 0) = 0, \quad \varphi_i'(\rho, 0) = 1, \tag{8}$$

and by $\psi_i(\rho, x)$, $i = \overline{1, r}$, the solutions of (1) satisfying the conditions

$$\psi_i(\rho, 0) = 1, \quad \psi_i'(\rho, 0) = 0. \tag{9}$$

Thus, for the solutions $s_i(\rho, x)$, $i = \overline{1, r}$, of (1) satisfying the conditions

$$s_i(\rho, 0) = 1, \quad s'_i(\rho, 0) = h_i,$$

we have

$$\begin{aligned} s_i(\rho, x) &= \psi_i(\rho, x) + h_i \varphi_i(\rho, x), \\ s'_i(\rho, x) &= \psi'_i(\rho, x) + h_i \varphi'_i(\rho, x). \end{aligned} \quad (10)$$

Then the solutions of (1) which satisfy the conditions (3) are

$$y_i(\rho, x) = A_i(\rho) s_i(\rho, x), \quad (11)$$

where $A_i(\rho)$ are constants. Substituting (11) into (2), we obtain the following equation for the eigenvalues of the problem L :

$$\varphi(\rho) \stackrel{\text{def}}{=} \phi(\rho) + \beta \chi(\rho) = 0,$$

where

$$\phi(\rho) \stackrel{\text{def}}{=} \sum_{i=1}^r s'_i(\rho, \pi) \times \prod_{j \neq i, j=1,2,\dots,r} s_j(\rho, \pi) \quad (12)$$

and

$$\chi(\rho) \stackrel{\text{def}}{=} \prod_{i=1}^r s_i(\rho, \pi). \quad (13)$$

Using a refined estimate for a fundamental pair of solutions of the equation (1) in [19], we get

$$\begin{aligned} s_i(\rho, \pi) &= \psi_i(\rho, \pi) + h_i \varphi_i(\rho, \pi) \\ &= \cos(\rho\pi) - c_{1,i} \frac{\cos(\rho\pi)}{\rho} + (b_{1,i} + h_i) \frac{\sin(\rho\pi)}{\rho} \\ &\quad + (d_{2,i} - h_i b_{1,i}) \frac{\cos(\rho\pi)}{\rho^2} + (d_{1,i} + h_i a_{1,i}) \frac{\sin(\rho\pi)}{\rho^2} \quad (14) \\ &\quad + o\left(\frac{e^{\tau\pi}}{\rho^2}\right) \end{aligned}$$

and

$$\begin{aligned} s'_i(\rho, \pi) &= \psi'_i(\rho, \pi) + h_i \varphi'_i(\rho, \pi) \\ &= -\rho \sin(\rho\pi) + (b_{1,i} + h_i) \cos(\rho\pi) + a_{1,i} \sin(\rho\pi) \\ &\quad + (e_{1,i} + h_i c_{1,i}) \frac{\cos(\rho\pi)}{\rho} + (e_{2,i} + h_i b_{1,i}) \frac{\sin(\rho\pi)}{\rho} \quad (15) \\ &\quad + o\left(\frac{e^{\tau\pi}}{\rho}\right), \end{aligned}$$

where $\tau = |\operatorname{Im} \rho|$.

Denote

$$\varphi_0(\rho) = -r\rho \sin(\rho\pi) \cos^{r-1}(\rho\pi). \quad (16)$$

Here, we point out that $\varphi_0(\rho)$ is the characteristic equation for $L(0, 0, 0)$ and its zeros, $\rho_n^{(0j)}$, $j = 1, 2, \dots, r$, are defined by

$$\rho_n^{(01)} = n, \quad n \in A_1,$$

and

$$\rho_n^{(0j)} = n + \frac{1}{2}, \quad n \in A_2, \quad j \neq 1.$$

Let the contour Γ_N , integer $N = 0, 1, 2, \dots \rightarrow \infty$, denote the following sequences of circular contours, traversed counterclockwise:

The contour Γ_N is the circle of radius $(N + \frac{1}{4})^2$ with its center at the origin.

