A Nonlinear Model Arising in the Buckling Analysis and its New Analytic Approximate Solution

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An analytical nonlinear buckling model where the rod is assumed to be an inextensible column and prismatic is studied. The dimensionless parameters reduce the constitutive equation to a nonlinear ordinary differential equation which is solved using the Adomian decomposition method (ADM) through Green's function technique. The nonlinear terms can be easily handled by the use of Adomian polynomials. The ADM technique allows us to obtain an approximate solution in a series form. Results are presented graphically to study the efficiency and accuracy of the method. To the author's knowledge, the current paper represents a new approach to the solution of the buckling of the rod problem. The fact that ADM solves nonlinear problems without using perturbations and small parameters can be judged as a lucid benefit of this technique over the other methods.

Key words: Adomian Decomposition Method; Adomian Polynomials; Green's Function; Buckling Phenomena.

1. Introduction

Buckling phenomena are widely used in wave propagation in nanostructures, nanobeams, nanoarches, nanorings, nanoplates, and nanoshells [1 – 6]. For example, buckling drastically cooperate the structural integrity of nanostructures. Two types of analysis are used: One of small deflections and the other of large deflections. Mostly, the analysis is done of small deflections because the evolution of nanostructures after a buckling behaviour can not be predicted in the case of large deflection. Recently, a significant number of nonlinear differential equations arising in the mathematical buckling model have been proposed [7 – 13]. These models have been used to explain different phenomena. One of these models is mentioned for the nonlocal elasticity theory [13]. The original idea of studied this model is based on Eringen's nonlocal elasticity and Timoshenko's beam model [9, 10]. The same model was then re-examined and re-solved by Xu et al. [13] for a buckling response. The complete dimensional governing equation can be found in the original manuscript of Xu et al. [13]. Here we present and analyze the corresponding nondimensional governing equation and boundary conditions which can be written as

$$\begin{aligned}\theta'' = &-\mu^2 \sin\theta + \mu^2\delta\left(\cos\theta\theta'' - \sin\theta\theta'^2\right) + \mu^2\chi^2 \\ &\cdot\cos^{-3}\theta\left(\theta'' + 3\tan\theta\theta'^2\right) - \delta\mu^2\chi^2 \\ &\cdot\cos^{-1}\theta\Big[\theta^{(4)} + 2\tan\theta\theta'\theta''' \\ &+ \left(1 + 2\tan^2\theta\right)\theta'^2\theta'' + \tan\theta\theta''^2\Big], \\ \theta(0) = 0,\ &\theta'(1) = 0,\ \theta(1) = \alpha.\end{aligned} \tag{1}$$

Various kinds of solution methods [13 – 15] were used to handle the buckling analysis. One of these methods is the Adomian decomposition method (ADM) proposed by Adomian [16] and further developed by many eminent researchers [17 – 26]. ADM is very well suited to physical problems since it does not require unnecessary linearization, discretization or other restrictive methods and assumptions which may change the problem to be solved, sometimes seriously. The

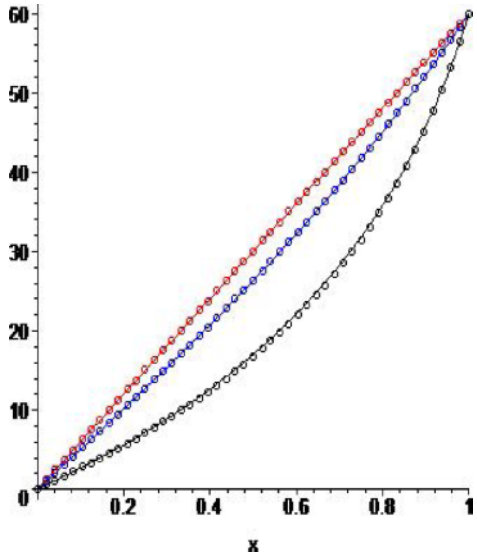

Fig. 1 (colour online). Analytic aproximate solutions ϕ_5: line, ϕ_4: circle. Parameters: $\mu = 0.1$, $\delta = 0$, $\alpha = 60$, $\chi = 0$ (red), 0.1 (blue), and 0.2 (black).

