On the Analytic Solution for a Steady Magnetohydrodynamic Equation

Babak Soltanalizadeh[a], Hadi Roohani Ghehsareh[b], Ahmet Yıldırım[c], and Saeid Abbasbandy[d]

[a] Young Researchers Club and Elites, Ardabil Branch, Islamic Azad University, Ardabil, Iran
[b] Young Researchers Club and Elites, Buinzahra Branch, Islamic Azad University, Buinzahra, Iran
[c] Ege University, Science Faculty, Department of Mathematics, 35100 Bornova Izmir, Turkey
[d] Department of Mathematics, Imam Khomeini International University, Ghazvin 34149, Iran

Reprint requests to B. S.; E-mail: babak.soltanalizadeh@gmail.com

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The purpose of this study is to apply the Laplace–Adomian Decomposition Method (LADM) for obtaining the analytical and numerical solutions of a nonlinear differential equation that describes a magnetohydrodynamic (MHD) flow near the forward stagnation point of two-dimensional and axisymmetric bodies. By using this method, the similarity solutions of the problem are obtained for some typical values of the model parameters. For getting computational solutions, we combined the obtained series solutions by LADM with the Padé approximation. The method is easy to apply and gives high accurate results. The presented results through tables and figures show the efficiency and accuracy of the proposed technique.

Key words: Laplace Transformation; Adomian Decomposition Method; Padé Approximation; Navier–Stokes Equations; Semi-Infinite Interval; MHD Flow.
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1. Introduction

Nonlinear phenomena, that appear in many areas of scientific fields such as solid state physics, plasma physics, fluid mechanics, population models, and chemical kinetics, can be defined by nonlinear differential equations. One of the most important kinds of these equations is the nonlinear differential equation that characterize boundary layer problems in unbounded domains.

Firstly, Sakiadis in 1961 [1] solved the problem of forced convection along an isothermal constantly moving plate which is a classical problem of fluid mechanics. Magnetohydrodynamics (MHD) is considering the interaction of conducting fluids with electromagnetic problems. The flow of an electrically conducting fluid within the magnetic field is one of the most applicable sections in various areas of engineering and technology. The viscous flow due to a stretching boundary is important in extrusion processes when sheet material is pulled out of an orifice with rasing velocity.

Therefore, since the numerical/analytical study of fluid flow across a thin liquid film is very important in many branches of science and technology, many authors paid much attention to considering the behaviour of this problem numerically and analytically. In the investigation of boundary layer problems, by applying a good variables transformation, we convert the system of the Navier–Stokes equations to a nonlinear ordinary boundary value problem with a semi-infinite interval. In [2], the infinite domain is replaced with $[-L,L]$ and the semi-infinite interval with $[0,L]$ by selecting a sufficiently large L. Guo [3] converted the problem of semi-infinite domains to a model of a bounded domain. Authors of [4 – 20] presented some other similar discussions.

The Adomian decomposition method (ADM) has been applied to a wide class of problems in physics, biology, and chemical reactions. The method provides the solution in a rapid convergent series with computable terms [21, 22]. Then by applying this method, the numerical solutions of some equations can be obtained. In this research, we will combine ADM with the Laplace transformation to get the similarity solution of an important nonlinear differential equation. This method is proposed in [23 – 25]. A combination of ADM with Padé approximations has been presented by Baker [26], and authors of [27 – 29] applied this method for obtaining the solution of some boundary layer problems which involve a boundary condition at infinity.

This paper has the following structure: converting the model of a system of nonlinear partial differential equations (PDEs) to a nonlinear ordinary differential equation is presented in Section 2. In Section 3, we apply LADM to the obtained ordinary equation from Section 2, and in Section 4, the combination of LADM with the Padé approximant is shown. Finally, the numerical results for the various values of parameters are reported by tables and figures.

2. Mathematical Formulation of the Problem

Consider an electrically conducting fluid with the transverse magnetic field $B(x)$ that is flowing past a flat plate stretched with a power-law velocity. According to the presented discussions in [30 – 32], suppose that (u,v) be the velocity components in (x,y) directions, respectively. Also σ and ν are the electrical conductivity and kinematic viscosity, respectively. Moreover, we know that $u_e(x) = cx^m$, $c > 0$, is the external velocity, and $B(x) = B_0 x^{\frac{m-1}{2}}$ is our magnetic field. Finally, suppose that m is the power-law velocity exponent, and ρ is the fluid density. Then based on the above assumptions, the corresponding phenomenon can be introduced as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e u_{ex} + \nu\frac{\partial^2 u}{\partial y^2} + \frac{\sigma B^2(x)}{\rho}(u_e - u), \tag{2}$$

subject to the following boundary conditions:

$$\begin{aligned} &u(x,0) = u_w(x) = ax^m, \; v(x,0) = v_w(x) = bx^{\frac{m-1}{2}}, \\ &u(x,\infty) = u_e(x), \end{aligned} \tag{3}$$

where a and b are constants, and $u_w(x)$ and $v_w(x)$ are the stretching and the suction (or injection) velocity.

