Approximate Functional Variable Separation for the Quasi-Linear Diffusion Equations with Weak Source

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As an extension to the functional variable separation approach, the approximate functional variable separation approach is proposed, and it is applied to study the quasi-linear diffusion equations with weak source. A complete classification of these perturbed equations which admit approximate functional separable solutions is obtained. As a result, the corresponding approximate functional separable solutions to the resulting perturbed equations are derived via examples.

Key words: Quasi-Linear Diffusion Equation; Approximate Functional Separable Solution; Approximate Generalized Conditional Symmetry.
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1. Introduction

A number of methods have been used to find symmetry reductions and construct solutions of partial differential equations (PDEs) [1]. These include the classical method [2], the differential Stäckel matrix approach [3], the ansatz-based method [4], the geometrical method [5], the formal variable separation approach [6, 7], the multi-linear variable separation approach [8], the functional variable separation approach [9 – 11], and the derivative-dependent functional variable separation approach [12 – 16], etc.

In the mean while, some nonlinear equations depending on a small parameter, or perturbed PDEs arising from various fields of science, technology, and engineering, have been attracting more and more attention. For decades, quite a few methods for tackling perturbed nonlinear evolution equations have been developed, such as the approximate Lie group theory [17], the approximate symmetry method [18], the approximate conditional symmetry method [19], the approximate potential symmetry method [20], the Lie group technique [21], the approximate generalized conditional symmetry approach [22], the approximate symmetry reduction for Cauchy problems of the perturbed PDEs [23], and so on.

In [9], the authors discussed the functional variable separation issue for the quasi-linear diffusion equations with nonlinear source. Now we intend to develop the functional variable separation approach to the perturbed case. The layout of the paper is as follows: In Section 2, we define the approximate functional separable solutions (AFSSs) to the perturbed equations and present the basic theory of the approximate functional variable separation (AFVS). In Section 3, we classify the quasi-linear diffusion equations with weak source which admit AFSSs. In Section 4, we illustrate the main operating procedure for the AFVS approach with some examples. The last section involve the concluding remarks.

2. Approximate Generalized Conditional Symmetries and Approximate Functional Separable Solutions for the Perturbed Evolution Equations

Consider a kth-order differential system $[E]$, which is perturbed up to the first order in the small parameter ε, viz.

$$\begin{aligned} &E^\beta(x,u,u_{(1)},\dots,u_{(k)};\varepsilon) \equiv E_0^\beta(x,u,u_{(1)},\dots,u_{(k)}) \\ &+\varepsilon E_1^\beta(x,u,u_{(1)},\dots,u_{(k)}) = 0\,, \\ &\beta = 1,\dots,q\,, \end{aligned} \tag{1}$$

where $x=(x^1,x^2,\dots,x^n)$, $u=(u^1,u^2,\dots,u^m)$, E_i^β are smooth functions in their arguments, ε a small parameter, $u_{(i)}$ $(i=1,\dots,k)$ is the collection of ith-order partial derivatives, and

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, \ldots, n,$$

denotes the operator of total derivative with respect to x^i.

Definition 1. An operator

$$\begin{aligned}\chi = \xi^i(x, u, u_{(1)}, \ldots, u_{(k)}; \varepsilon) \frac{\partial}{\partial x^i} \\ + \eta^\alpha(x, u, u_{(1)}, \ldots, u_{(k)}; \varepsilon) \frac{\partial}{\partial u^\alpha}\end{aligned} \tag{2}$$

(summation on i and α) is the first-order approximate generalized conditional symmetry (AGCS) of (1), if

$$\chi^{[k]}(E^\beta)|_{[W]\bigcap[E]} = O(\varepsilon^2), \tag{3}$$

where

$$\chi = X_0 + \varepsilon X_1, \quad \chi^{[k]} = X_0^{[k]} + \varepsilon X_1^{[k]}, \tag{4}$$

and

$$\begin{aligned} X_b &= \xi_b^i \frac{\partial}{\partial x^i} + \eta_b^\alpha \frac{\partial}{\partial u^\alpha}, \\ X_b^{[k]} &= X_b + \zeta_{b,i}^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{b,i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} \\ &\quad + \cdots + \zeta_{b,i_1 i_2 \cdots i_k}^\alpha \frac{\partial}{\partial u_{i_1 i_2 \cdots i_k}^\alpha}, \quad b = 0, 1, \end{aligned} \tag{5}$$