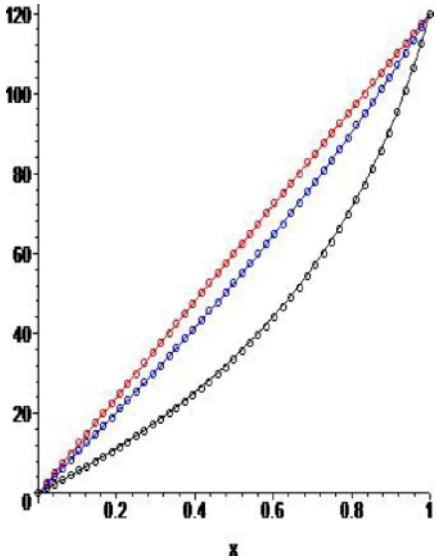

Fig. 2 (colour online). Analytic aproximate solutions ϕ_5: line, ϕ_4: circle. Parameters: $\mu = 0.05$, $\delta = 0$, $\alpha = 120$, $\chi = 0$ (red), 0.1 (blue), and 0.2 (black).

basic motivation of the present study is to propose a new approach to develop an approximate solution for the buckling phenomena equations. Inspired and motivated by the ongoing research in this area, we apply the ADM with the Green function technique for solving the governing problem. The ADM is much easier to implement as compared with the homotopy perturbation method (HPM) where huge complexities are involved. To the best of our knowledge, it seems to me that no attempt is available in the literature with the help of ADM through the Green function technique to solve a governing nonlinear model. The fact that ADM solves nonlinear problems without using perturbation theory [27–35] can be considered as a clear advantage of this technique over the perturbation method.

2. Description of the Method

In the beginning of the 1980's, Adomian [16] proposed a new and fruitful method (hereafter called the Adomian decomposition method or ADM) for solving linear and nonlinear (algebraic, differential, partial differential, integral, etc.) equations. It has been shown that this method yields a rapid convergence of the solution series to linear and nonlinear deterministic and stochastic equations. In order to elucidate the solution procedure of the ADM through the Green function technique, we consider the general nonlinear differential equation

$$\begin{aligned} &\theta''(x) + g(x,\theta) = f(x)\,, \quad a \le x \le b\,, \\ &\theta(a) = \alpha\,, \quad \theta(b) = \beta\,, \quad \alpha, \beta \in \mathbb{R}\,, \end{aligned} \tag{2}$$

where $\theta = \theta(x)$, $g(x,\theta)$ is a linear or nonlinear function of θ, and $f(x)$ is a continuous function defined in the interval. We are seeking for the solution θ satisfying (2) and assume that (2) has an unique solution.

Applying the decomposition method as in [16], (2) can be written as

$$L\theta = f(x) - N\theta\,, \tag{3}$$

where $L = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ is the linear operator and $N\theta = g(x,\theta)$ is the nonlinear operator. Consequently,

$$\theta = h(x) + \int_a^b G(x,\xi)\Big\{f(\xi) - N\theta\Big\}\,\mathrm{d}\xi\,, \tag{4}$$

where $h(x)$ is the solution of $L\theta = 0$ with the boundary conditions, and $G(x,\xi)$ is the Green function given by

$$G(x,\xi) = \begin{cases} g_1(x,\xi) & \text{if } a \le \xi \le x \le b\,, \\ g_2(x,\xi) & \text{if } a \le x \le \xi \le b\,. \end{cases} \tag{5}$$

The Adomian technique consists in approximating the solution of (4) as an infinite series

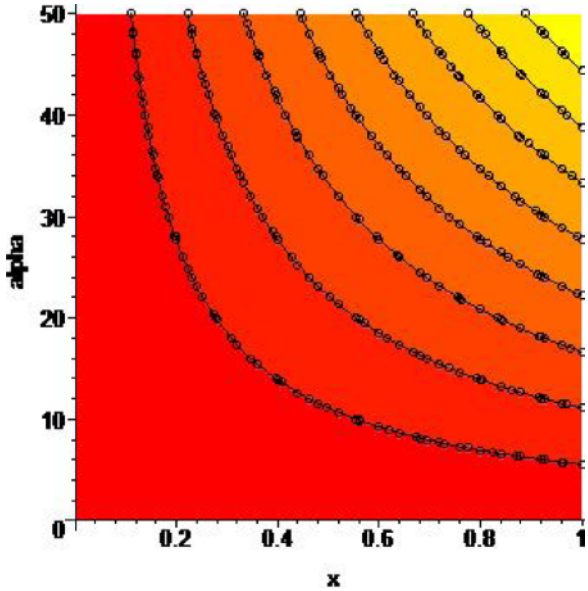

Fig. 3 (colour online). Analytic aproximate solutions ϕ_4: line, ϕ_3: circle. Parameters: $\mu = 0.1$, $\delta = 0$, $\chi = 0$.