We define the velocity components u and v by

$$(u,v) = \left(\frac{\partial \varphi}{\partial y}, -\frac{\partial \varphi}{\partial x}\right). \tag{4}$$

By using (4) in (1), (2), and (3), we have

$$\begin{aligned} &\frac{\partial \varphi}{\partial y}\frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial \varphi}{\partial x}\frac{\partial^2 \varphi}{\partial y^2} \\ &= u_e u_{ex} + \nu\frac{\partial^3 \varphi}{\partial y^3} + \frac{\sigma B^2(x)}{\rho}(u_e - u), \end{aligned} \tag{5}$$

subject to the boundary conditions

$$\begin{aligned} &\frac{\partial \varphi}{\partial y}(x,0) = ax^m, \; \frac{\partial \varphi}{\partial x}(x,0) = -bx^{\frac{m-1}{2}}, \\ &\frac{\partial \varphi}{\partial y}(x,\infty) = cx^m. \end{aligned} \tag{6}$$

We introduce $\varphi(x,y)$ as the stream function and τ as a variable given below:

$$\varphi(x,y) = x^{\frac{m+1}{2}} f(\tau)\sqrt{\nu c}, \; \tau = x^{\frac{m-1}{2}} y\sqrt{\frac{\nu}{c}}. \tag{7}$$

Now, by applying (7) in (5) and (6), we have

$$\begin{aligned} &f'''(\tau) + \frac{m+1}{2} f(\tau) f''(\tau) + m(1 - f'^2(\tau)) \\ &+ M^2(1 - f'(\tau)) = 0 \end{aligned} \tag{8}$$

with boundary conditions

$$f(0) = \alpha, \; f'(0) = \beta, \; \lim_{\tau\to\infty} f'(\tau) = 1, \tag{9}$$

where

$$\alpha = \frac{2b}{(m+1)\sqrt{\nu c}}, \; \beta = \frac{a}{c}, \; M = B_0\sqrt{\frac{\sigma}{c\rho}},$$

and M^2 is the Hartmann number. The analytical discussions of the above equation have been presented in [32]. However, to our knowledge, this paper may be the first attempt to apply a numerical method for obtaining some numerical results of (8) and (9).

3. Application of the Laplace Adomian Decomposition Method

In this section, firstly the Laplace transform algorithm will be employed to the nonlinear ordinary differential equation (8) with boundary conditions (9). For this purpose, we take the Laplace transformation ($\mathcal{L}$) on both sides of (8) in the presence of the boundary conditions (9). Then, we get

$$\mathcal{L}\{f(\tau)\} = \frac{1}{s^3 - M^2 s}\Big(\alpha(s^2 - M^2) + f''(0) + s\beta - \frac{m+M^2}{s} + m\mathcal{L}\{f'(\tau)^2\} - \frac{m+1}{2}\mathcal{L}\{f(\tau)f''(\tau)\}\Big). \tag{10}$$

Our main aim is now to determine the value of $f''(0)$ for different values of the parameters α, β, m, and M. Then, if we define $f''(0) = \gamma$, we can solve equation (8) subject to the initial value conditions

$$f(0) = \alpha\,,\ \ f'(0) = \beta\,,\ \ f''(0) = \gamma, \tag{11}$$

where γ is an unknown constant that had to be determined. Now, by applying the conditions (11) into (10), we get

$$\mathcal{L}\{f(\tau)\} = \frac{1}{s^3 - M^2 s}\Big(\alpha(s^2 - M^2) + \gamma + s\beta - \frac{m+M^2}{s} + m\mathcal{L}\{f'(\tau)^2\} - \frac{m+1}{2}\mathcal{L}\{f(\tau)f''(\tau)\}\Big). \tag{12}$$

By using the Laplace decomposition method [33, 34], we will be able to obtain an analytic solution of (12) in the form of an infinite series as follow:

$$f(\tau) = \sum_{n=0}^{\infty} f_n(\tau). \tag{13}$$

The components $f_n(\tau)$, for $n = 0, 1, 2, \ldots$, will be determined by an iterative algorithm. Moreover, we decomposed the nonlinear terms $f'(\tau)^2$ and $f(\tau)f''(\tau)$ by using the infinite series of the so-called Adomian polynomials [21, 22]:

$$N_1(f) = f'(\tau)^2 = \left(\sum_{n=0}^{\infty} f_n(\tau)\right)'^2 = \sum_{n=0}^{\infty} A_n\,,$$

$$N_2(f) = f(\tau)f''(\tau) = \left(\sum_{n=0}^{\infty} f_n(\tau)\right)\left(\sum_{n=0}^{\infty} f_n(\tau)\right)'' = \sum_{n=0}^{\infty} B_n\,,$$

where the Adomian polynomials A_n and B_n can be shown in the from of

$$A_n = \frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\xi^n} N_1\left(\sum_{i=0}^{\infty} \xi^i f_i(\tau)\right)\right]_{\xi=0},$$

$$B_n = \frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\xi^n} N_2\left(\sum_{i=0}^{\infty} \xi^i f_i(\tau)\right)\right]_{\xi=0}.$$

Some of the components of the above Adomian polynomials can be given by the following formulas:

$$A_0 = f_0'^2(\tau)\,,\ \ A_1 = 2f_0'(\tau)f_1'(\tau)\,,$$
$$A_2 = f_1'^2(\tau) + 2f_0'(\tau)f_2'(\tau)\,,$$

and

$$B_0 = f_0(\tau)f_0''(\tau)\,,\ \ B_1 = f_1(\tau)f_0''(\tau) + f_0(\tau)f_1''(\tau)\,,$$
$$B_2 = f_0(\tau)f_2''(\tau) + f_2(\tau)f_0''(\tau) + f_1(\tau)f_1''(\tau)\,.$$

