Elastic and Annihilation Solitons of the $(3+1)$-Dimensional Generalized Shallow Water Wave System

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With the help of the symbolic computation system Maple, the mapping approach, and a linear variable separation method, a new exact solution of the $(3+1)$-dimensional generalized shallow water wave (GSWW) system is derived. Based on the obtained solitary wave solution, some novel soliton excitations are investigated.

Key words: Mapping Approach; GSWW System; Exact Solutions; Solution Excitations.
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1. Introduction

Many dynamical problems in physics and other natural fields are usually characterized by the nonlinear evolution of partial differential equations known as governing equations. Searching for an analytical exact solution to a nonlinear system has long been an important and interesting topic in nonlinear science both for physicists and mathematicians, and various methods for obtaining exact solutions of a nonlinear system have been proposed, for example, the bilinear method, the standard Painlevé truncated expansion, the method of 'coalescence of eigenvalue' or 'wavenumbers', the homogenous balance method, the hyperbolic function method, the Jacobian elliptic method, the variable separation method, the (G'/G)-expansion method [1 – 12], and the mapping method [13 – 15], etc. The mapping approach is a kind of classic, efficient, and well-developed method to solve nonlinear evolution equations. The remarkable characteristic of which is that we can have many different ansatzes and, therefore, a large number of solutions [16 – 21]. In this paper, with the mapping approach and a linear variable separation approach, a new family of exact solutions with arbitrary functions of the $(3+1)$-dimensional generalized shallow water wave (GSWW) system is derived. Based on the derived solitary wave solution, we study some novel soliton excitations such as elastic and annihilation solitons.

The GSWW system is given by

$$u_{xxxy} + 3u_x u_{xy} + 3u_y u_{xx} - u_{yt} - u_{zx} = 0 . \qquad (1)$$

The shallow water wave system was first derived by Boiti et al. [22] as a compatibility for a 'weak' lax pair. In [23], Paquin and Winternitz showed that the symmetry algebra of a water wave system is infinite-dimensional and has a Kac–Moody–Virasoro structure. Some special similarity solutions are also given in [23] by using symmetry algebra and the classical theoretical analysis. The more general symmetry algebra, W_∞, is given in [24]. In [25], the soliton-type solutions for (1) were constructed by using a generalized tanh algorithm with symbolic computation. In [26], the travelling-wave solutions of (1) expressed by hyperbolic, trigonometric, and rational functions were established with the (G'/G)-expansion method. In [27], based on the Grammian and Pfaffian derivative formulae, Grammian and Pfaffian solutions of (1) are obtained.

2. Exact Solutions of the GSWW System

As is well known, to search for the solitary wave solutions for a nonlinear physical model, we can apply different approaches. One of the most efficient methods of finding soliton excitations of a physical model is the so-called mapping approach. The basic idea of the algorithm is as follows. For a given nonlinear partial differential equation (NPDE) with the independent