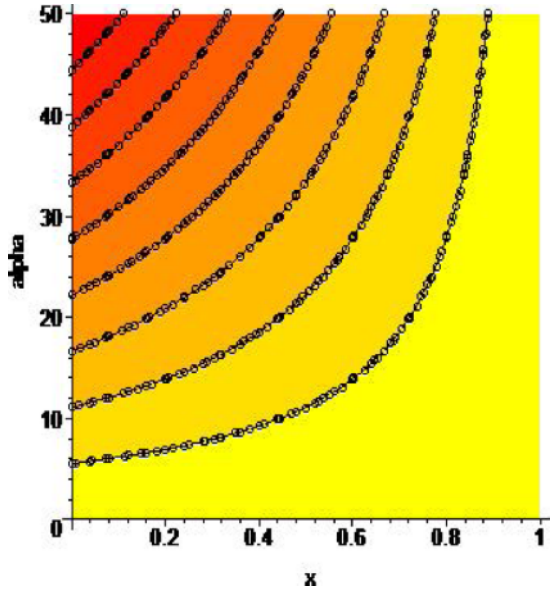

Fig. 4 (colour online). Analytic aproximate solutions $\phi_4 - \alpha$: line, $\phi_3 - \alpha$: circle. Parameters: $\mu = 0.1$, $\chi = 0$.

$$\theta = \sum_{n=0}^{\infty} \theta_n \,, \tag{6}$$

and decomposing the nonlinear operator $N\theta$ as

$$N\theta = \sum_{n=0}^{\infty} A_n \,, \tag{7}$$

where A_n are polynomials of $\theta_0, \ldots, \theta_n$ (called Adomian's polynomials [16]) given by

$$A_n = \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \left[N \left(\sum_{i=0}^{\infty} \lambda^i y_i \right) \right]_{\lambda=0} , \quad n = 0, 1, 2, \ldots . \tag{8}$$

The proofs of the convergence of the series $\sum_{n=0}^{\infty} \theta_n$ and $\sum_{n=0}^{\infty} A_n$ are given in [17]. Substituting (6) and (7) into (4) yields

$$\sum_{n=0}^{\infty} \theta_n = h(x) + \int_a^b G(x,\xi) \left\{ f(\xi) - \sum_{n=0}^{\infty} A_n \right\} \mathrm{d}\xi \,. \tag{9}$$

Thus, we can identify

$$\begin{aligned} \theta_0 &= h(x) + \int_a^b G(x,\xi) f(\xi) \, \mathrm{d}\xi \,, \\ \theta_{n+1} &= -\int_a^b G(x,\xi) A_n \, \mathrm{d}\xi \,, \quad n = 0, 1, 2, \ldots . \end{aligned} \tag{10}$$

Now all components of θ can be calculated once the A_n are given. We then define the n-term approximant to the solution θ by $\phi_n[\theta] = \sum_{i=0}^{n-1} \theta_i$ with $\lim_{n\to\infty} \phi_n[\theta] = \theta$.

3. Numerical Application

In this section, we apply the Adomian decomposition method through the Green function technique for finding the approximate solution of the studied model.

Case 1. In this case, we use the assumption

$$\sin\theta \approx \theta \,, \quad \cos\theta \approx 1 \,, \quad \tan\theta \approx \theta \,, \tag{11}$$

then (1) becomes

$$\begin{aligned} \theta'' = {} & -\mu^2\theta + \mu^2\delta \left(\theta'' - \theta\theta'^2\right) + \mu^2\chi^2 \\ & \cdot \left(\theta'' + 3\theta\theta'^2\right) - \delta\mu^2\chi^2 \Big[\theta^{(4)} \\ & + 2\theta\theta'\theta''' + \left(1 + 2\theta^2\right)\theta'^2\theta'' + \theta\theta''^2 \Big] . \end{aligned} \tag{12}$$

