Blow-Up of Solutions for a System of Petrovsky Equations with an Indirect Linear Damping

Wenjun Liu

College of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China

Reprint requests to W. L.; E-mail: wjliu@nuist.edu.cn

Z. Naturforsch. **68a,** 343 – 349 (2013) / DOI: 10.5560/ZNA.2012-0116
Received August 6, 2012 / revised November 7, 2012 / published online January 23, 2013

In this paper, we consider a coupled system of Petrovsky equations in a bounded domain with clamped boundary conditions. Due to several physical considerations, a linear damping which is distributed everywhere in the domain under consideration appears only in the first equation whereas no damping term is applied to the second one (this is indirect damping). Many studies show that the solution of this kind of system has a polynomial rate of decay as time tends to infinity, but does not have exponential decay. For four different ranges of initial energy, we show here the blow-up of solutions and give the lifespan estimates by improving the method of Wu (Electron. J. Diff. Equ. 105, 1 (2009)) and Li et al. (Nonlin. Anal. 74, 1523 (2011)).

From the applications point of view, our results may provide some qualitative analysis and intuition for the researchers in other fields such as engineering and mechanics when they study the concrete models of Petrovsky type.

Key words: Petrovsky Systems; Blow-Up; Indirect Damping; Lifespan Estimates.
Mathematics Subject Classification 2000: 35L05, 35L57, 35B44

1. Introduction

We consider the following coupled system of Petrovsky equations in a bounded domain with clamped boundary conditions:

$$\begin{aligned}
&u_{tt}+\Delta^2 u+u_t=F_u(u,v)\,, && (x,t)\in\Omega\times[0,T)\,,\\
&v_{tt}+\Delta^2 v=F_v(u,v)\,, && (x,t)\in\Omega\times[0,T)\,,\\
&u(x,0)=u_0(x)\,,u_t(x,0)=u_1(x)\,, && x\in\Omega\,,\\
&v(x,0)=v_0(x)\,,v_t(x,0)=v_1(x)\,, && x\in\Omega\,,\\
&u(x,t)=\partial_\nu u(x,t)=0\,, && (x,t)\in\partial\Omega\times[0,T)\,,\\
&v(x,t)=\partial_\nu v(x,t)=0\,, && (x,t)\in\partial\Omega\times[0,T)\,,
\end{aligned}\tag{1}$$

where $\Omega\subset\mathbb{R}^n$ $(n\geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, ν is the unit normal vector pointing toward the exterior of Ω, $T>0$, and $F:\mathbb{R}^2\to\mathbb{R}$ is a C^1 function given by

$$F(u,v)=\alpha\,|u+v|^{r+1}+2\beta\,|uv|^{\frac{r+1}{2}}\,,$$

where $r\geq 3$, $\alpha>1$, and $\beta>0$, which implies

$$\begin{aligned}
&F_u(u,v)=(r+1)\Big[\alpha\,|u+v|^{r-1}(u+v)\\
&\qquad+\beta\,|u|^{\frac{r-3}{2}}\,|v|^{\frac{r+1}{2}}\,u\Big]\,,\\
&F_v(u,v)=(r+1)\Big[\alpha\,|u+v|^{r-1}(u+v)\\
&\qquad+\beta\,|v|^{\frac{r-3}{2}}\,|u|^{\frac{r+1}{2}}\,v\Big]\,,\\
&uF_u(u,v)+vF_v(u,v)=(r+1)F(u,v)\\
&\text{for all }\ (u,v)\in\mathbb{R}^2\,.
\end{aligned}\tag{2}$$

The physical origin of (1) lies in the study of beam and plate, and it falls within the framework of indirect damping mechanisms developed by Russell [1] in the early nineties. What makes the problem to be discussed interesting is the fact that, due to several physical considerations, the linear damping which is distributed everywhere in the domain Ω appears only in the first equation of problem (1) whereas no damping term is applied to the second one (this is the so-called indirect damping, see also [2]). Indirect damping of reversible systems occurs in many applications in engineering and mechanics. Indeed, it arises whenever it is impossible or too expensive to damp all the components of the state. Many studies show that the solution of this kind of system has a polynomial rate of decay as time tends to infinity, but does not have exponential decay (see [3–8] and references therein). Our main purpose in this work is to investigate the blow-up properties of solutions of problem (1).