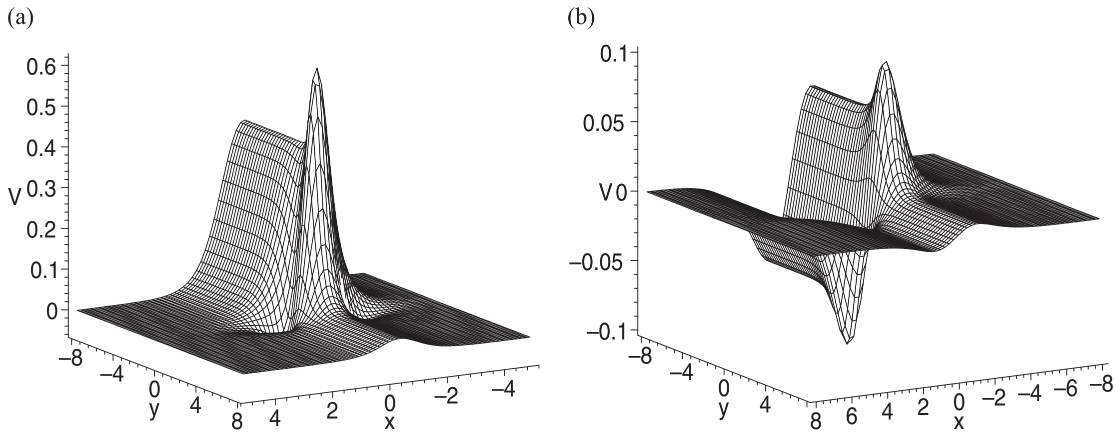

Fig. 1. Two types of soliton localized excitation for the physical quantity V given by the solution (11) with the choices (12) and (13) at time $t = 0$.

variables $x = (x_0 = t, x_1, x_2, \ldots, x_m)$ and the dependent variable u, in the form

$$P(u, u_t, u_{x_i}, u_{x_i x_j}, \ldots) = 0, \tag{2}$$

where P is in general a polynomial function of its arguments and the subscripts denote the partial derivatives, the solution can be assumed to be in the form

$$u = A(x) + \sum_{i=1}^{n} B_i(x)\phi^i q(x) + \frac{C_i(x)}{\phi^i q(x)} \tag{3}$$

with

$$\phi' = \sigma + \phi^2, \tag{4}$$

where σ is a constant and the prime denotes the differentiation with respect to q. To determine u explicitly, one may substitute (3) and (4) into the given NPDE and collect coefficients of polynomials of ϕ, then eliminate each coefficient to derive a set of partial differential equations of A, B_i, C_i, and q, and solve this system of partial differential equations to obtain A, B_i, C_i, and q. Finally, (4) possesses the general solution

$$\phi = \begin{cases} -\sqrt{-\sigma}\coth(\sqrt{-\sigma}q), & \sigma < 0, \\ -\sqrt{\sigma}\cot(\sqrt{\sigma}q), & \sigma > 0. \end{cases} \tag{5}$$

Substituting A, B_i, C_i, q, and (5) into (3), one can obtain the exact solutions of the given NPDE.

Now we apply the mapping approach to (1). By the balancing procedure, ansatz (3) becomes

$$u = f + g\phi(q) + \frac{h}{\phi(q)}, \tag{6}$$

where f, g, h, and q are functions of (x, y, t) to be determined. Substituting (6) and (4) for (1) and collecting the coefficients of the polynomials of ϕ, then setting each coefficient to zero, we have

$$\begin{aligned} f &= -\frac{1}{3}\int \Big(q_{xxx}q_y - 3q_{xy}q_{xx} - q_t q_y - 16q_x^3 q_y \sigma \\ &\quad + 3q_{xxy}q_x - q_x q_z\Big)(q_x q_y)^{-1}\,\mathrm{d}x, \\ g &= -2q_x, \quad h = 2q_x\sigma \end{aligned} \tag{7}$$

with

$$q = \chi(x,t) + \varphi(y) + bz, \tag{8}$$

where $\chi \equiv \chi(x,t)$ and $\varphi \equiv \varphi(y)$ are two arbitrary variable separation functions of (x,t) and of y, and b is an arbitrary constant. Based on the solutions of (4), we can derive the following exact solutions of (1):

Case 1. For $\sigma < 0$, we can get the following solitary wave solution of (1):

$$\begin{aligned} u_1 &= \frac{1}{3}\int \frac{\chi_t\varphi_y - \chi_{xxx}\varphi_y + 16\chi_x^3\varphi_y\sigma + b\chi_x}{\chi_x\varphi_y}\,\mathrm{d}x \\ &\quad + 2\chi_x\sqrt{-\sigma}\coth\Big(\sqrt{-\sigma}(\chi + \varphi + bz)\Big) \\ &\quad + \frac{2\chi_x\sqrt{-\sigma}}{\coth\Big(\sqrt{-\sigma}(\chi + \varphi + bz)\Big)}. \end{aligned} \tag{9}$$

Case 2. For $\sigma > 0$, we obtain the following periodic wave solution of (1):