By inserting the above results and Adomian polynomials into (12), we get

$$\mathcal{L}\left\{\sum_{n=0}^{\infty} f_n(\tau)\right\} = \frac{1}{s^3 - M^2 s}\Bigg(\alpha(s^2 - M^2) + \gamma + s\beta - \frac{m+M^2}{s} + m\mathcal{L}\left\{\sum_{n=0}^{\infty} A_n\right\} - \frac{m+1}{2}\mathcal{L}\left\{\sum_{n=0}^{\infty} B_n\right\}\Bigg). \tag{14}$$

Now notice that in the form of (14), a lot of work has to be done to compute the components $f_n(\tau)$. Therefore, we rewrite this equation to the following case, and then we will do our arithmetics based on the following formula:

$$\begin{aligned}\mathcal{L}\left\{\sum_{n=0}^{\infty} f_n(\tau)\right\} &= \frac{1}{s^3}\Bigg(\alpha(s^2 - M^2) + \gamma + s\beta - \frac{m+M^2}{s} + m\mathcal{L}\left\{\sum_{n=0}^{\infty} A_n\right\} - \frac{m+1}{2}\mathcal{L}\left\{\sum_{n=0}^{\infty} B_n\right\} + M^2 s\mathcal{L}\left\{\sum_{n=0}^{\infty} f_n(\tau)\right\}\Bigg), \\ &= K(s) + \frac{1}{s^3}\Bigg(m\mathcal{L}\left\{\sum_{n=0}^{\infty} A_n\right\} - \frac{m+1}{2} \cdot \mathcal{L}\left\{\sum_{n=0}^{\infty} B_n\right\} + M^2 s\mathcal{L}\left\{\sum_{n=0}^{\infty} f_n(\tau)\right\}\Bigg).\end{aligned} \tag{15}$$

On the other hand, if we let $K(s) = \frac{\alpha(s^2 - M^2)}{s^3} + \frac{\gamma}{s^3} + \frac{\beta}{s^2} - \frac{m+M^2}{s^4}$ represent the term arising from prescribed initial conditions, then based on the modified Laplace decomposition method [35], the function $K(s)$ can be decomposed into four parts named as $K(s) = K_0(s) + K_1(s) + K_2(s) + K_3(s)$. Hence, for obtaining the f_n,

$n \geq 0$, firstly we compare both sides of (15) and then use the inverse Laplace transform $\mathcal{L}^{-1}$. In this way, an iterative process will be obtained that gives us the values of f_n for $n = 0, 1, 2, \ldots$:

$$
\begin{aligned}
f_0 &= \mathcal{L}^{-1}\left\{\frac{1}{s^3}\left(\alpha\left(s^2 - M^2\right)\right)\right\},\\
f_1 &= \mathcal{L}^{-1}\left\{\frac{1}{s^3}\left(\beta s + M^2 s\mathcal{L}\{f_0\} + m\mathcal{L}\{A_0\}\right.\right.\\
&\quad \left.\left. - \frac{m+1}{2}\mathcal{L}\{B_0\}\right)\right\},\\
f_2 &= \mathcal{L}^{-1}\left\{\frac{1}{s^3}\left(\gamma + M^2 s\mathcal{L}\{f_1\} + m\mathcal{L}\{A_1\}\right.\right.\\
&\quad \left.\left. - \frac{m+1}{2}\mathcal{L}\{B_1\}\right)\right\},\\
f_3 &= \mathcal{L}^{-1}\left\{\frac{1}{s^3}\left(-\frac{m+M^2}{s} + M^2 s\mathcal{L}\{f_1\} + m\mathcal{L}\{A_1\}\right.\right.\\
&\quad \left.\left. - \frac{m+1}{2}\mathcal{L}\{B_1\}\right)\right\},\\
f_i &= \mathcal{L}^{-1}\left\{\frac{1}{s^3}\left(M^2 s\mathcal{L}\{f_{i-1}\} + m\mathcal{L}\{A_{i-1}\}\right.\right.\\
&\quad \left.\left. - \frac{m+1}{2}\mathcal{L}\{B_{i-1}\}\right)\right\},\\
i &= 4, 5, \ldots\,.
\end{aligned}
\tag{16}
$$

By using the inverse Laplace transform in (16), we can obtain the initial term f_0. Now, we can compute the value of f_1 by using the known value of f_0. By continuing this process, we can find the successive terms. Thus we have

$$
\begin{aligned}
f_0 &= \alpha - \frac{1}{2}M^2\alpha\tau^2,\\
f_1 &= \beta\tau + \frac{1}{2}\alpha M^2\tau^2 + \frac{1}{240}M^2\alpha(20\alpha m + 20\alpha)\tau^3\\
&\quad - \frac{1}{24}M^4\alpha\tau^4 + \frac{1}{240}M^2\alpha(3\alpha m M^2 - M^2\alpha)\tau^5,\\
f_2 &= \frac{1}{2}\gamma\tau^2 + \left(\frac{1}{6}M^2\beta - \frac{1}{12}M^2 m\alpha^2 - \frac{1}{12}M^2\alpha^2\right)\tau^3\\
&\quad + \left(-\frac{1}{96}M^2\alpha^3 - \frac{1}{16}M^2\alpha m\beta + \frac{1}{24}M^4\alpha\right.\\
&\quad \left. + \frac{1}{48}M^2\beta\alpha - \frac{1}{48}M^2\alpha^3 m - \frac{1}{96}M^2\alpha^3 m^2\right)\tau^4\\
&\quad + \left(\frac{1}{60}\alpha^2 M^4 - \frac{1}{60}\alpha^2 m M^4\right)\tau^5 + \left(\frac{1}{576}\alpha^3 M^4\right.\\
&\quad \left. - \frac{1}{480}\alpha^3 m M^4 - \frac{1}{720}M^6\alpha - \frac{11}{2880}\alpha^3 m^2 M^4\right)\tau^6\\
&\quad + \left(\frac{1}{840}M^6\alpha^2 m - \frac{1}{1260}M^6\alpha^2\right)\tau^7 + \left(-\frac{11}{161280}\right.\\
&\quad \left. \cdot\alpha^3 M^6 - \frac{3}{17920}M^6\alpha^3 m^2 + \frac{1}{3840}\alpha^3 m M^6\right)\tau^8.
\end{aligned}
$$