We should mention that the initial-boundary value problem for Petrovsky or wave equation with linear or nonlinear damping term has been studied by many authors. For the single initial-boundary value problem

$$\begin{aligned} &u_{tt}+\Delta^2 u+g(u_t)=\beta\,|u|^{r-1}u\,, \quad (x,t)\in\Omega\times[0,T)\,,\\ &u(x,0)=u_0(x)\,,\ u_t(x,0)=u_1(x)\,,\ x\in\Omega\,,\\ &u(x,t)=\partial_\nu u(x,t)=0\,, \qquad (x,t)\in\partial\Omega\times[0,T)\,, \end{aligned} \tag{3}$$

we refer to [9–12] and the references therein. In [10], Messaoudi studied problem (3) with $g(u_t)=\alpha\,|u_t|^{p-1}u_t$ and showed that the solution blows up in finite time if $r>p$ and the energy is negative, while the solution is global if $p\geq r$. Then Wu and Tsai in [12] showed that the solution is global under some conditions without any relation between p and r. They also proved the local solution blows up in finite time if $r>p$ and the initial energy is nonnegative. In [9], Amroun and Benaissa proved the global existence of the solutions by means of the stable set method in $H_0^2(\Omega)$ combined with the Faedo–Galerkin procedure. They also studied the asymptotic behaviour of solutions when the nonlinear dissipative term g does not necessarily have a polynomial growth near the origin. For other related results of individual Petrovsky or wave equation, we refer the reader to [13–18] and the references therein. For the study of the system of nonlinear wave equations

$$\begin{aligned} &u_{tt}-\Delta u+|u_t|^{p-1}u_t=F_u(u,v)\,,\\ &v_{tt}-\Delta v+|v_t|^{q-1}v_t=F_v(u,v)\,, \end{aligned} \tag{4}$$

where $p,q\geq 1$, the reader can see [19–25] for examples. Recently, Li et al. [26] investigated global existence, uniform decay, and blow-up of solutions for the coupled system of Petrovsky equations with linear or nonlinear damping terms in both equations.

In this paper, we are interested in the blow-up behaviour of solutions for (1) in a bounded domain. For four different ranges of initial energy, we show the blow-up of solutions and give the lifespan estimates by improving the method of [25, 26]. Therefore, this work improves an earlier work [26], in which similar results have been established for (1) but in the presence of the damping terms in both equations. From the applications point of view, our results may provide some qualitative analysis and intuition for the researchers in other fields such as engineering and mechanics when they study the concrete models of Petrovsky type.

Our paper is organized as follows. In Section 2, we present some notation and state the main result. The proof of the main result is given in Section 3.

2. Preliminaries

In this section, we present some notation and state the main result. We use the standard Lebesgue space $L^p(\Omega)$ and the Sobolev space $H_0^2(\Omega)$ with their usual scalar product and norms.

We define the following functionals:

$$\begin{aligned} J(t):=J(u(t),v(t))=\frac{1}{2}\int_\Omega\Big[&|\Delta u|^2+|\Delta v|^2\\ &-2\alpha\,|u+v|^{r+1}-4\beta\,|uv|^{\frac{r+1}{2}}\Big]\,\mathrm{d}x\,, \end{aligned} \tag{5}$$