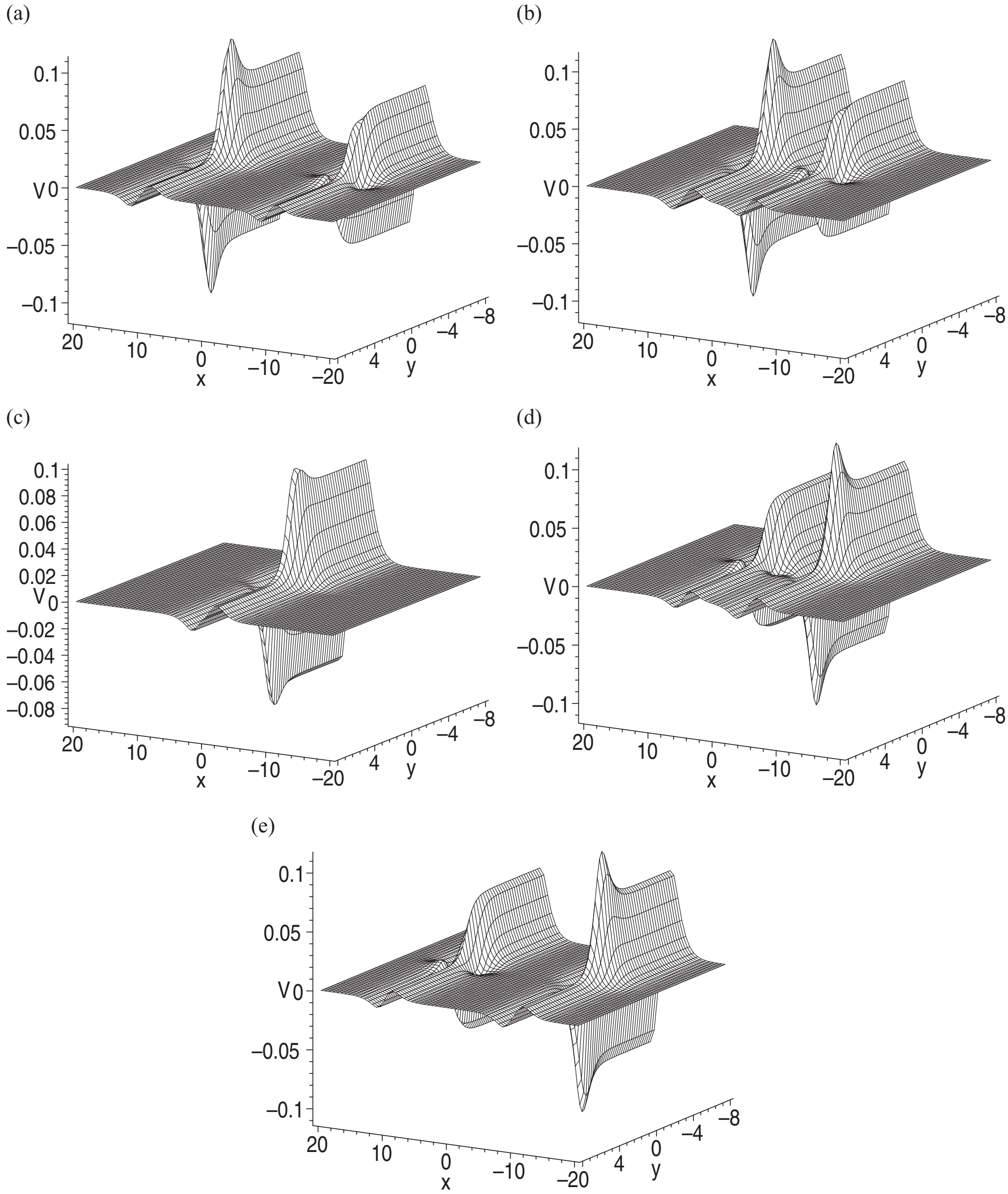

Fig. 2. Evolutional profile of two solitary waves for the solution V defined by (11) with condition (14) at different times: (a) $t=-10$; (b) $t=-5$; (c) $t=0$; (d) $t=5$; (e) $t=10$.