Obviously, $\rho_n^{(0j)}$ don't lie on the contour Γ_N . Substituting the expressions (14) and (15) into (12) and (13), we have on the contour Γ_N

$$\begin{aligned} \frac{\phi(\rho)}{\varphi_0(\rho)} &= \frac{1}{r} \sum_{i=1}^r \left[1 - \frac{b_{1,i} + h_i}{\rho} \cot(\rho\pi) - \frac{a_{1,i}}{\rho} \right. \\ &\quad \left. - (e_{1,i} + h_i c_{1,i}) \frac{\cot(\rho\pi)}{\rho^2} - \frac{e_{2,i} + h_i b_{1,i}}{\rho^2} + o\left(\frac{1}{\rho^2}\right) \right] \\ &\quad \cdot \prod_{l \neq i} \left[1 + \frac{T_l}{\rho} + \frac{V_l}{\rho^2} + o\left(\frac{1}{\rho^2}\right) \right], \end{aligned}$$

where

$$\begin{aligned} T_l &= -c_{1,l} + (h_l + b_{1,l}) \tan(\rho\pi) \\ V_l &= d_{2,l} - h_l b_{1,l} + (d_{1,l} + h_l a_{1,l}) \tan(\rho\pi). \end{aligned}$$

Moreover, by calculation, we obtain

$$\begin{aligned} \frac{\phi(\rho)}{\varphi_0(\rho)} &= \frac{1}{r} \sum_{i=1}^r \left[1 - \frac{b_{1,i} + h_i}{\rho} \cot(\rho\pi) - \frac{a_{1,i}}{\rho} \right. \\ &\quad \left. - (e_{1,i} + h_i c_{1,i}) \frac{\cot(\rho\pi)}{\rho^2} - \frac{(e_{2,i} + h_i b_{1,i})}{\rho^2} + o\left(\frac{1}{\rho^2}\right) \right] \\ &\quad \cdot \left[1 + \frac{\sum_{l \neq i} T_l}{\rho} + \frac{\sum_{l \neq i} V_l + \sum_{i_1 < i_2 \neq i} T_{i_1} T_{i_2}}{\rho^2} + o\left(\frac{1}{\rho^2}\right) \right] \\ &= 1 - \frac{\sum_{i=1}^r (b_{1,i} + h_i)}{r} \times \frac{\cot(\rho\pi)}{\rho} + \frac{r-1}{r} \sum_{i=1}^r \frac{T_i}{\rho} \\ &\quad - \frac{1}{r} \sum_{i=1}^r \frac{a_{1,i}}{\rho} + \frac{r-1}{r} \sum_{i=1}^r \frac{V_i}{\rho^2} + \frac{r-2}{r} \sum_{i_1 < i_2} \frac{T_{i_1} T_{i_2}}{\rho^2} \\ &\quad - \frac{\sum_{i=1}^r (b_{1,i} + h_i) \sum_{i=1}^r T_i - \sum_{i=1}^r (b_{1,i} + h_i) T_i}{r} \cdot \frac{\cot(\rho\pi)}{\rho^2} \quad (17) \end{aligned}$$

$$-\frac{\sum_{i=1}^r a_{1,i} \sum_{i=1}^r T_i - \sum_{i=1}^r a_{1,i} T_i}{r} \cdot \frac{1}{\rho^2} - \frac{\sum_{i=1}^r (e_{1,i} + h_i c_{1,i})}{r} \\ \cdot \frac{\cot(\rho\pi)}{\rho^2} - \frac{\sum_{i=1}^r (e_{2,i} + h_i b_{1,i})}{r} \cdot \frac{1}{\rho^2} + o\left(\frac{1}{\rho^2}\right),$$