in which ξ_b^i and η_b^α are functions of $(x, u, u_{(1)}, \ldots, u_{(k)})$. The additional coefficients are determined by

$$\begin{aligned}\zeta_{b,i_1 i_2 \cdots i_s}^\alpha = D_{i_1} D_{i_2} \cdots D_{i_s}(W_b^\alpha) \\ + \xi_b^j u_{j i_1 i_2 \cdots i_s}^\alpha, \quad s = 1, \ldots, n,\end{aligned} \tag{6}$$

where W_b^α is the characteristic defined by

$$W_b^\alpha = \eta_b^\alpha - \xi_b^j u_j^\alpha, \quad \alpha = 1, \ldots, m. \tag{7}$$

X_b are the generalized symmetry operators. Moreover, $[E]$ is the solution manifold of (1), and $[W]$ denotes the following system, namely

$$\begin{aligned} W_0^\alpha + \varepsilon W_1^\alpha &= (\eta_0^\alpha - \xi_0^i u_i^\alpha) + \varepsilon(\eta_1^\alpha - \xi_1^i u_i^\alpha) \\ &= O(\varepsilon^2), \end{aligned} \tag{8}$$

$$\partial_{i_1} \cdots \partial_{i_s}(W_0^\alpha + \varepsilon W_1^\alpha) = O(\varepsilon^2), \tag{9}$$

where $\partial_{i_s} = \partial/\partial x^{i_s}$, $i_s = 1, \ldots, n$. Expression (8) is the invariant surface condition of the system $[E]$, while the set of surface conditions (9) are just different-order derivatives of (8).

Suppose (1) admits the AGCS generated by χ, then the solution

$$u^i \approx U_0^i + \varepsilon U_1^i, \quad i = 1, \ldots, m, \tag{10}$$

is an approximate invariant solution of (1) under a one-parameter subgroup generated by χ if system $[W]$ holds together with (1). Thus an approximate solution can be determined by solving the invariant surface conditions (8), (9), and (1).

In particular, for a perturbed $(1+1)$-dimensional nonlinear evolution equation, we have the following definition.

Definition 2. The evolutionary vector field

$$V = \eta \frac{\partial}{\partial u} \equiv \eta(x, t, u; \varepsilon) \frac{\partial}{\partial u} \tag{11}$$

or $\eta = \eta(x, t, u; \varepsilon)$ is said to be an AGCS of the perturbed nonlinear evolution equation

$$u_t = K(x, t, u; \varepsilon) \tag{12}$$

if and only if

$$V^{(k)}(u_t - K(x, t, u; \varepsilon))|_{[W]\bigcap[E]} = O(\varepsilon^2), \tag{13}$$

whenever $u_t = K(x, t, u; \varepsilon)$, where $V^{(k)}$ denotes the kth-order prolongation to (11), K and η are differentiable functions of t, x and $u, u_x, u_{xx}, \ldots$, $[W]$ indicates the set of all differential consequences of $\eta = O(\varepsilon^2)$ with respect to x, that is, $D_x^j \eta = O(\varepsilon^2)$, $j = 0, 1, 2, \ldots$.

Proposition 1. Equation (12) admits the AGCS (11) if there exists a function $S(x, t, u, \eta)$ such that

$$\begin{aligned}\frac{\partial \eta}{\partial t} + [K, \eta] = S(x, t, u, \eta) + O(\varepsilon^2), \\ S(x, t, u, O(\varepsilon^2)) = O(\varepsilon^2),\end{aligned} \tag{14}$$

where $[K, \eta] = \eta' K - K' \eta$, the prime denotes the Fréchet derivative, and S is an analytic function of x, t, u, u_1, ..., and η, $D_x \eta$, $D_x^2 \eta$, ...