By obtaining the components $f_i(\tau)$ for $i = 0, 1, 2, 3, \ldots$, the approximate analytic solution of the equation can be found from (13). The approximate analytic solution for the second iteration process is

$$
\begin{aligned}
f(\tau) &= \sum_{n=0}^{2} f_n(\tau) = \alpha + \beta\tau + \frac{1}{2}\gamma\tau^2 + \frac{1}{6}M^2\tau^3\beta\\
&\quad + \left(-\frac{1}{96}M^2\alpha^3 m^2 - \frac{1}{96}M^2\alpha^3 + \frac{1}{48}M^2\beta\alpha\right.\\
&\quad \left. - \frac{1}{16}M^2\alpha m\beta - \frac{1}{48}M^2\alpha^3 m\right)\tau^4 + \left(\frac{1}{80}\alpha^2 M^4\right.\\
&\quad \left. - \frac{1}{240}\alpha^2 m M^4\right)\tau^5 + \left(\frac{1}{576}\alpha^3 M^4 - \frac{1}{480}\alpha^3 m M^4\right.\\
&\quad \left. - \frac{1}{720}M^6\alpha - \frac{11}{2880}\alpha^3 m^2 M^4\right)\tau^6 + \left(\frac{1}{840}M^6\alpha^2 m\right.\\
&\quad \left. - \frac{1}{1260}M^6\alpha^2\right)\tau^7 + \left(-\frac{11}{161280}\alpha^3 M^6\right.\\
&\quad \left. - \frac{3}{17920}M^6\alpha^3 m^2 + \frac{1}{3840}\alpha^3 m M^6\right)\tau^8.
\end{aligned}
\tag{17}
$$

From (17), it is evident that the obtained analytic solutions through LADM are power series in the independent variable. But these solutions have not the correct behaviour at infinity according tothe boundary condition $f'(\infty) = 1$, and these solutions cannot be directly applied. Hence, it is essential to combine the series solutions, obtained by LADM, with the Padé approximants to overcome this problem.

4. The LADM-Padé Approximation

As mentioned in the previous section, the obtained series solutions by the LADM (17) has not the correct behaviour at infinity according to the boundary condition $f'(\infty) = 1$, and this power series solution cannot be directly applied. To overcome this problem, here the obtained power series (17) will be approximated by a rational function, called Padé approximation. To this end, we approximate the power series (17) obtained by the Laplace–Adomian decomposition method by a rational function as follows:

$$
[S/N](\tau) = \frac{\sum_{j=0}^{S} a_j\tau^j}{1 + \sum_{j=1}^{N} b_j\tau^j}\,. \tag{18}
$$

The rational function (18) has $S + N + 1$ coefficients that we will determine. We know that, when $[S/N](\tau)$

is exactly a Padé approximation of the series solution $f(\tau)$ given by (17), then $f(\tau)-[S/N](\tau)=O(\tau^{S+N+1})$. So we can obtain the coefficients a_j and b_j by the following relations:

$$\sum_{i=0}^{j} b_i f_{j-i} = a_j\,,\quad j=0,\ldots,S\,, \tag{19}$$

$$\sum_{i=0}^{j} b_i f_{j-i} = 0\,,\quad j=S+1,\ldots,S+N\,, \tag{20}$$

where $a_k - b_k = 0$ if $k > N$. From (19) and (20), we can achieve the values for $a_i (0 \le i \le S)$ and $b_j (1 \le j \le N)$.

Note that every term of the series solution $f(\tau)$ given by (17) depends on the unknown value $\gamma = f''(0)$; so its Padé approximant depends on γ, too. Hence, to compute an accurate analytical solution of the governing problem, firstly the missing value γ should be determined with high accuracy. To get the missing value γ, we will employ the infinity boundary condition $f'(\infty)=1$. For this purpose, $f'(\tau)$ of the series solution given by (17) is approximated by the diagonal Padé approximations $[N/N](\tau)$, $(f'(\tau) \approx [N/N](\tau))$. Then, based on the infinity boundary condition, we should have $f'(\infty) \approx [N/N](\infty) = 1$. Therefore, the subtraction of the highest power in the numerator a_N and denominator b_N of the diagonal Padé approximations $[N/N](\tau)$ should vanish $(a_N - b_N = 0)$. So we would expect that there is a sequence of roots $D^N = \{a_N - b_N, N = 1,2,3,\ldots\}$ that converges towards the actual value of $\gamma = f''(0)$. The equation D^N exhibits many roots and their number increases with N. If we compare the roots of two successive sequences, we can identify the sequence of roots that converges towards the actual value of $f''(0)$. The presented approach determine the missing value $\gamma = f''(0)$ with high accuracy.

After computing the missing value γ, an accurate approximated semi-analytical solution for the governing problem can be given as Padé approximation of the series solution given by (17). Now, notice that based on the presented results in [32], the uniqueness and convergence solution of (8) subject to the boundary conditions (9) depends on the values of m, β, and M. Then recall the obtained results as follows.