where

$$\begin{aligned} \sum_{i=1}^r T_i &= -\sum_{i=1}^r c_{1,i} + \sum_{i=1}^r (b_{1,i} + h_i) \tan(\rho\pi), \\ \sum_{i=1}^r V_i &= \sum_{i=1}^r (d_{2,i} - h_i b_{1,i}) + \sum_{i=1}^r (d_{1,i} + h_i a_{1,i}) \tan(\rho\pi), \\ \sum_{i_1 < i_2} T_{i_1} T_{i_2} &= \sum_{i_1 < i_2} [(b_{1,i_1} + h_{i_1})(b_{1,i_2} + h_{i_2}) \tan^2(\rho\pi) \\ &\quad - (b_{1,i_1} + h_{i_1})c_{1,i_2} \tan(\rho\pi) - (b_{1,i_2} + h_{i_2})c_{1,i_1} \tan(\rho\pi) \\ &\quad + c_{1,i_1}c_{1,i_2}], \\ \sum_{i=1}^r (b_{1,i} + h_i) T_i &= \sum_{i=1}^r [(b_{1,i} + h_i)^2 \tan(\rho\pi) \\ &\quad - (b_{1,i} + h_i)c_{1,i}], \end{aligned}$$

and

$$\sum_{i=1}^r a_{1,i} T_i = \sum_{i=1}^r [a_{1,i}(b_{1,i} + h_i) \tan(\rho\pi) - a_{1,i}c_{1,i}].$$

Also, on the contour Γ_N , it yields

$$\begin{aligned} \frac{\beta\chi(\rho)}{\varphi_0(\rho)} &= -\frac{\beta}{r\rho} \left[\cot(\rho\pi) - c_{1,1} \frac{\cot(\rho\pi)}{\rho} + \frac{b_{1,1} + h_1}{\rho} \right. \\ &\quad \left. + O\left(\frac{1}{\rho^2}\right) \right] \cdot \prod_{i=2}^r \left[1 - \frac{c_{1,i}}{\rho} + (b_{1,i} + h_i) \frac{\tan(\rho\pi)}{\rho} \right. \\ &\quad \left. + O\left(\frac{1}{\rho^2}\right) \right] = -\frac{\beta}{r\rho} \cot(\rho\pi) + \frac{\beta}{r} \sum_{i=1}^r c_{1,i} \frac{\cot(\rho\pi)}{\rho^2} \quad (18) \\ &\quad - \frac{\beta}{r} \sum_{i=1}^r \frac{b_{1,1} + h_1}{\rho^2} + O\left(\frac{1}{\rho^3}\right). \end{aligned}$$

Therefore, on the contour Γ_N , the following equation holds:

$$\frac{\varphi(\rho)}{\varphi_0(\rho)} = 1 + \frac{\omega_1}{\rho} + \frac{\omega_2}{\rho^2} + o\left(\frac{1}{\rho^2}\right),$$

where

$$\begin{aligned} \omega_1 &= -\frac{\sum_{i=1}^r (b_{1,i} + h_i)}{r} \cot(\rho\pi) + \frac{r-1}{r} \sum_{i=1}^r T_i \\ &\quad - \frac{1}{r} \sum_{i=1}^r a_{1,i} - \frac{\beta}{r} \cot(\rho\pi) \end{aligned}$$

$$\begin{aligned} \text{and} \\ \omega_2 &= \frac{r-1}{r} \sum_{i=1}^r V_i + \frac{r-2}{r} \sum_{i_1 < i_2} T_{i_1} T_{i_2} \\ &\quad - \frac{\sum_{i=1}^r (b_{1,i} + h_i) \sum_{i=1}^r T_i}{r} \cot(\rho\pi) \\ &\quad + \frac{\sum_{i=1}^r (b_{1,i} + h_i) T_i}{r} \cot(\rho\pi) \\ &\quad - \frac{\sum_{i=1}^r a_{1,i} \sum_{i=1}^r T_i - \sum_{i=1}^r a_{1,i} T_i}{r} \\ &\quad - \frac{\sum_{i=1}^r (e_{1,i} + h_i c_{1,i})}{r} \cot(\rho\pi) - \frac{\sum_{i=1}^r (e_{2,i} + h_i b_{1,i})}{r} \\ &\quad + \frac{\beta}{r} \sum_{i=1}^r c_{1,i} \cot(\rho\pi) - \frac{\beta}{r} \sum_{i=1}^r (b_{1,i} + h_i). \end{aligned}$$