$$\begin{aligned} E(t)&=\frac{1}{2}\left(\|u_t(t)\|_2^2+\|v_t(t)\|_2^2\right)+J(u(t),v(t))\\ &=\frac{1}{2}\int_\Omega\Big[u_t^2+v_t^2+|\Delta u|^2+|\Delta v|^2\\ &\qquad-2\alpha\,|u+v|^{r+1}-4\beta\,|uv|^{\frac{r+1}{2}}\Big]\,\mathrm{d}x\,, \end{aligned} \tag{6}$$

$$\begin{aligned} I(t):=I(u(t),v(t))&=\int_\Omega\Big[|\Delta u|^2+|\Delta v|^2-(r+1)\\ &\quad\cdot\alpha\,|u+v|^{r+1}-2(r+1)\beta\,|uv|^{\frac{r+1}{2}}\Big]\,\mathrm{d}x\\ &=\|\Delta u\|_2^2+\|\Delta v\|_2^2-(r+1)\int_\Omega F(u,v)\,\mathrm{d}x\,. \end{aligned} \tag{7}$$

We denote

$$d=\inf_{(u,v)\in H_0^2(\Omega)\times H_0^2(\Omega),(u,v)\neq(0,0)}\ \sup_{\lambda\geq 0}J(\lambda(u,v))\,, \tag{8}$$

and define

$$\begin{aligned} W_1=\Big\{(u,v)\,\Big|\,&(u,v)\in H_0^2(\Omega)\times H_0^2(\Omega)\,,\\ &I(u,v)>0\Big\}\cup\{(0,0)\}\,, \end{aligned}$$

$$W_2 = \left\{(u,v)\,\middle|\,(u,v)\in H_0^2(\Omega)\times H_0^2(\Omega)\,,I(u,v)<0\right\},$$
$$N = \left\{(u,v)\,\middle|\,(u,v)\in H_0^2(\Omega)\times H_0^2(\Omega)\,,(u,v)\neq(0,0),\ I(u,v)=0\right\}.$$

The structure of the functional J allows us to easily show that (see [20])

$$d = \inf_{(u,v)\in N} J(u,v)\,.$$

We then state a local existence theorem which can be established by combining arguments of [10, 19, 27].

Theorem 1 (Local existence)**.** *Assume that $u_0,v_0\in H_0^2(\Omega)$, $u_1,v_1\in L^2(\Omega)$, and*

$$3\le r<+\infty \ \text{ if } \ n=1,2,3,4$$
$$\text{or } \ 3\le r\le(3n-10)/(n-4) \ \text{ if } \ n\ge 5\,,$$

then there exists a unique local solution (u,v) of (1) defined on $[0,T)$, for some $T>0$. In addition, the solution satisfies

$$u,v\in C([0,T],H_0^2(\Omega)),\ \ u_t,v_t\in C([0,T],L^2(\Omega))\,.$$

Moreover, at least one of the following statements holds true:

1. $\|u_t\|_2^2+\|v_t\|_2^2+\|\Delta u\|_2^2+\|\Delta v\|_2^2\to\infty$ as $t\to T^-$,
2. $T=\infty$.

Now we are in a position to state our main result.

Theorem 2. *Assume that $u_0,v_0\in H_0^2(\Omega)$, $u_1,v_1\in L^2(\Omega)$, and*

$$3\le r<+\infty \ \text{ if } \ n=1,2,3,4$$
$$\text{or } \ 3\le r\le(3n-10)/(n-4) \ \text{ if } \ n\ge 5\,.$$

Suppose that any one of the following statements is satisfied:

(i) $E(0)<0$,
(ii) $E(0)=0$ and $\int_\Omega(u_0u_1+v_0v_1)\,\mathrm{d}x>0$,
(iii) $0<E(0)<d$ and $I(u_0,v_0)<0$,
(iv) $d\le E(0)\le\Lambda$ for

$$\Lambda=\min\left\{\frac{[\int_\Omega(u_0u_1+v_0v_1)\,\mathrm{d}x]^2}{2\left[(T_1+1)\|u_0\|_2^2+\|v_0\|_2^2\right]},\frac{r+3}{2r_2(r+1)}\cdot\left[2\int_\Omega(u_0u_1+v_0v_1)\,\mathrm{d}x-r_2\left(2\|u_0\|_2^2+\|v_0\|_2^2\right)\right]\right\},$$

where $r_2=2\sqrt{r+3}/\left(\sqrt{r+3}+\sqrt{r-1}\right)$ and T_1 is a certain constant appearing in (31) below.