Expanding and collecting the terms with the same coefficients, we get

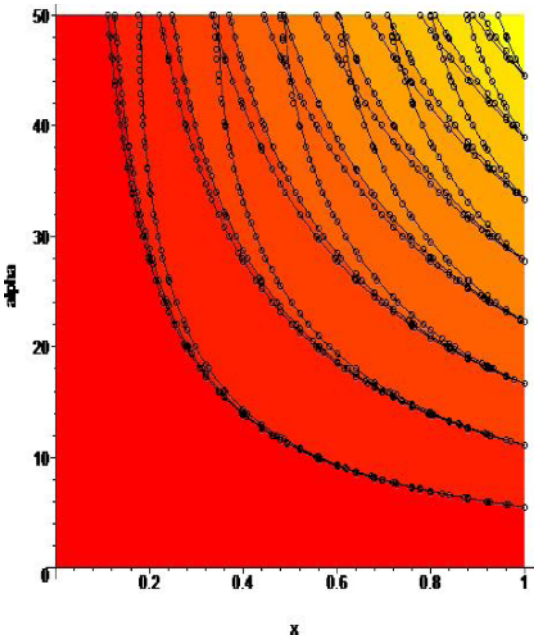

Fig. 5 (colour online). Analytic aproximate solutions ϕ_4: line, ϕ_3: circle. Parameters: $\mu = 0.1$, $\delta = 0$, $\chi = 0, 0.1, 0.2$.

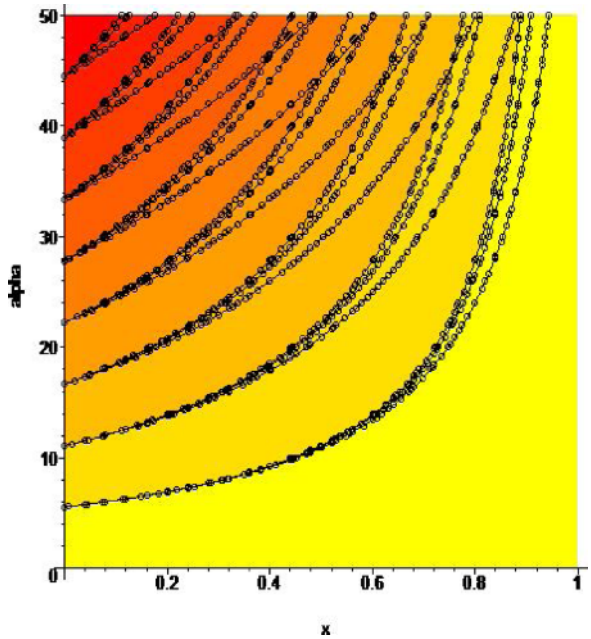

Fig. 6 (colour online). Analytic aproximate solutions $\phi_4 - \alpha$: line, $\phi_3 - \alpha$: circle. Parameters: $\mu = 0.1$, $\delta = 0$, $\chi = 0, 0.1, 0.2$.

$$\begin{aligned}\theta'' = &-\frac{\mu^2}{(1-\mu^2\delta-\mu^2\chi^2)}\theta - \frac{\mu^2}{(1-\mu^2\delta-\mu^2\chi^2)} \\ &\cdot\Big[\left(\delta-3\chi^2\right)\theta\theta'^2 + \delta\chi^2\left(\theta^{(4)}+2\theta\theta'\theta'''\right) \\ &+\delta\chi^2\left(\theta'^2\theta''+2\theta^2\theta'^2\theta''+\theta\theta''^2\right)\Big].\end{aligned} \tag{13}$$

In view of (3), (13) can be written as

$$\begin{aligned}L\theta = &-\frac{\mu^2}{(1-\mu^2\delta-\mu^2\chi^2)}\theta - \frac{\mu^2}{(1-\mu^2\delta-\mu^2\chi^2)} \\ &\cdot\Big[\left(\delta-3\chi^2\right)N_1\theta + \delta\chi^2\left(\theta^{(4)}+2N_2\theta\right) \\ &+\delta\chi^2\left(N_3\theta+2N_4\theta+N_5\theta\right)\Big],\end{aligned} \tag{14}$$

where $L = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ is the linear operator and

$$\begin{aligned}&N_1\theta = \theta\theta'^2,\ N_2\theta = \theta\theta'\theta''',\ N_3\theta = \theta'^2\theta'', \\ &N_4\theta = \theta^2\theta'^2\theta'',\ N_5\theta = \theta\theta''^2\end{aligned} \tag{15}$$

are the nonlinear operators.