Next, the power series expansion implies

$$\ln \frac{\varphi(\rho)}{\varphi_0(\rho)} = \frac{\omega_1}{\rho} + \frac{\omega_2 - \frac{1}{2}\omega_1^2}{\rho^2} + o\left(\frac{1}{\rho^2}\right), \quad (19)$$

where

$$\begin{aligned} -\frac{1}{2}\omega_1^2 &= -\frac{(\sum_{i=1}^r (b_{1,i} + h_i))^2}{2r^2} \cot^2(\rho\pi) \\ &\quad - \frac{\beta}{r^2} \sum_{i=1}^r (b_{1,i} + h_i) \cot^2(\rho\pi) - \frac{\beta^2}{2r^2} \cot^2(\rho\pi) \\ &\quad + \frac{r-1}{r^2} \sum_{i=1}^r (b_{1,i} + h_i) \sum_{i=1}^r T_i \cot(\rho\pi) \\ &\quad + \frac{\beta(r-1)}{r^2} \sum_{i=1}^r T_i \cot(\rho\pi) - \frac{\beta}{r^2} \sum_{i=1}^r a_{1,i} \cot(\rho\pi) \quad (20) \\ &\quad - \frac{1}{r^2} \sum_{i=1}^r a_{1,i} \sum_{i=1}^r (b_{1,i} + h_i) \cot(\rho\pi) - \frac{(r-1)^2}{2r^2} \sum_{i=1}^r T_i^2 \\ &\quad - \frac{(r-1)^2}{r^2} \sum_{i_1 < i_2} T_{i_1} T_{i_2} - \frac{(\sum_{i=1}^r a_{1,i})^2}{2r^2} \\ &\quad + \frac{r-1}{r^2} \sum_{i=1}^r T_i \sum_{i=1}^r a_{1,i}. \end{aligned}$$

The residue theorem in complex analysis tells us, on the contour Γ_N ,

$$\begin{aligned} \sum_{n \in A_N} \left[(\rho_n^{(1)})^2 - (\rho_n^{(01)})^2 \right] + \sum_{n \in A_{N'}} \sum_{i=2}^r \left[(\rho_n^{(i)})^2 - (\rho_n^{(0i)})^2 \right] \\ = \frac{1}{2\pi i} \oint_{\Gamma_N} \rho^2 \left[\frac{\varphi'(\rho)}{\varphi(\rho)} - \frac{\varphi'_0(\rho)}{\varphi_0(\rho)} \right] d\rho \quad (21) \\ = -\frac{1}{2\pi i} \oint_{\Gamma_N} 2\rho \ln \frac{\varphi(\rho)}{\varphi_0(\rho)} d\rho, \end{aligned}$$

where $\rho_n^{(i)}, \rho_n^{(0i)}$ are the zeros of the entire functions $\varphi(\rho), \varphi_0(\rho)$ inside the contour Γ_N listed with multiplicity, respectively, and

$$\begin{aligned} A_N &= \{\pm 0, \pm 1, \pm 2, \dots, \pm N\}, \\ A_{N'} &= \{0, 1, 2, \dots, N-1, -1, -2, \dots, -N\}. \end{aligned}$$

Direct calculations yield that

$$\frac{1}{2\pi i} \oint_{\Gamma_N} \frac{\tan(\rho\pi)}{\rho} d\rho = \frac{1}{2\pi i} \oint_{\Gamma_N} \frac{\cot(\rho\pi)}{\rho} d\rho = 0 \quad (22)$$

and

$$\begin{aligned} \frac{1}{2\pi i} \oint_{\Gamma_N} \tan(\rho\pi) d\rho &= -\frac{2N_0}{\pi}, \\ \frac{1}{2\pi i} \oint_{\Gamma_N} \cot(\rho\pi) d\rho &= \frac{2N+1}{\pi}. \end{aligned} \quad (23)$$