Lemma 1. *Let α be any real number and $\beta > 1$. Then there exists a unique concave solution of the problem (8) and (9) in the two following cases:*

Case 1. $-1 < m \le 0$ *and* $M^2 > -m(\beta+1)$.

Case 2. $m > 0$ *and* $M^2 > -2m$.

Moreover, there exists $\alpha < l < \sqrt{\alpha^2 + 4\frac{\beta-1}{m+1}}$ such that for all $\tau \ge 0$, we have $\tau + \alpha \le f(\tau) \le \tau + l$.

Proof. See [32]. □

Lemma 2. *Let α be any real number and $0 \le \beta < 1$. Then there exists an unique convex solution of the problem (8) and (9) in the two following cases:*

Case 3. $-1 < m \le 0$ *and* $M^2 > -2m$.

Case 4. $m \ge 0$ *and* $M^2 > -m(\beta+1)$.

Proof. See [32]. □

In this section, based on the above lemmas, we will obtain the numerical result of problem (8) and (9) in the above four cases.

Example 1. Consider (8) and (9) when the model parameters satisfy Case 1. In this example, we consider two types of these model parameters. In Case 1, let $\alpha = 1$, $m = -\frac{1}{2}$, and $\beta = 2$, and on this assumption, we must have $M^2 > \frac{3}{2}$. The obtained computational values of the missing value $f''(0)$ by using the LADM–Padé approximation are presented in Table 1. Further, some of the computed similarity solutions using LADM combined with [16/16]-Padé approximations for $f(\tau)$ and $f'(\tau)$ for the viscous values of the model parameters M are shown in Figures 1 and 2, respectively. For another case of Example 1, consider the governing problem with $\alpha = -2$, $m = -\frac{3}{4}$, and $\beta = 3$. Under this condition, we must have $M^2 > 3$. The obtained results are plotted in Figures 3 and 4 for $f(\tau)$ and $f'(\tau)$, respectively.

Example 2. Consider (8) and (9) when the parameters satisfy Case 2: $\alpha = 1$, $m = 2$, and $\beta = 4$. In this case, we must have $M^2 \ge -4$. The obtained computational values of $f''(0)$ by using the LADM–Padé approximation for the viscous values of the model parameters M are reported in Table 2. Additionally, some

Table 1. Computational values of $f''(0)$ for Example 1 by choosing $\alpha = 1$, $m = -\frac{1}{2}$, $\beta = 2$, and different values of M.

	$M=\sqrt{2}$	$M=\sqrt{3}$	$M=2$	$M=3$	$M=4$	$M=\sqrt{20}$
[10, 10]	– 1.09958191	– 1.513306711	– 1.83357682	– 2.9368438037	– 3.9852628901	– 4.4743012634
[12, 12]	– 1.09942565	– 1.513282601	– 1.83357206	– 2.9368438328	– 3.9852629506	– 4.4724422437
[14, 14]	– 1.09787349	– 1.513275018	–	– 2.9368438181	– 3.9852629459	– 4.4724416716
[15, 15]	– 1.09778324	– 1.513275016	– 1.83357534	– 2.9368438173	– 3.9852629442	– 4.4724416694
[16, 16]	– 1.09778363	– 1.513275015	– 1.83357238	– 2.9368438174	– 3.9852629442	– 4.4724416693

Table 2. Numerical results of Example 2 for $f''(0)$ and several values of M.

	$M=1$	$M=\sqrt{3}$	$M=2$	$M=3$	$M=5$	$M=10$
[10, 10]	– 12.65484970	– 13.46365794	– 13.85268351	– 15.6408674939	– 20.8486905354	– 33.8353658883
[12, 12]	– 12.69014233	– 13.47234618	– 13.85317133	– 15.6402269974	– 20.2212915102	– 33.8353660148
[14, 14]	– 12.62562003	– 13.45567862	– 13.84791606	– 15.2273846080	– 20.2212926665	– 33.8353659259
[15, 15]	– 12.62619367	– 13.45613634	– 13.84847111	– 15.6402608728	– 20.2212916630	– 33.8353659256
[16, 16]	–	– 13.45681634	– 13.84931284	– 15.6402610818	– 20.2212915978	– 33.8353659256

Table 3. Numerical results of $f''(0)$ for Example 3 and several values of M.

	$M=\sqrt{2}$	$M=2$	$M=3$	$M=5$	$M=7$	$M=10$
[10, 10]	0.6269014122	0.9666115894	1.5001960642	2.525568688048	3.5362029386543	5.0440809374667
[12, 12]	0.6265451802	0.9666107317	1.5001960870	2.525568711608	3.5362029305870	5.0441208018978
[14, 14]	0.6266928866	0.9666107591	1.5001960835	2.525569165810	3.5362029305948	5.0441208020202
[16, 16]	0.6267354530	0.9666108835	1.5001960835	2.525568711197	3.5362029305861	5.0441208020159

Table 4. Numerical results of $f''(0)$ for Example 4 and several values of M.