Consequently,

$$\begin{aligned}\theta = ax &- \frac{\mu^2}{(1-\mu^2\delta-\mu^2\chi^2)}\int_0^1 G(x,\xi)\theta(\xi)\,\mathrm{d}\xi \\ &- \frac{\mu^2}{(1-\mu^2\delta-\mu^2\chi^2)}\int_0^1 G(x,\xi)\Big\{\left(\delta-3\xi^2\right) \\ &\cdot N_1\theta + \delta\chi^2\left(\theta^{(4)}+2N_2\theta\right) + \delta\chi^2(N_3\theta \\ &+2N_4\theta+N_5\theta)\Big\}\,\mathrm{d}\xi\,,\end{aligned} \tag{16}$$

where $G(x,\xi)$ is the Green function given by

$$G(x,\xi) = \begin{cases}(x-1)\xi & \text{if } 0 \le \xi \le x \le 1\,, \\ (\xi-1)x & \text{if } 0 \le x \le \xi \le 1\,.\end{cases} \tag{17}$$

Firstly, we set

$$\begin{aligned}&N_1\theta = \theta\theta'^2 = A_{1,n}\,,\ N_2\theta = \theta\theta'\theta''' = A_{2,n}\,, \\ &N_3\theta = \theta'^2\theta'' = A_{3,n}\,, \\ &N_4\theta = \theta^2\theta'^2\theta'' = A_{4,n}\,, \\ &N_5\theta = \theta\theta''^2 = A_{5,n}\,.\end{aligned} \tag{18}$$

Substituting (6) and (18) in (16), the iterations are then determined in the following recursive way:

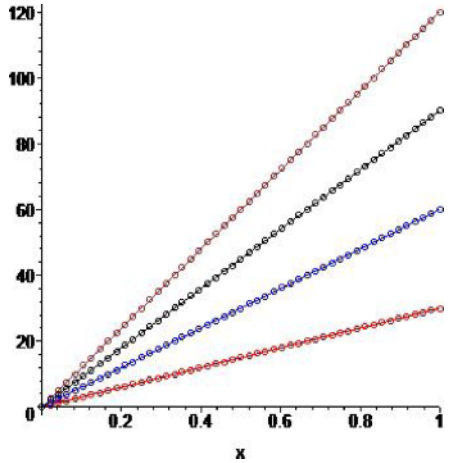

Fig. 7 (colour online). Analytic aproximate solutions ϕ_5: line, ϕ_4: circle. Parameters: $\mu = 0.1$, $\delta = 0.05$, $\chi = 0$, $\alpha = 30, 60, 90, 120$.

$$
\begin{aligned}
\theta_0 &= ax\,,\\
\theta_{n+1} &= -\frac{\mu^2}{(1-\mu^2\delta-\mu^2\chi^2)}\int_0^1 G(x,\xi)\theta_n(\xi)\,\mathrm{d}\xi\\
&\quad -\frac{\mu^2}{(1-\mu^2\delta-\mu^2\chi^2)}\int_0^1 G(x,\xi)\\
&\quad \Big\{\left(\delta-3\chi^2\right)A_{1,n}+\delta\chi^2\left(\theta_n^{(4)}+2A_{2,n}\right)\\
&\quad +\delta\chi^2\left(A_{3,n}+2A_{4,n}+A_{5,n}\right)\Big\}\,\mathrm{d}\xi\,,\\
&\quad n=0,1,2,\ldots.
\end{aligned}
\tag{19}
$$

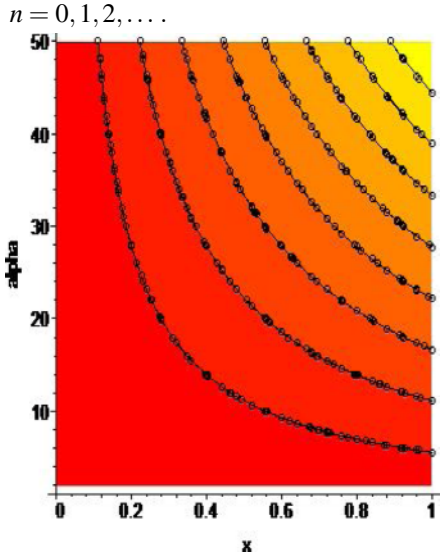

Fig. 8 (colour online). Analytic aproximate solutions ϕ_4: line, ϕ_3: circle. Parameters: $\mu = 0.1$, $\delta = 1, 2, 3$, $\chi = 0$.