Using (22), (23), and (19), we have

$$\begin{aligned} &-\frac{1}{2\pi i} \oint_{\Gamma_N} 2\rho \ln \frac{\varphi(\rho)}{\varphi_0(\rho)} d\rho \\ &= -\frac{1}{2\pi i} \oint_{\Gamma_N} \left[2\omega_1 + \frac{2\omega_2 - \omega_1^2}{\rho} + o\left(\frac{1}{\rho}\right) \right] d\rho \quad (24) \\ &= \frac{4Nr+2}{\pi r} \sum_{i=1}^r (b_{1,i} + h_i) + \frac{4N_0+2}{r} \beta - \sum_{i=1}^r (2d_{2,i} + b_{1,i}^2) \\ &\quad + \frac{2}{r} \sum_{i=1}^r (d_{2,i} + e_{2,i}) - \sum_{i=1}^r h_i^2 - \frac{\beta^2}{r^2} + \omega_3 + o(1), \end{aligned}$$

where

$$\begin{aligned} \omega_3 &= \frac{2}{r^2} \sum_{i_1 < i_2} c_{1,i_1} c_{1,i_2} + \frac{2}{r} \sum_{i=1}^r a_{1,i} c_{1,i} - \frac{2}{r^2} \sum_{i=1}^r a_{1,i} \sum_{i=1}^r c_{1,i} \\ &\quad + \frac{(r-1)^2}{r^2} \sum_{i=1}^r c_{1,i}^2 + \frac{1}{r^2} \left(\sum_{i=1}^r a_{1,i} \right)^2 \quad (25) \\ &= \frac{1}{r^2} \left[\sum_{i=1}^r (a_{1,i} - c_{1,i}) \right]^2 + \frac{2}{r} \sum_{i=1}^r a_{1,i} c_{1,i} + \frac{r-2}{r} \sum_{i=1}^r c_{1,i}^2. \end{aligned}$$

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Substituting (24) into (21) yields

$$\begin{aligned} &\sum_{n \in A_N} \left[(\rho_n^{(1)})^2 - (\rho_n^{(01)})^2 - \frac{2}{\pi r} \sum_{i=1}^r (b_{1,i} + h_i) - \frac{2\beta}{\pi r} \right] \\ &\quad + \sum_{n \in A_{N'}} \left\{ \sum_{i=2}^r \left[(\rho_n^{(i)})^2 - (\rho_n^{(0i)})^2 \right] - \frac{2(r-1)}{\pi r} \sum_{i=1}^r (b_{1,i} + h_i) \right\} \\ &= -\frac{2}{\pi r} \sum_{i=1}^r (b_{1,i} + h_i) - \frac{2\beta}{\pi r} - \sum_{i=1}^r (2d_{2,i} + b_{1,i}^2) \\ &\quad + \frac{2}{r} \sum_{i=1}^r (d_{2,i} + e_{2,i}) - \sum_{i=1}^r h_i^2 - \frac{\beta^2}{r^2} + \omega_3 + o(1). \end{aligned} \quad (26)$$

Letting $N \rightarrow \infty$ in (26), we obtain

$$\begin{aligned} &\sum_{n \in A_1} \left[(\rho_n^{(1)})^2 - (\rho_n^{(01)})^2 - \frac{2}{\pi r} \sum_{i=1}^r (b_{1,i} + h_i) - \frac{2\beta}{\pi r} \right] \\ &\quad + \sum_{n \in A_2} \left\{ \sum_{i=2}^r \left[(\rho_n^{(i)})^2 - (\rho_n^{(0i)})^2 \right] - \frac{2(r-1)}{\pi r} \sum_{i=1}^r (b_{1,i} + h_i) \right\} \\ &= -\frac{2}{\pi r} \sum_{i=1}^r (b_{1,i} + h_i) - \frac{2\beta}{\pi r} - \sum_{i=1}^r (2d_{2,i} + b_{1,i}^2) \\ &\quad + \frac{2}{r} \sum_{i=1}^r (d_{2,i} + e_{2,i}) - \sum_{i=1}^r h_i^2 - \frac{\beta^2}{r^2} + \omega_3. \end{aligned}$$

The proof of theorem is finished.

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