$$u_2 = \frac{1}{3}\int \frac{\chi_t\varphi_y - \chi_{xxx}\varphi_y + 16\chi_x^3\varphi_y\sigma + b\chi_x}{\chi_x\varphi_y}\,\mathrm{d}x + 2\chi_x\sqrt{\sigma}\cot\left(\sqrt{\sigma}(\chi+\varphi+bz)\right) - \frac{2\chi_x\sqrt{\sigma}}{\cot\left(\sqrt{\sigma}(\chi+\varphi+bz)\right)}. \tag{10}$$

3. Localized Excitations of the GSWW System

In this section, we mainly discuss the solitary solutions, namely Case 1. Owing to the arbitrariness of the functions $\chi(x,t)$ and $\varphi(y)$ included in this case, the physical quantities u may possess rich localized structures. For simplicity in the following discussion, we

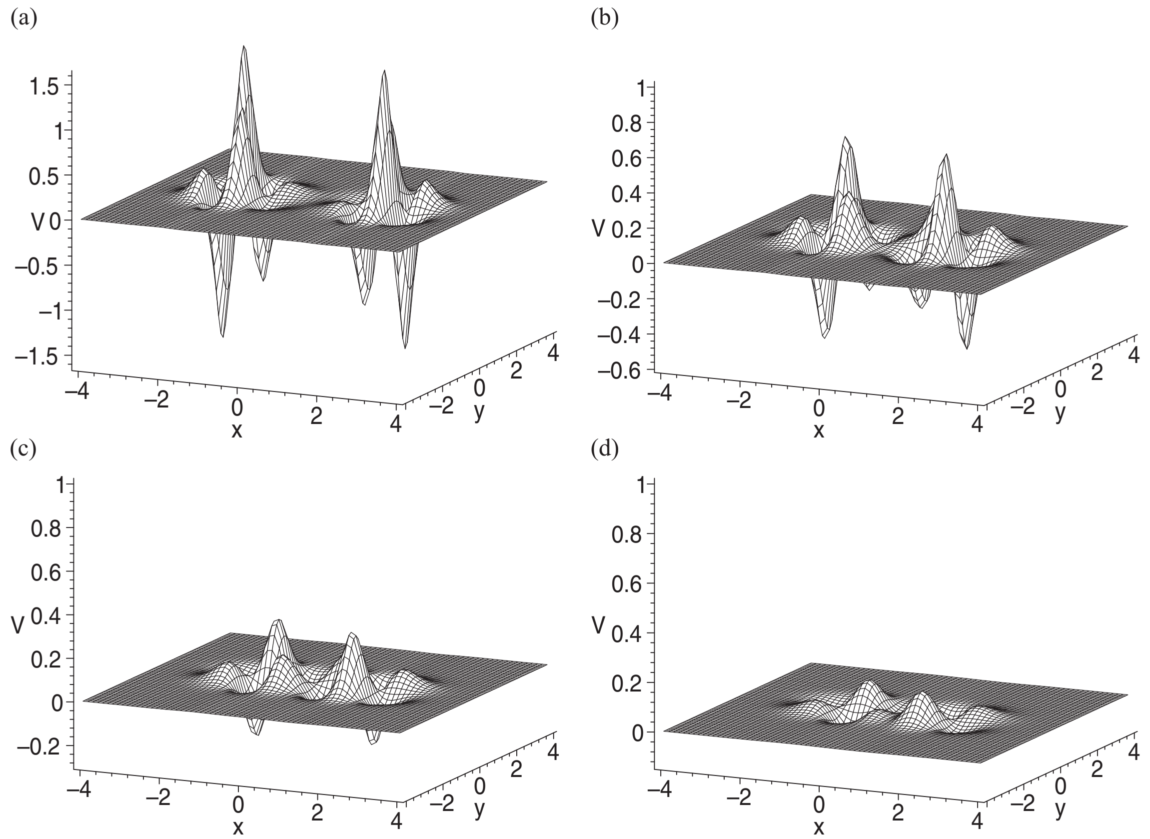

Fig. 3. A plot of the annihilation of solitons for the solution V by (11) under the condition (15) at different times: (a) $t = -3$; (b) $t = -1$; (c) $t = 0$; (d) $t = 1$.