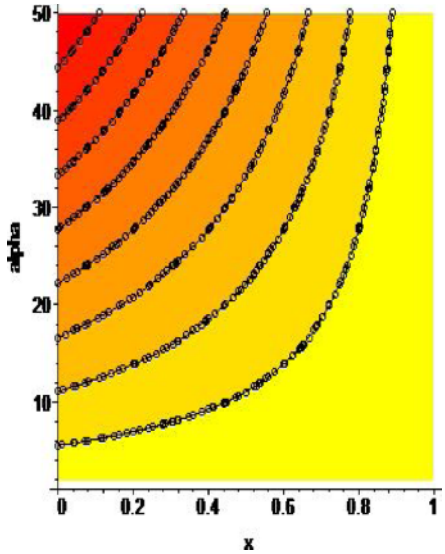

Fig. 9 (colour online). Analytic aproximate solutions $\phi_4 - \alpha$: line, $\phi_3 - \alpha$: circle. Parameters: $\mu = 0.1$, $\delta = 1, 2, 3$, $\chi = 0$.

That is, we use the functional iteration with analytical integration to compute $\theta_n(x)$. To obtain the sequence $\{\theta_n(x)\}_{n=0}^{\infty}$, we also calculate $\phi_n(x)$ in ordinary form, i. e., $\phi_n(x) = \sum_{i=0}^{n-1}\theta_i(x)$.

Case 2. In this case, we choose $\chi = 0$, and we will not consider the assumptions defined in (11):

$$
\begin{aligned}
\theta'' &= -\mu^2\sin\theta+\mu^2\delta\left(\cos\theta\theta''-\sin\theta\theta'^2\right)\\
&\quad +\mu^2\chi^2\cos^{-3}\theta\left(\theta''+3\tan\theta\theta'^2\right)\\
&\quad -\delta\mu^2\chi^2\cos^{-1}\theta\Big[\theta^{(4)}+2\tan\theta\theta'\theta'''\\
&\quad +\left(1+2\tan^2\theta\right)\theta'^2\theta''+\tan\theta\theta''^2\Big]\,,
\end{aligned}
\tag{20}
$$

subject to the same boundary conditions defined in (1),

$$
\begin{aligned}
\theta'' &= -\mu^2\sin\theta+\mu^2\delta\left(\cos\theta\theta''-\sin\theta\theta'^2\right)+\mu^2\chi^2\\
&\quad \cdot\left(\cos^{-3}\theta\theta''+3\cos^{-4}\theta\sin\theta\theta'^2\right)-\delta\mu^2\chi^2\\
&\quad \cdot\Big[\cos^{-1}\theta\theta^{(4)}+2\cos^{-2}\theta\sin\theta\theta'\theta'''\\
&\quad +\cos^{-1}\theta\left(1+2\tan^2\theta\right)\theta'^2\theta''\\
&\quad +\cos^{-2}\theta\sin\theta\theta''^2\Big]\,.
\end{aligned}
\tag{21}
$$

Applying the ADM as in [16], (21) can be written as

$$
\begin{aligned}
L\theta = &-\mu^2 N_1\theta + \mu^2\delta(N_2\theta - N_3\theta) \\
&+ \mu^2\chi^2(N_4\theta + 3N_5\theta) \\
&- \delta\mu^2\chi^2\left[N_6\theta + 2N_7\theta + N_8\theta + N_9\theta\right],
\end{aligned} \tag{22}
$$

where $L = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ is the linear operator and the nonlinear term can be decomposed as

$$
\begin{aligned}
&N_1\theta = \sin\theta\,,\ N_2\theta = \cos\theta\theta''\,,\ N_3\theta = \sin\theta\theta'^2\,, \\
&N_4\theta = \cos^{-3}\theta\theta''\,,\ N_5\theta = \cos^{-4}\theta\sin\theta\theta'^2\,, \\
&N_6\theta = \cos^{-1}\theta\theta^{(4)}\,,\ N_7\theta = \cos^{-2}\theta\sin\theta\theta'\theta'''\,, \\
&N_8\theta = \cos^{-1}\theta\left(1+2\tan^2\theta\right)\theta'^2\theta''\,, \\
&N_9\theta = \cos^{-2}\theta\sin\theta\theta''^2\,.
\end{aligned} \tag{23}
$$