Then, the solution $(u(t),v(t))$ blows up at a finite time T^* in the sense of

$$\lim_{t\to T^{*-}}\left\{\int_\Omega\left(u^2+v^2\right)\mathrm{d}x+\int_0^t\|u\|_2^2\,\mathrm{d}t\right\}=\infty\,. \quad (9)$$

In case (i),

$$T^*\le t_0-\frac{Y(t_0)}{Y'(t_0)}\,.$$

Furthermore, if $Y(t_0)<\min\left\{1,\sqrt{-\frac{a}{b}}\right\}$, we have

$$T^*\le t_0+\frac{1}{\sqrt{-b}}\ln\frac{\sqrt{-\frac{a}{b}}}{\sqrt{-\frac{a}{b}}-Y(t_0)}\,,$$

where

$$a=\kappa^2Y^{2+\frac{2}{\kappa}}(t_0)\left[\left(a'(t_0)-\|u_0\|_2^2\right)^2-8E(0)Y^{\frac{-1}{\kappa}}(t_0)\right]>0\,,\ \ \kappa=\frac{r-1}{4}\,, \quad (10)$$

$$b=\frac{(r-1)^2}{2}E(0)<0\,. \quad (11)$$

In case (ii),

$$T^*\le t_0+\frac{Y(t_0)}{\sqrt{a}}\,,$$

where a and b are defined as (10) and (11), respectively.

In cases (iii) and (iv),

$$T^*\le\frac{Y(t_0)}{\sqrt{a}}\ \text{ or }\ T^*\le t_0+2^{(3\kappa+1)/2\kappa}\left(\frac{a}{b}\right)^{2+\frac{1}{\kappa}}\cdot\frac{\kappa}{\sqrt{a}}\left[1-(1+cY(t_0))^{-1/2\kappa}\right].$$

Furthermore, in case (iii)

$$a=\kappa^2Y^{2+\frac{2}{\kappa}}(t_0)\left[\left(a'(t_0)-\|u_0\|_2^2\right)^2+\frac{2c}{1+2\kappa}Y^{\frac{-1}{\kappa}}(t_0)\right]>0\,, \quad (12)$$

$$b=-\frac{2c\kappa^2}{1+2\kappa}\,, \quad (13)$$

and in case (iv) a and b are defined as (10) and (11), respectively.

Here $t_0=t^*$ is given by (24) in case (i), $t_0=t_*$ is given by (26) in cases (iii), and $t_0=0$ in case (ii) and (iv), and $Y(\cdot)$ is the function defined in (31) below.

3. Blow-Up of Solutions

In this section, we shall discuss the blow-up property of solutions for (1). Before doing this, let us give the following lemmas that will be used later.

Lemma 1 ([23]). *Let us have $\kappa > 0$ and let $B(t) \in C^2(0,\infty)$ be a nonnegative function satisfying*

$$B''(t) - 4(\kappa+1)B'(t) + 4(\kappa+1)B(t) \geq 0. \quad (14)$$

If

$$B'(0) > r_2 B(0) + K_0 \quad (15)$$

with $r_2 = 2(\kappa+1) - 2\sqrt{(\kappa+1)\kappa}$, then $B'(t) > K_0$ for $t > 0$, where K_0 is a constant.