It follows from (14) that (12) admits AGCS (11) if and only if

$$D_t \eta|_{[W]\bigcap[E]} = O(\varepsilon^2). \tag{15}$$

Definition 3. The approximate solution $u = u(x, t; \varepsilon)$ of (12) is said to be an approximate functional separable solution (AFSS) if there exist some functions f, g, ψ, ϕ, ω, and θ of their arguments such that

$$f(u)+\varepsilon g(u)=\psi(x)+\phi(t)+\varepsilon(\omega(x)+\theta(t))+O(\varepsilon^2). \tag{16}$$

For brevity, we set $f \equiv f(u)$, $g \equiv g(u)$. Suppose $|\varepsilon f'/g'|<1$, differentiating (16) with respect to x and t, then expanding it into power series in ε, we get

$$u_{xt}+\left[(\ln(f'))'+\varepsilon\left(\frac{g'}{f'}\right)'\right]u_x u_t=O(\varepsilon^2), \tag{17}$$

where the prime denotes first-order derivative with respect to u. Then we have the following statement:

Theorem 1. *Equation (12) possesses AFSS (16) if and only if it admits the AGCS*

$$V=\eta\frac{\partial}{\partial u}\equiv\left[u_{xt}+(p(u)+\varepsilon q(u))\,u_x u_t\right]\frac{\partial}{\partial u}, \tag{18}$$

where

$$p(u)=(\ln(f'))',\quad q(u)=\left(\frac{g'}{f'}\right)'. \tag{19}$$

To perform the approximate functional variable separation (AFVS) approach, as an application, we are mainly concerned with the $(1+1)$-dimensional quasi-linear diffusion equation with weak source

$$u_t=(A(u)u_x)_x+\varepsilon F(u), \tag{20}$$

where $A(u)\neq 0$ and $F(u)\neq 0$ are arbitrary functions to be fixed, ε is a small parameter. First, we classify (20) which admits AGCS in the form

$$\eta=u_{xt}+(p(u)+\varepsilon q(u))u_x u_t\,. \tag{21}$$

Then we show how to construct AFSSs to the resulting perturbed quasi-linear diffusion equations with its AGCSs in the classification theorem by way of examples.

If (20) admits AFSS (16), then the following perturbed equations

$$v_t=L(v)v_{xx}+Q(v)v_x^2+\varepsilon M(v) \tag{22}$$

and

$$w_t=G(w)w_{xx}+\varepsilon H(w) \tag{23}$$

also have AFSSs. In fact, (20), (22), and (23) are related as follows:

If we put $u=u(v)$ in (20), by comparison with (22) and calculation, we get the following relation between (20) and (22):

$$\begin{aligned} u(v)&=\int^v\left[\frac{1}{L(v)}\exp\left(\int\left(\frac{Q(v)}{L(v)}\right)\mathrm{d}v\right)\right]\mathrm{d}v\,,\\ A(u)&=L(v)\,,\quad F(u)=M(v)\frac{\mathrm{d}u}{\mathrm{d}v}\,.\end{aligned} \tag{24}$$

If we substitute $w=w(v)$ into (23), using (22) and calculating, we obtain the relation between (22) and (23) as

$$\begin{aligned} w(v)&=\int^v\left[\exp\left(\int\left(\frac{Q(v)}{L(v)}\right)\mathrm{d}v\right)\right]\mathrm{d}v\,,\\ G(w)&=L(v)\,,\quad H(w)=M(v)\frac{\mathrm{d}w}{\mathrm{d}v}\,.\end{aligned} \tag{25}$$

In the same way, it is possible to relate (23) and (20) with

$$\begin{aligned} w(u)&=\int^u A(u)\,\mathrm{d}u\,,\quad G(w)=A(u)\,,\\ H(w)&=F(u)A(u)\,,\quad w=w(u)\,.\end{aligned} \tag{26}$$

Moreover, if (22) admits AFSSs for any function $v=k(u)$, then the perturbed equation

$$u_t=\tilde{L}(u)u_{xx}+\tilde{Q}(u)u_x^2+\varepsilon\tilde{M}(u)$$

also possesses AFSSs, where

$$\begin{aligned}\tilde{L}(u)&=L(k(u))\,,\\ \tilde{Q}(u)&=\frac{L(k(u))\ddot{k}+Q(k(u))\dot{k}^2}{\dot{k}}\,,\\ \tilde{M}(u)&=\frac{M(k(u))}{\dot{k}}\,,\end{aligned}$$

and $\dot{k}=\mathrm{d}k/\mathrm{d}u$, $\ddot{k}=\mathrm{d}^2k/\mathrm{d}u^2$.