From (22), we have

$$
\begin{aligned}
\theta = ax + \int_0^1 G(x,\xi)\Big\{ &-\mu^2 N_1\theta + \mu^2\delta(N_2\theta - N_3\theta) \\
&+ \mu^2\chi^2(N_4\theta + 3N_5\theta) - \delta\mu^2\chi^2\Big[N_6\theta \\
&+ 2N_7\theta + N_8\theta + N_9\theta\Big]\Big\}\,\mathrm{d}\xi\,.
\end{aligned} \tag{24}
$$

The first few components of the Adomian polynomials, for example, are given by

$$
\begin{aligned}
&N_1\theta = \sin\theta = B_{1,n}\,,\ N_2\theta = \cos\theta\theta'' = B_{2,n}\,, \\
&N_3\theta = \sin\theta\theta'^2 = B_{3,n}\,, \\
&N_4\theta = \cos^{-3}\theta\theta'' = B_{4,n}\,, \\
&N_5\theta = \cos^{-4}\theta\sin\theta\theta'^2 = B_{5,n}\,, \\
&N_6\theta = \cos^{-1}\theta\theta^{(4)} = B_{6,n}\,, \\
&N_7\theta = \cos^{-2}\theta\sin\theta\theta'\theta''' = B_{7,n}\,, \\
&N_8\theta = \cos^{-1}\theta\left(1+2\tan^2\theta\right)\theta'^2\theta''' = B_{8,n}\,, \\
&N_9\theta = \cos^{-2}\theta\sin\theta\theta''^2 = B_{9,n}\,.
\end{aligned} \tag{25}
$$

It is clear from (24), that the recursive relation is

$$
\begin{aligned}
&\theta_0 = \alpha x\,, \\
&\theta_{n+1} = \int_0^1 G(x,\xi)\Big\{ -\mu^2 B_{1,n} + \mu^2\delta\left(B_{2,n} - B_{3,n}\right) \\
&\qquad + \mu^2\chi^2\left(B_{4,n} + 3B_{5,n}\right) - \delta\mu^2\chi^2\Big[B_{6,n} \\
&\qquad + 2B_{7,n} + B_{8,n} + B_{9,n}\Big]\Big\}\,\mathrm{d}\xi\,, \\
&n = 0, 1, 2, \ldots\,.
\end{aligned} \tag{26}
$$

4. Results and Discussion

Equation (1) subject to the boundary conditions is solved analytically using the Adomian decomposition method through the Green function technique for two different cases. The results in the form of the different physical parameters μ, δ, χ, and α for two different cases are shown in Figures 1–9. From Figures 1 and 2, one can easily observe that the increasing value of free end slope α and dimensionless parameter χ reduces the buckling load, while the buckling load increases curvedly with respect to different values of α and $\chi = 0$ in Figures 3 to 4. Figures 5 to 6 demonstate the similar effect for different values of χ. The effect of α for Case 2 is shown in Figure 7. It presents a quite opposite behaviour to Figures 1 to 2, while the Figures 8 and 9 show similar behaviour for the second case as discussed in Figures 3–6. The buckling response becomse more noteworthy as the parameters μ, δ, χ, and α become larger and the magnitude of the post-buckling load remains permanent.

5. Conclusion

We have derived an analytic-approximate solution of a nonlinear buckling model. This particular problem has received a great deal of interest both from the analysis and numerical communities. However, we believe that this is the first time that an ADM solution through Green's function has been presented. The ADM procedure is straightforward to implement and provides only with a few terms a reliable analytic-approximate solution. It also avoids the difficulties and massive computational work as compared to other analytical and numerical methods. The method is applied here in a direct manner without the use of linearization, transformation, discretization or other restrictive assumptions. The analytic-approximate solution obtained by ADM are proven to be convergent and uniformly valid. This study shows that ADM coupled with the Green function technique suits for other dynamics models arising in applied sciences and engineering.

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