Lemma 2 ([23]). *If $Y(t)$ is a nonincreasing function on $[t_0,\infty)$ and satisfies the differential inequality*

$$Y'(t)^2 \geq a + bY(t)^{2+\frac{1}{\kappa}} \quad \text{for } t \geq t_0, \quad (16)$$

where $a > 0$, $b \in R$, then there exists a finite time T^* such that

$$\lim_{t \to T^{*-}} Y(t) = 0.$$

Upper bounds for T^* are estimated as follows:

(i) If $b < 0$, then $T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{-\frac{a}{b}}}{\sqrt{-\frac{a}{b}} - Y(t_0)}$.

(ii) If $b = 0$, then $T^* \leq t_0 + \frac{Y(t_0)}{Y'(t_0)}$.

(iii) If $b > 0$, then $T^* \leq \frac{Y(t_0)}{\sqrt{a}}$ or $T^* \leq t_0 + 2^{(3\kappa+1)/2\kappa} \frac{\kappa c}{\sqrt{a}} \left[1 - (1 + cY(t_0))^{-1/2\kappa}\right]$,

where $c = \left(\frac{a}{b}\right)^{2+\frac{1}{\kappa}}$.

Lemma 3. *$E(t)$ is a nonincreasing function for $t \geq 0$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) = -\|u_t(t)\|_2^2. \quad (17)$$

Proof. Multiplying the first and the second equations of (1) by u_t and v_t, respectively, integrating them over Ω, adding the results together, and then integrating by parts, we obtain

$$E(t) - E(0) = -\int_0^t \|u_t\|_2^2 \,\mathrm{d}t \quad \text{for } t \geq 0. \quad (18)$$

Being the primitive of an integrable function, $E(t)$ is absolutely continuous and equality (17) is satisfied. □

Lemma 4 ([9, 20, 26]). *Suppose that $u_0, v_0 \in H_0^2(\Omega)$, $u_1, v_1 \in L^2(\Omega)$, and*

$$3 \leq r < +\infty \text{ if } n = 1,2,3,4$$
$$\text{or } 3 \leq r \leq (3n-10)/(n-4) \text{ if } n \geq 5.$$

Suppose further that $(u_0, v_0) \in W_2$ and $E(0) < d$. Then we have $(u(t), v(t)) \in W_2$ for all $t \in [0,T)$, and

$$\int_\Omega \left(|\Delta u|^2 + |\Delta v|^2\right) \mathrm{d}x > \frac{2(r+1)}{r-1} d. \quad (19)$$

Let

$$a(t) = \int_\Omega \left(u^2 + v^2\right) \mathrm{d}x + \int_0^t \|u\|_2^2 \,\mathrm{d}t \quad \text{for } t \geq 0. \quad (20)$$

To prove Theorem 2, we need to introduce the following two lemmas by modifying and improving the method of [25, 26].

Lemma 5. *Suppose that $u_0, v_0 \in H_0^2(\Omega)$ and $u_1, v_1 \in L^2(\Omega)$, and $\kappa = \frac{r-1}{4}$, then we have*

$$\begin{aligned} &a''(t) - 4(\kappa+1) \int_\Omega \left(u_t^2 + v_t^2\right) \mathrm{d}x \\ &\geq (-4-8\kappa)E(0) + (4+8\kappa) \quad (21) \\ &\cdot \int_0^t \|u_t\|_2^2 \,\mathrm{d}t + 4\kappa\left(\|\triangle u\|_2^2 + \|\triangle v\|_2^2\right). \end{aligned}$$

Proof. By (20), we have

$$a'(t) = 2\int_\Omega (uu_t + vv_t) \,\mathrm{d}x + \|u\|_2^2, \quad (22)$$

and

$$\begin{aligned} a''(t) = {} & 2\int_\Omega \left(u_t^2 + v_t^2\right) \mathrm{d}x - 2\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\right) \quad (23) \\ & + 2(r+1)\alpha \|u+v\|_{r+1}^{r+1} + 4(r+1)\beta \|uv\|_{\frac{r+1}{2}}^{\frac{r+1}{2}}. \end{aligned}$$