So it is sufficient to study the AFSSs to (20).

3. Classification of (20) which admits AGCS (21)

Now we apply the AFVS approach to deal with the classification problem of (20) which admits AGCS (21). The algorithm for calculating AGCSs of nonlinear evolution equations can be found in [22]. By the definition of AGCS and (15), after straightforward calculation, we find that (20) admits AGCS (21) if and only if

$$\begin{aligned}D_t\eta|_{[W]\bigcap[E]} = \varepsilon\Big[&\Omega_0 u_x u_{xx}^2 + (\Omega_1 u_x^3 + \Omega_2 u_x)u_{xx}\\ &+ \Omega_3 u_x^5 + \Omega_4 u_x^3\Big] + \Lambda_0 u_x u_{xx}^2\\ &+ \Lambda_1 u_x^3 u_{xx} + \Lambda_2 u_x^5 = O(\varepsilon^2),\end{aligned} \tag{27}$$

where $\Omega_i \equiv \Omega_i(u)$, $\Lambda_j \equiv \Lambda_j(u)$ $(i = 0,1,\ldots,4,\ \ j = 0,1,2)$ depend on $A(u)$, $F(u)$, $p(u)$, $q(u)$, and their derivatives with respect to u. Decomposing (27) yields the following over-determined system of ordinary differential equations (ODEs):

$$\Omega_0 = -3qA^2A' + 2(2pq - q')A^3 = O(\varepsilon), \tag{28}$$

$$\begin{aligned}\Omega_1 = &-qA^2A'' - 4qA(A')^2 + 2(3pq - 2q')A^2A'\\ &+ (2p'q + 2pq' - q'')A^3 = O(\varepsilon),\end{aligned} \tag{29}$$

$$\begin{aligned}\Omega_2 = &\ 3FAA'' - 3F(A')^2 - (4pF + F')AA'\\ &+ (F'' + pF' + F(2p^2 - p'))A^2 = O(\varepsilon),\end{aligned} \tag{30}$$

$$\begin{aligned}\Omega_3 = &-qAA'A'' - q(A')^3 + 2(pq - q')A(A')^2\\ &+ (2p'q + 2pq' - q'')A^2A' = O(\varepsilon),\end{aligned} \tag{31}$$

$$\begin{aligned}\Omega_4 = &\ FAA''' - (pA + A')FA'' - (F' + 2pF)(A')^2\\ &+ (F'' + pF' + (p^2 - p')F)AA'\\ &+ (2pp' - p'')FA^2 = O(\varepsilon),\end{aligned} \tag{32}$$

$$\begin{aligned}\Lambda_0 = &\ 3A^2A'' - 3A(A')^2 - 3pA^2A'\\ &+ 2(p^2 - p')A^3 = O(\varepsilon^2),\end{aligned} \tag{33}$$

$$\begin{aligned}\Lambda_1 = &\ A^2A''' + (2A' - pA)AA'' - 3(A')^3\\ &- 4pA(A')^2 + (3p^2 - 4p')A^2A'\\ &+ (2pp' - p'')A^3 = O(\varepsilon^2),\end{aligned} \tag{34}$$

$$\begin{aligned}\Lambda_2 = &\ AA'A''' - (pA + A')A'A'' - p(A')^3\\ &+ (p^2 - 2p')A(A')^2\\ &+ (2pp' - p'')A^2A' = O(\varepsilon^2),\end{aligned} \tag{35}$$

where the primes denote different-order derivatives with respect to u, respectively.

Solving (28)–(35) for unknown functions $A(u)$, $F(u)$, $p(u)$, and $q(u)$, we finally obtain the complete classification of (20) which admits AGCS (21).