	$M=0$	$M=\sqrt{2}$	$M=3$	$M=5$	$M=7$	$M=10$
[10, 10]	1.02114743	1.19990732	1.35348958	2.3884155638	3.1484795288	4.3189689080
[12, 12]	1.01399769	1.19944920	1.66833288	2.3865157673	3.1484793135	4.3192026648
[14, 14]	1.02482320	1.20023671	1.66833102	2.3865157257	3.1484794833	4.3192027855
[15, 15]	1.01831010	1.19977241	1.66833136	2.3865157307	3.1484794831	4.3192027854

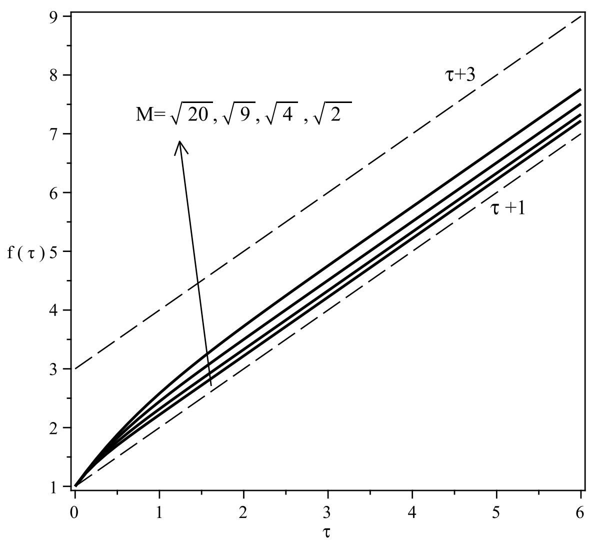

Fig. 1. Plot of LADM-Padé approximate solution of $f(\tau)$ for Example 1 by choosing $\alpha = 1$, $m = -\frac{1}{2}$, $\beta = 2$, and different values of M.

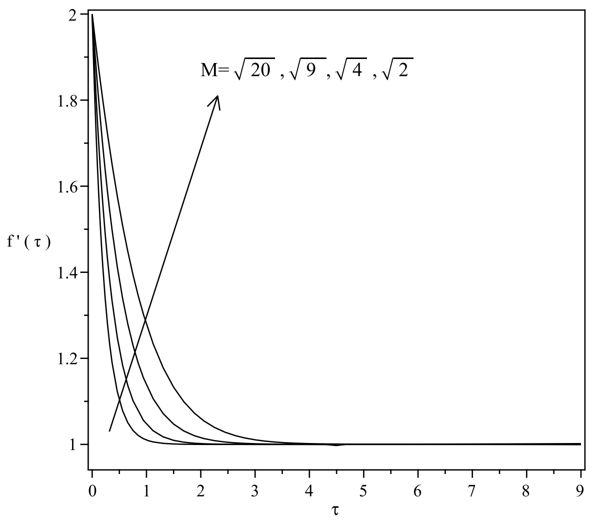

Fig. 2. Plot of LADM-Padé approximate solution of $f'(\tau)$ for Example 1 by choosing $\alpha = 1$, $m = -\frac{1}{2}$, $\beta = 2$, and different values of M.

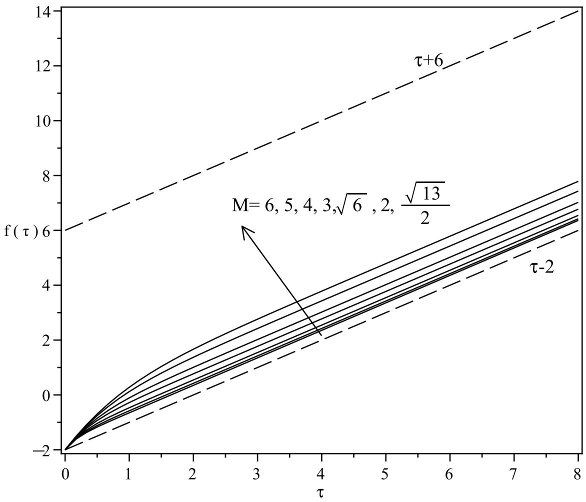

Fig. 3. Plot of LADM-Padé approximate solution of $f(\tau)$ for Example 1 with $\alpha=-2$, $m=-\frac{3}{4}$, $\beta=3$, and several values of M.

of the computed similarity solutions by using LADM coupled with $[16/16]$-Padé approximations for $f'(\tau)$ and $f(\tau)$ for the different values of M are plotted in Figures 5 and 6, respectively.

Example 3. Consider (8) and (9) when the parameters satisfy Case 3: $\alpha=1$, $m=-\frac{1}{2}$, and $\beta=\frac{1}{2}$. On this assumption, we must have $M^2\geq 1$. Some of the numerical results of $f''(0)$ by applying the LADM–Padé approximation for the model parameters M are

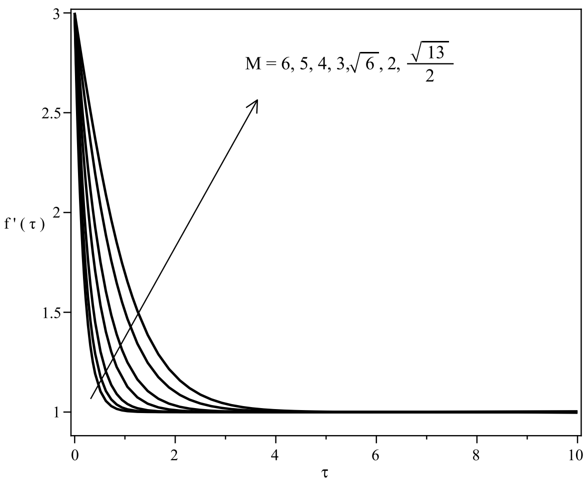

Fig. 4. Plot of LADM-Padé approximate solution of $f'(\tau)$ for Example 1 with $\alpha=-2$, $m=-\frac{3}{4}$, $\beta=3$, and several values of M.