Then from (23), (6), and (18), we obtain (21). □

Lemma 6. *Under the conditions of Theorem 2, we have $a'(t) > \|u_0\|_2^2$ for $t > t_0$, where $t_0 = t^*$ is given by (24) in case (i), $t_0 = t_*$ is given by (26) in case (iii), and $t_0 = 0$ in cases (ii) and (iv).*

Proof. (i) If $E(0) < 0$, then from (21), we have

$$a'(t) \geq a'(0) - 4(1+2\kappa)E(0)t > a'(0) \quad \text{for } t > 0.$$

Thus we get $a'(t) > \|u_0\|_2^2$ for $t > t^*$, where

$$t^* = \max\left\{\frac{a'(0) - \|u_0\|_2^2}{4(1+2\kappa)E(0)}, 0\right\}. \quad (24)$$

(ii) If $E(0) = 0$, then from (21), we obtain $a''(t) \geq 0$ for $t \geq 0$. If $a'(0) > \|u_0\|_2^2$, then we have $a'(t) > \|u_0\|_2^2$, $t \geq 0$.

(iii) If $0 < E(0) < d$ and $I(u_0, v_0) < 0$, then from (21) and by Lemma 4, we get

$$a''(t) \geq (-4-8\kappa)E(0) + 4\kappa\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\right) \geq (4+8\kappa)\,(d - E(0)) := c > 0\,. \tag{25}$$

Thus, we obtain $a'(t) > \|u_0\|_2^2$ for $t > t_*$, where

$$t_* = \max\left\{\frac{\|u_0\|_2^2 - a'(0)}{c}, 0\right\}. \tag{26}$$

(iv) For the case $E(0) \geq d$, we first note that

$$2\int_0^t \int_\Omega u u_t \,\mathrm{d}x\,\mathrm{d}t = \|u\|_2^2 - \|u_0\|_2^2\,. \tag{27}$$

By Hölder's inequality and Young's inequality, we obtain from (27),

$$\|u\|_2^2 \leq \|u_0\|_2^2 + \int_0^t \|u\|_2^2\,\mathrm{d}t + \int_0^t \|u_t\|_2^2\,\mathrm{d}t\,. \tag{28}$$

By Hölder's inequality and Young's inequality again and using (22), (20), and (28), we get

$$a'(t) \leq a(t) + \|u_0\|_2^2 + \int_\Omega \left(u_t^2 + v_t^2\right)\mathrm{d}x + \int_0^t \|u_t\|_2^2\,\mathrm{d}t\,. \tag{29}$$

Hence by (21) and (29), we have

$$a''(t) - 4(\kappa+1)a'(t) + 4(\kappa+1)a(t) + \Big[(4+8\kappa)E(0) + 4(\kappa+1)\,\|u_0\|_2^2\Big] \geq 4\kappa\int_0^t \|u_t\|_2^2\,\mathrm{d}t \geq 0\,.$$

Let

$$k(t) = a(t) + \frac{(4+8\kappa)E(0) + 4(\kappa+1)\|u_0\|_2^2}{4(\kappa+1)} \text{ for } t > 0.$$

Then $k(t)$ satisfies Lemma 1. We see that if

$$a'(0) > r_2\left[a(0) + \frac{(4+8\kappa)E(0) + 4(\kappa+1)\,\|u_0\|_2^2}{4(1+\kappa)}\right] + \|u_0\|_2^2\,, \tag{30}$$

then $a'(t) > \|u_0\|_2^2$, $t \geq 0$, where r_2 is given in Lemma 1. Moreover, (30) is equal to

$$E(0) \leq \frac{(1+\kappa)}{r_2(1+2\kappa)}\left[2\int_\Omega (u_0u_1 + v_0v_1) - r_2\left(2\|u_0\|_2^2 + \|v_0\|_2^2\right)\right].$$

□

Now, we show the proof of Theorem 2.