Theorem 2. *Suppose $A(u)F(u) \neq 0$, then the perturbed equation*

$$u_t = (A(u)u_x)_x + \varepsilon F(u) \tag{36}$$

admits AGCS (21) if and only if it is equivalent to one of the following equations, up to first-order in ε:

$$\begin{aligned}(1)\ \ u_t = &\ c_1u^{\alpha}u_{xx} + c_1\alpha u^{\alpha-1}u_x^2\\ &+ \varepsilon(c_2u + c_3u^{\alpha+1}),\ \ \alpha \neq -\frac{6}{5},\end{aligned} \tag{37}$$

$$\eta = u_{xt} - u^{-1}u_xu_t = O(\varepsilon^2); \tag{38}$$

$$(2)\ \ u_t = (A(u)u_x)_x + \varepsilon F(u), \tag{39}$$

$$\eta = u_{xt} + p(u)u_xu_t = O(\varepsilon^2), \tag{40}$$

where $A = A(u)$, $F = F(u)$, and $p = p(u)$ satisfy the following ODEs:

$$p' - p^2 + c_1A = 0, \tag{41}$$

$$3AA'' - 3(A' + pA)A' - 2(p' - p^2)A^2 = 0, \tag{42}$$

$$AF'' + (pA - A')F' + (p'A - pA')F = 0; \tag{43}$$

$$(3)\ \ u_t = u^{-\frac{6}{5}}u_{xx} - \frac{6}{5}u^{-\frac{11}{5}}u_x^2 + \varepsilon\left(c_2u + c_3u^{-\frac{1}{5}}\right), \tag{44}$$

$$\eta = u_{xt} + \left(-u^{-1} + \varepsilon c_1u^{-\frac{1}{5}}\right)u_xu_t = O(\varepsilon^2); \tag{45}$$

$$(4)\ \ u_t = c_1u_{xx} + \varepsilon(c_2u + c_3), \tag{46}$$

$$\eta = u_{xt} + \varepsilon c_4u_xu_t = O(\varepsilon^2),\ \ c_4 \neq 0; \tag{47}$$

$$(5)\ \ u_t = c_1u_{xx} + \varepsilon(c_2u + c_3u\ln(u)), \tag{48}$$

$$\eta = u_{xt} + (u^{-1} + \varepsilon c_4u^{-2})u_xu_t = O(\varepsilon^2), \tag{49}$$

where c_i, $i = 1,\ldots,4$ are arbitrary constants, and $c_1 \neq 0$, $|c_2| + |c_3| \neq 0$.

Remark 1. By the transformations (24)–(26), we can also obtain the corresponding classification theorems for perturbed (22) and (23) which admit AFSSs.

4. Construction of AFSSs for the Resulting Equations

To construct AFSSs to the equations listed in Theorem 2, we should take three main steps:

(i) In terms of $p(u)$ and $q(u)$ from the corresponding AGCS listed in different cases of Theorem 2, we can get $f(u)$ and $g(u)$ by solving (19).

(ii) Substituting $u = u_0 + \varepsilon u_1$ into the perturbed equation and its ansatz (16), expanding them into power series in ε respectively, and equating the coefficients of ε^0 and ε^1, then the resulting two expressions can be reduced to a system of four ODEs.

(iii) Solving that system for unknown functions $\psi(x)$, $\phi(t)$, $\omega(x)$, and $\theta(t)$, an AFSS to the perturbed equation can be finally obtained via (16).

We show the way by some examples.

Example 1. To obtain an AFSS to (44), one generally intends to solve (44) with AGCS (45) and the ansatz (16).

Firstly, comparing AGCS (45) with (21), we have

$$p(u) = -u^{-1}\,, \quad q(u) = c_1 u^{-\frac{1}{5}}\,. \tag{50}$$

Substituting (50) into (19) and solving them, we find that

$$\begin{aligned} f(u) &= s_2 \ln(u) + s_1\,, \\ g(u) &= \frac{25}{16} c_1 s_2 u^{\frac{4}{5}} + s_3 \ln(u) + s_4\,, \quad s_2 \neq 0\,. \end{aligned} \tag{51}$$

Therefore, after substitution of (51), an AFSS in the form (16) reads

$$\begin{aligned} &s_2 \ln(u) + s_1 + \varepsilon\left(\frac{25}{16} c_1 s_2 u^{\frac{4}{5}} + s_3 \ln(u) + s_4\right) \\ &= \psi(x) + \phi(t) + \varepsilon(\omega(x) + \theta(t)) + O(\varepsilon^2)\,. \end{aligned} \tag{52}$$