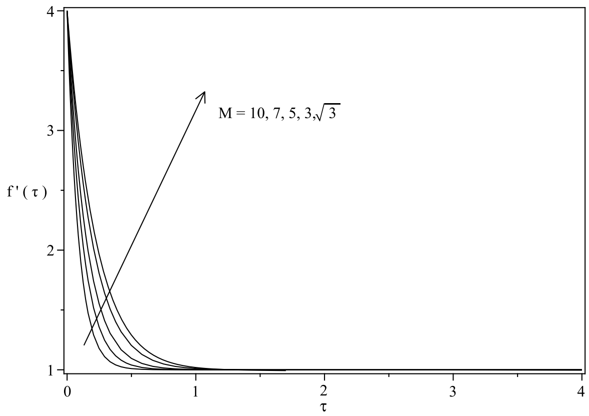

Fig. 5. Plot of LADM-Padé approximate solution of $f'(\tau)$ for Example 2 and several values of M.

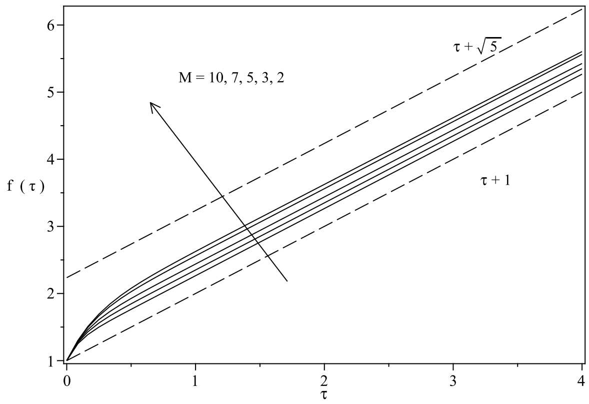

Fig. 6. Plot of LADM-Padé approximate solution of $f(\tau)$ for Example 2 and several values of M.

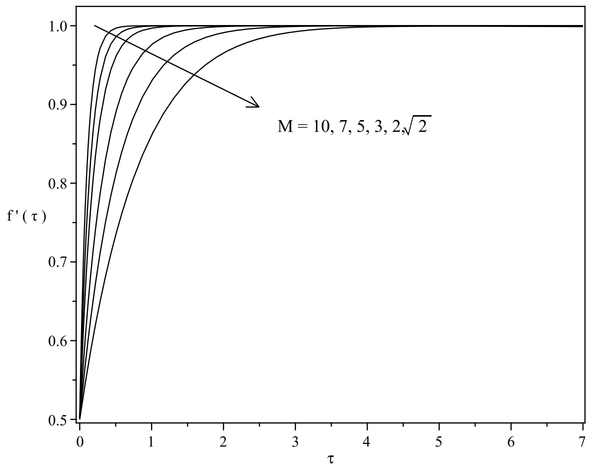

Fig. 7. Plot of LADM-Padé approximate solution of $f'(\tau)$ for Example 3 and various values of M.

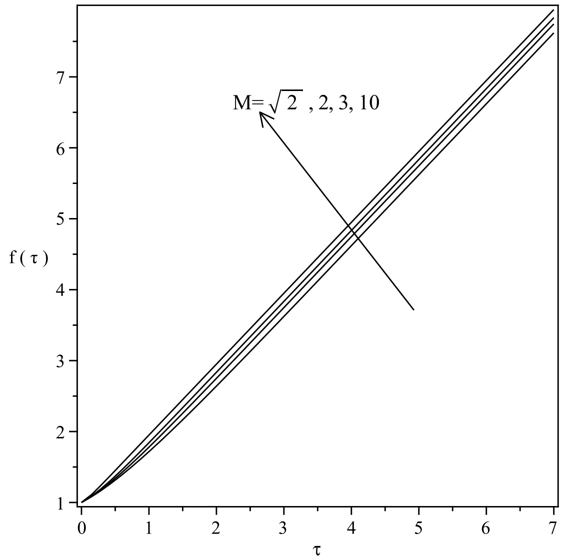

Fig. 8. Plot of LADM-Padé approximate solution of $f(\tau)$ for Example 3 and various values of M.

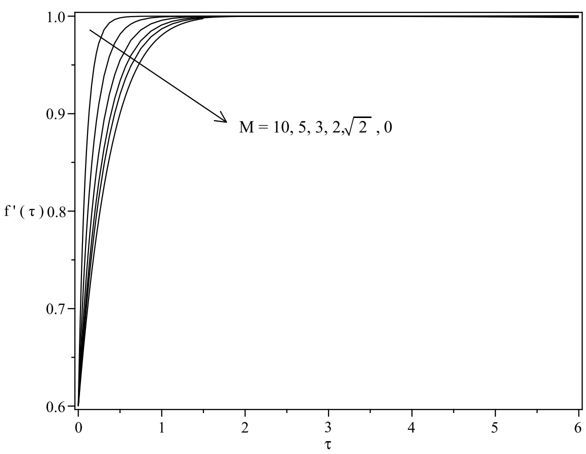

Fig. 9. Plot of LADM-Padé approximate solution of $f'(\tau)$ for Example 4 and different values of M.

reported in Table 3. Further, the computed similarity solutions for $f'(\tau)$ and $f(\tau)$ for the various values of M are shown in Figures 7 and 8.