Proof of Theorem 2. Let

$$Y(t) = \Big[a(t) + (T_1 - t)\,\|u_0\|_2^2\Big]^{-\kappa}, \quad t \in [0, T_1]\,, \tag{31}$$

where $T_1 > 0$ is a certain constant which will be specified later. Then we get

$$Y'(t) = -\kappa\Big[a(t) + (T_1 - t)\,\|u_0\|_2^2\Big]^{-\kappa-1}\Big(a'(t) - \|u_0\|_2^2\Big) = -\kappa Y^{1+\frac{1}{\kappa}}(t)\left(a'(t) - \|u_0\|_2^2\right), \tag{32}$$

and

$$Y''(t) = -\kappa Y^{1+\frac{2}{\kappa}}(t)a''(t)\Big[a(t) + (T_1 - t)\,\|u_0\|_2^2\Big] + \kappa Y^{1+\frac{2}{\kappa}}(t)(1+\kappa)\left(a'(t) - \|u_0\|_2^2\right)^2.$$

We set

$$V(t) = a''(t)\Big[a(t) + (T_1 - t)\,\|u_0\|_2^2\Big] - (1+\kappa)\left(a'(t) - \|u_0\|_2^2\right)^2, \tag{33}$$

then

$$Y''(t) = -\kappa Y^{1+\frac{2}{\kappa}}(t)V(t)\,. \tag{34}$$

For simplicity of calculation, we denote

$$P_u = \int_\Omega u^2\,\mathrm{d}x\,,\ P_v = \int_\Omega v^2\,\mathrm{d}x\,,\ Q_u = \int_0^t \|u\|_2^2\,\mathrm{d}t\,,$$
$$R_u = \int_\Omega u_t^2\,\mathrm{d}x\,,\ R_v = \int_\Omega v_t^2\,\mathrm{d}x\,,\ S_u = \int_0^t \|u_t\|_2^2\,\mathrm{d}t\,,$$

From (23), (27), and Hölder's inequality, we define

$$a'(t) = 2\int_\Omega (uu_t + vv_t)\,\mathrm{d}x + \|u_0\|_2^2 + 2\int_0^t\int_\Omega uu_t\,\mathrm{d}x\,\mathrm{d}t \leq 2\left(\sqrt{R_uP_u} + \sqrt{Q_uS_u} + \sqrt{R_vP_v}\right) + \|u_0\|_2^2\,. \tag{35}$$

For the case (i) or (ii), by (21), we have

$$a''(t) \geq (-4-8\kappa)E(0) + 4(1+\kappa)\,(R_u + R_v + S_u)\,. \tag{36}$$

Thus, from (33), (35), (36), and (31), we obtain

$$V(t) \geq [(-4-8\kappa)E(0) + 4(1+\kappa)\,(R_u + R_v + S_u)] \cdot Y^{\frac{-1}{\kappa}}(t) - 4(1+\kappa)\left(\sqrt{R_uP_u} + \sqrt{R_vP_v} + \sqrt{Q_uS_u}\right)^2.$$

From

$$a(t) = \int_\Omega \left(u^2+v^2\right) \mathrm{d}x + \int_0^t \|u\|_2^2 \,\mathrm{d}t = P_u + P_v + Q_u$$

and (31), we get

$$V(t) \geq (-4-8\kappa)E(0)Y^{\frac{-1}{\kappa}}(t) + 4(1+\kappa) \cdot \left[(R_u+R_v+S_u)(T_1-t)\|u_0\|_2^2 + \Phi(t)\right],$$

where

$$\Phi(t) = (R_u+R_v+S_u)(P_u+P_v+Q_u) - \left(\sqrt{R_uP_u}+\sqrt{R_vP_v}+\sqrt{Q_uS_u}\right)^2.$$