Secondly, substituting $u = u_0 + \varepsilon u_1$ into (52), expanding it into power series in ε and vanishing of coefficients of ε^0 and ε^1, we have

$$s_2 \ln(u_0) + s_1 - \psi(x) - \phi(t) = 0\,, \tag{53}$$

$$\begin{aligned} &\frac{25}{16} c_1 s_2 u_0^{\frac{4}{5}} + s_2 \frac{u_1}{u_0} + s_3 \ln(u_0) \\ &+ s_4 - \omega(x) - \theta(t) = 0\,. \end{aligned} \tag{54}$$

Similarly, after substituting $u = u_0 + \varepsilon u_1$ into (44) and expanding it into power series in ε, the vanishing of coefficients of ε^0 and ε^1 gives, respectively, the original unperturbed equation of u_0 and the equation of u_0 and u_1 as

$$u_{0t} = u_0^{-\frac{6}{5}} u_{0xx} - \frac{6}{5} u_0^{-\frac{11}{5}} u_{0x}^2\,, \tag{55}$$

$$\begin{aligned} u_{1t} &= u_0^{-\frac{6}{5}} u_{1xx} - \frac{12}{5} u_0^{-\frac{11}{5}} u_{0x} u_{1x} \\ &+ \frac{6}{5} u_0^{-\frac{11}{5}} \left(\frac{11}{5} u_0^{-1} u_{0x}^2 - u_{0xx}\right) u_1 + c_3 u_0^{-\frac{1}{5}} + c_2 u_0\,. \end{aligned} \tag{56}$$

Solving u_0 and u_1 from (53) and (54), and substituting them into (55) and (56), making full use of the usual variable separation method, by some detailed reasoning and calculation, we attain the following ODEs regarding $\psi(x)$, $\phi(t)$, $\omega(x)$, and $\theta(t)$:

$$\left(\psi'(x)\right)^2 = \frac{1}{2}\left(\rho\, \mathrm{e}^{\frac{6}{5}\frac{\psi(x)-s_1}{s_2}} + \nu\, \mathrm{e}^{\frac{2}{5}\frac{\psi(x)-s_1}{s_2}}\right)\,, \tag{57}$$

$$5 s_2 \phi'(t) = \rho\, \mathrm{e}^{-\frac{6}{5}\frac{\phi(t)}{s_2}}\,, \quad \rho \neq 0\,, \tag{58}$$

$$\begin{aligned} &40\big(5 s_2^3 \omega''(x) - 2 s_2^2 \psi'(x) \omega'(x) + s_2 s_3 \psi'(x)^2 \\ &+ 5 c_3 s_2^4\big)\, \mathrm{e}^{\frac{6}{5}\frac{s_1-\psi(x)}{s_2}} - 48\rho\big(s_2 \omega(x) \\ &- s_3 \psi(x) - s_2 s_4 + s_1 s_3\big) = \mu\,, \end{aligned} \tag{59}$$

$$\begin{aligned} &200 s_2^3 (\theta'(t) - c_2 s_2)\, \mathrm{e}^{\frac{6}{5}\frac{\theta(t)}{s_2}} + 48\rho(s_2 \theta(t) \\ &- s_3 \phi(t)) + 75 c_1 s_2^2 \nu\, \mathrm{e}^{\frac{4}{5}\frac{\theta(t)}{s_2}} = \mu\,. \end{aligned} \tag{60}$$

Lastly, solving (57)–(60), we find that $\psi(x)$, $\phi(t)$, $\omega(x)$, and $\theta(t)$ are determined by