Example 4. Consider (8) and (9) when the parameters satisfy Case 4: $\alpha = 1$, $m = \frac{3}{2}$, and $\beta = \frac{3}{5}$. For having an unique solution, we must select $M > -2.4$. The

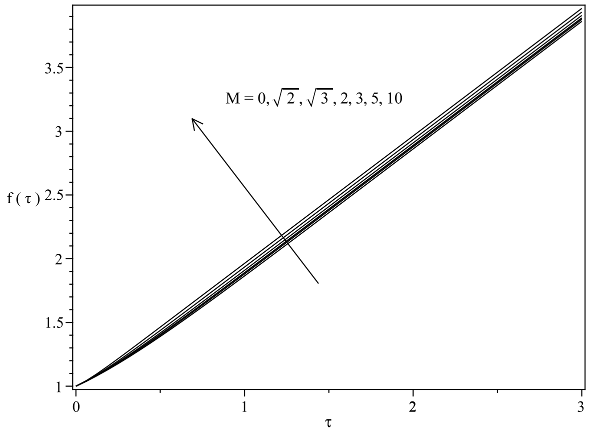

Fig. 10. Plot of LADM-Padé approximate solution of $f(\tau)$ for Example 4 and different values of M.

obtained computational values of $f''(0)$ are reported in Table 4. Also some of the computed similarity solutions by using the LADM–Padé approximations for $f'(\tau)$ and $f(\tau)$ for the viscous values of the model parameters M are plotted in Figures 9 and 10, respectively.

5. Conclusions

In this article, one of the third-order nonlinear autonomous equations subject to a boundary condition which is defined at infinity, is considered. The similarity solution is obtained by using the Laplace–Adomian decomposition method. Then, the computational results for various values of the parameters of the equation are obtained by combining LADM with Padé approximants. Based on our knowledge, this paper is the first one trying to obtain the computational results for the presented equation. The numerical results arranged in tables and figures show the accuracy of the presented process. It is evidence that this method gives high accurate results in very few iterations and can be applied to other similar equations.

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[1] B. C. Sakiadis, AIChE J. **7**, 221 (1961).
[2] J. P. Boyd, Chebyshev and Fourier Spectral Methods, second edition, Dover, New York 2000.
[3] B. Y. Guo, J. Math. Anal. Appl. **243**, 373 (2000).
[4] T. Hayat, T. Javed, and M. Sajid, Phys. Lett. A **372**, 3264 (2008).
[5] K. V. Prasad, D. Pal, and P. S. Datti, Commun. Nonlin. Sci. Numer. Simul. **14**, 2178 (2009).
[6] K. Parand, A. R. Rezaei, and S. M. Ghaderi, Commun. Nonlin. Sci. Numer. Simul. **16**, 274 (2011).
[7] K. Parand and M. Shahini, Phys. Lett. A **373**, 210 (2009).
[8] S. Abbasbandy and T. Hayat, Commun. Nonlin. Sci. Numer. Simul. **16**, 3140 (2011).
[9] R. Codina and N. Hernandez, J. Comput. Phys. **230**, 1281 (2011).
[10] F. Li and L. Xu, J. Comput. Phys. **231**, 2655 (2012).
[11] F. Li, L. Xu, and S. Yakovlev, J. Comput. Phys. **230**, 4828 (2011).
[12] J. P. Boyd, J. Comput. Phys. **69**, 112 (1987).
[13] B. Raftari and A. Yıldırım, Comput. Math. Appl. **61**, 1676 (2011).
[14] H. R. Ghehsareh, S. Abbasbandy, and B. Soltanalizadeh, Z. Naturforsch. **67a**, 248 (2012).
[15] B. Soltanalizadeh and A. Yıldırım, Z. Naturforsch. **67a**, 160 (2012).
[16] B. Soltanalizadeh, Opt. Commun. **284**, 2109 (2011).
[17] S. Abbasbandy and E. Shivanian, Commun. Nonlin. Sci. Numer. Simul. **16**, 2745 (2011).
[18] A. Yıldırım and T. Ozis, Phys. Lett. A **369**, 70 (2007).
[19] S. Abbasbandy and H. R. Ghehsareh, Int. J. Numer. Meth. Heat Fluid Flow **23**, 388 (2013).
[20] S. Abbasbandy and H. R. Ghehsareh, Int. J. Numer. Meth. Fluid **70**, 1324 (2012).
[21] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic Publishers, Dordrecht, Boston 1994.
[22] G. Adomian and R. Rach, Math. Comput. Model. **24**, 39 (1996).
[23] S. A. Khuri, J. Math. Appl. **4**, 141 (2001).
[24] M. I. Syam and A. Hamdan, Appl. Math. Comput. **176**, 704 (2006).
[25] M. Y. Ongun, Math. Comp. Model. **53**, 597 (2011).
[26] G. A. Baker, Essentials of Padé Approximants, Academic Press, London 1975.
[27] A. M. Wazwaz, Appl. Math. Comput. **177**, 737 (2006).
[28] M. Khan and M. Hussain, Numer. Algor. **56**, 211 (2011).
[29] M. Khan and M. A. Gondal, World Appl. Sci. J. **12**, 2309 (2011).
[30] P. S. Lawrence and B. N. Rao, Acta Mech. **113**, 1 (1995).
[31] J. A. Shercliff, A Textbook of Magnetohydrodynamics, Pergamon Press, Oxford 1965.
[32] J. D. Hoernel, Commun. Nonlin. Sci. Numer. Simul. **13**, 1353 (2008).
[33] S. A. Khuri, J. Appl. Math. **1**, 141 (2001).
[34] E. Yusufoglu, Appl. Math. Comput. **177**, 572 (2006).
[35] M. Khan and M. Hussain, Appl. Math. Sci. **4**, 1769 (2010).