merely analyze the potential u_{1x} ($\sigma = -1$) determined by (9), namely

$$\begin{aligned} V &= u_{2x} \\ &= -\frac{1}{3}\left(4\chi_x^2 - \frac{b}{\varphi_y} + \frac{\chi_{xxx}}{\chi_x} - \frac{\chi_t}{\chi_x}\right) + 2\chi_{xx} \\ &\quad \cdot \coth(\chi+\varphi+bz) - 2\chi_x^2 \coth^2(\chi+\varphi+bz) \\ &\quad + \frac{2\chi_{xx}}{\coth(\chi+\varphi+bz)} - \frac{2\chi_{x^2}}{\coth^2(\chi+\varphi+bz)}\,. \end{aligned} \tag{11}$$

According to the solution V in (11), we first discuss its soliton excitations. For instance, if we choose χ and φ as

$$\chi = 1 + \tanh(x+t)\,, \quad \varphi = 1 + \tanh(y)\,, \tag{12}$$

we can obtain a soliton excitation for the physical quantity V of (11) presented in Figure 1a with fixed parameters $\sigma = -1$, $b = 1$, and $z = 0.1$ at time $t = 0$.

Furthermore, if we choose χ and φ as

$$\chi = 1 + \operatorname{sech}(x+t)\,, \quad \varphi = 1 + \tanh(y)\,, \tag{13}$$

we can obtain another soliton structure for the physical quantity V of (11) presented in Figure 1b with fixed parameters $\sigma = -1$, $b = 1$, and $z = 0.1$ at time $t = 0$.

The interactions between soliton solutions of integrable models are usually considered to be completely elastic. For instance, if we choose χ and φ as

$$\begin{aligned} \chi &= 1 + \operatorname{sech}(x+t) + \operatorname{sech}(x-t)\,, \\ \varphi &= 1 + \tanh(y)\,, \end{aligned} \tag{14}$$

we can derive the time evolution of the solitary waves for the physical quantity V as presented in Figure 2 with fixed parameters $\sigma = -1$, $b = 1$, $z = 0.1$ at different times: (a) $t = -10$; (b) $t = -5$; (c) $t = 0$; (d) $t = 5$; (e) $t = 10$. From Figure 2, one finds that the interactions of the two solitary waves are completely elastic since their amplitudes, velocities, and wave shapes do not undergo any change after their collision.

However, the interactions among solitons also can be inelastic. For example, when choosing χ and φ in solution (11) to be

$$\chi = 1 + \operatorname{sech}(x^2 + t), \quad \varphi = 1 + \tanh(y^2), \qquad (15)$$

we can see that the annihilation of solitons for the physical quantity V of (11) under the condition (15) presented in Figure 3 with fixed parameters $\sigma = -1$, $b = 0.1$, and $z = 0.01$ at different times: (a) $t = -3$; (b) $t = -1$; (c) $t = 0$; (d) $t = 1$. From Figure 3, we find that the amplitude and shape of the solitons become smaller and smaller after the interaction, finally, they reduce to zero.

4. Summary and Discussion

In this paper, via the mapping approach and a linear variable separation method, we found new exact solutions of the $(3+1)$-dimensional generalized shallow water wave system. Based on the derived solitary wave solution, we studied the two types of soliton localized excitation, the completely elastic interactions between two solitons and the annihilation phenomena of solitons. Although we gave out some soliton elastic interaction and annihilation phenomena in the $(3+1)$-dimensional case, it is obvious that there are still many significant and interesting problems to be further discussed. As the authors of [28] have pointed out in $(1+1)$-dimensional cases: What are soliton elastic and nonelastic interactions? What is the general equation for the distribution of the energy and momentum after soliton interaction? How can we use the soliton annihilation of integrable models to investigate the observed soliton annihilation in the experiments? These are all pending issues to be further studied.

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