$$\begin{aligned} &\pm\sqrt{2}\int^{\psi(x)} \frac{1}{\sqrt{\rho\, \mathrm{e}^{\frac{6}{5}\frac{\xi-s_1}{s_2}} + \nu\, \mathrm{e}^{\frac{2}{5}\frac{\xi-s_1}{s_2}}}}\, \mathrm{d}\xi = x + a_1\,, \\ &\phi(t) = -\frac{5}{6}\ln\left(\frac{25}{6}\frac{s_2^2}{\rho(t+a_2)}\right)\,, \\ &40\bigg(5 s_2^3 \omega''(x) - 2 s_2^2 \psi'(x) \omega'(x) + s_2 s_3 \psi'(x)^2 \\ &+ 5 c_3 s_2^4\bigg)\, \mathrm{e}^{\frac{6}{5}\frac{s_1-\psi(x)}{s_2}} - 48\rho\big(s_2 \omega(x) \\ &- s_3 \psi(x) - s_2 s_4 + s_1 s_3\big) - \mu = 0\,, \\ &\theta(t) = -\frac{3}{80} \times 30^{\frac{2}{3}} c_1 \nu (s_2 \rho)^{-\frac{1}{3}} (t+a_2)^{\frac{2}{3}} \\ &\quad - \frac{5}{6} s_3 \left[\ln\left(\frac{25}{6}\frac{s_2^2}{\rho(t+a_2)}\right) + 1\right] \\ &\quad + \bigg(\frac{1}{2} c_2 s_2 t^2 + a_2 c_2 s_2 t - \frac{1}{48}\mu (s_2\rho)^{-1} t \\ &\quad + a_3\bigg)(t+a_2)^{-1}\,. \end{aligned}$$

Thus, we obtain an explicit AFSS from (16) by substituting the above expressions for functions $\psi(x)$, $\phi(t)$, $\omega(x)$, and $\theta(t)$ into (52) and solving it for u. To rule out trivial AFSSs, where and hereafter we assume that $\psi'(x)\phi'(t)\omega'(x)\theta'(t) \neq 0$.

In the same way, using the AFVS approach, some AFSSs to other equations in Theorem 2 can be determined. We display some results below.

Example 2. Equation (37) enjoys AFSSs (16), with

$$f(u) = k_2 \ln(u) + k_1\,, \quad g(u) = k_4 \ln(u) + k_3\,,$$

where $\psi(x)$, $\phi(t)$, $\omega(x)$, and $\theta(t)$ satisfy

(i) $\alpha \neq -\frac{6}{5}, 0, -2.$

$$\pm c_1(\sqrt{\alpha+2}) \int^{\psi(x)} \left(e^{\frac{2\xi(1+\alpha)}{k_2}} \right) \cdot \left(\sqrt{b_2^{\frac{1}{k_2}} + 2\lambda\, e^{\frac{\alpha k_1 + (\alpha+2)\xi}{k_2}}} \right)^{-1} d\xi = x + b_3,$$

$$\phi(t) = \frac{k_2}{\alpha} \ln \left(-\frac{k_2^2}{\alpha\lambda(t+b_1)} \right),$$

$$\Big[c_1 k_2^3 \omega''(x) + 2c_1 k_2^2 (1+\alpha) \psi'(x) \omega'(x) - c_1 k_2 k_4 (1+\alpha)(\psi'(x))^2 + c_3 k_2^4 \Big] e^{\frac{\alpha(\psi(x)-k_1)}{k_2}} - \lambda\alpha\,(k_4 \psi(x) - k_4 k_1 - k_2 \omega(x) + k_2 k_3) + \gamma = 0,$$

$$\theta(t) = \frac{k_4}{\alpha} \ln \left(-\frac{k_2^2}{\alpha\lambda(t+b_1)} \right) + \Big[2^{-1} c_2 k_2 t^2 + \left(c_2 k_2 b_1 + \alpha^{-1} \lambda^{-1} (k_4 + \gamma) \right) t + b_1 k_4 \alpha^{-1} + b_4 \Big] \times (t+b_1)^{-1}, \quad b_2 k_2 \lambda \neq 0.$$

(ii) $\alpha = -2.$

$$\pm \sqrt{\frac{k_2}{2}} \int^{\psi(x)} \frac{e^{\frac{k_1-\xi}{k_2}}}{\sqrt{\lambda\xi + v_2 k_2}} d\xi = x + v_3,$$

$$\phi(t) = -\frac{1}{2} \ln \left(\frac{k_2^2}{2c_1\lambda(t+v_1)} \right) k_2,$$

$$k_2 \Big[c_1 \left(k_2^2 \omega''(x) - 2k_2 \psi'(x) \omega'(x) + k_4 \psi''(x) \right) + c_3 k_2^3 \Big] e^{\frac{2(k_1 - \psi(x))}{k_2}} - 2c_1 k_4 \lambda (k_1 - \psi(x)) + 2c_1 k_2 \lambda (k_3 - \omega(x)) + \gamma = 0,$$

$$\theta(t) = -\frac{k_4}{2} \ln \left(\frac{k_2^2}{c_1\lambda(t+v_1)} \right) + \Big[2^{-1} c_2 k_2 t^2 + \left(2^{-1}(\ln 2 - 1) k_4 + c_2 k_2 v_1 \right) t + v_4 - 2^{-1} k_4 v_1 \Big] \times (t+v_1)^{-1}, \quad k_2 \lambda \neq 0.$$

Note that for $\alpha = 0$, (37) is linear, we needn't discuss it here.

Example 3. An AFSS to (46) is determined by (16), with

$$g(u) = \frac{1}{2} c_4 r_1 u^2 + (c_4 r_2 + r_3) u + r_4, \quad f(u) = r_1 u + r_2,$$

where $\psi(x)$, $\phi(t)$, $\omega(x)$, and $\theta(t)$ are expressed by

$$\psi(x) = \frac{1}{2} \beta c_1^{-1} x^2 + h_2 x + h_3, \quad \phi(t) = \beta t + h_1,$$

$$\omega(x) = \frac{1}{24} \beta (c_1^2 r_1)^{-1} (2c_4\beta - c_2 r_1) x^4 + \frac{1}{6} h_2 (c_1 r_1)^{-1} \cdot (2c_4\beta - c_2 r_1) x^3 + \frac{1}{2} (c_1 r_1)^{-1} \Big(c_1 c_4 h_2^2 + c_2 r_1 r_2 - c_2 r_1 h_3 - c_3 r_1^2 - \delta \Big) x^2 + h_5 x + h_6,$$

$$\theta(t) = \frac{1}{2} c_2 \beta t^2 + r_1^{-1} (c_2 r_1 h_1 - \delta) t + h_4, \quad r_1 \beta \neq 0.$$

Example 4. Some AFSSs to (48) are given by (16), with

$$f(u) = b_2 \ln(u) + b_1, \quad g(u) = b_3 \ln(u) + b_2 c_4 u^{-1} + b_4,$$

where $\psi(x)$, $\phi(t)$, $\omega(x)$, and $\theta(t)$ are determined by

(i) $$\psi(x) = -mx + b_2 \ln \left(\frac{l_3 - l_2\, e^{\frac{2mx}{b_2}}}{2m} \right),$$

$$\phi(t) = \frac{c_1 m^2}{b_2} t + l_1,$$

$$c_1 b_2^2 \omega''(x) + 2c_1 b_2 \psi'(x) \omega'(x) - c_1 b_3 (\psi'(x))^2 + c_3 b_2^2 \psi(x) - c_3 b_1 b_2^2 + c_2 b_2^3 - \sigma = 0,$$

$$\theta(t) = \frac{c_1 c_3 m^2}{2b_2} t^2 + \left(c_3 l_1 + \frac{\sigma}{b_2^2} \right) t + l_4,$$

$b_2 m \neq 0;$

(ii) $$\psi(x) = b_2 \ln \left(\frac{l_3 \cos(\frac{mx}{b_2}) - l_2 \sin(\frac{mx}{b_2})}{m} \right),$$

$$\phi(t) = -\frac{c_1 m^2}{b_2} t + l_1,$$

$$c_1 b_2^2 \omega''(x) + 2c_1 b_2 \psi'(x) \omega'(x) - c_1 b_3 (\psi'(x))^2 + c_3 b_2^2 \psi(x) - c_3 b_1 b_2^2 + c_2 b_2^3 - \sigma = 0,$$

$$\theta(t) = -\frac{c_1 c_3 m^2}{2b_2} t^2 + \left(c_3 l_1 + \frac{\sigma}{b_2^2} \right) t + l_4,$$

$b_2 m \neq 0.$

5. Concluding Remarks

In summary, we have presented the AFVS approach for the perturbed nonlinear evolution equations which admit AFSSs. By using the approach, we have classified the quasi-linear diffusion equation with weak source which admits AFSSs and shown the main solving procedure by way of examples. In general, these results cannot be obtained by the other symmetry reduction methods. This extends the scope of the approximate symmetry and the perturbation theory in some manner. It is interesting to investigate other types of perturbed PDEs in terms of the AFVS approach, and some new results will be achieved sooner or later.

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