By the Schwarz inequality, we know that $\Phi(t)$ is non-negative. Hence, we have

$$V(t) \geq (-4-8\kappa)E(0)Y^{\frac{-1}{\kappa}}(t), \quad t \geq t_0. \tag{37}$$

Therefore, by (34) and (37), we get

$$Y''(t) \leq 4\kappa(1+2\kappa)E(0)Y^{1+\frac{1}{\kappa}}(t), \quad t \geq t_0. \tag{38}$$

By Lemma 6, we know that $Y'(t) < 0$ for $t \geq t_0$. Multiplying (38) by $Y'(t)$ and integrating it from t_0 to t, we get

$$Y'^2(t) \geq a + bY^{2+\frac{1}{\kappa}}(t)$$

for $t \geq t_0$, where a, b are defined as (10) and (11), respectively.

For the case (iii), we obtain from (21) and (25) that

$$a''(t) \geq c + 4(1+\kappa)(R_u+R_v+S_u).$$

By the steps in case (i), we get

$$Y''(t) \leq -\kappa c Y^{1+\frac{1}{\kappa}}(t), \quad t \geq t_0,$$

and then

$$Y'^2(t) \geq a + bY^{2+\frac{1}{\kappa}}(t),$$

where a, b are defined as (12) and (13), respectively.

For the case (iv), by the steps in case (i), we get (37) and (38) if and only if

$$E(0) < \frac{\left(a'(0)-\|u_0\|_2^2\right)^2}{8\left(a(0)+T_1\|u_0\|_2^2\right)} = \frac{\left[\int_\Omega (u_0u_1+v_0v_1)\,\mathrm{d}x\right]^2}{2\left[(T_1+1)\|u_0\|_2^2+\|v_0\|_2^2\right]}.$$

Therefore, by Lemma 2, there exists a finite time T^* such that $\lim_{t\to T^*-} Y(t) = 0$ and the upper bound of T^* is estimated according to the sign of $E(0)$. This means that (9) holds. □

Remark 1. The choice of T_1 in (31) is possible provided that $T_1 \geq T^*$, we refer the reader to [25] for details.

4. Concluding Remarks

In this paper, we have investigated a coupled system of Petrovsky equations in a bounded domain with clamped boundary conditions, the physical origin of which lies in the study of beam and plate. Due to several physical considerations, a linear damping which is distributed everywhere in the domain under consideration appears only in the first equation whereas no damping term is applied to the second one (this is indirect damping). Indirect damping of reversible systems occurs in many applications in engineering and mechanics. Indeed, it arises whenever it is impossible or too expensive to damp all the components of the state. Many studies show that the solution of this kind of system has a polynomial rate of decay as time tends to infinity, but does not have exponential decay. For four different ranges of initial energy, we have showed here the blow-up of solutions and give the lifespan estimates by improving the method of [25, 26].

From the applications point of view, our results may provide some qualitative analysis and intuition for the researchers in other fields such as engineering and mechanics when they study the concrete models of Petrovsky type.

Acknowledgements

This work was partly supported by the National Natural Science Foundation of China (Grant No. 11126291, 40975002, 41174165) and the Natural Science Foundation of the Jiangsu Higher Education Institutions (Grant No. 09KJB110005). The author would like to thank Professor Shun-Tang Wu for his valuable discussion of this paper.

[1] D. L. Russell, J. Math. Anal. Appl. **173**, 339 (1993).
[2] K. Liu, SIAM J. Control Optim. **35**, 1574 (1997).
[3] F. Alabau, P. Cannarsa, and V. Komornik, J. Evol. Equ. **2**, 127 (2002).
[4] F. Alabau–Boussouira, SIAM J. Control Optim. **41**, 511 (2002).
[5] A. Beyrath, Bol. Soc. Parana. Mat. **22**, 17 (2004).

[6] A. Borichev, and Y. Tomilov, Math. Ann. **347**, 455 (2010).
[7] M. L. Santos, M. P. C. Rocha, and S. C. Gomes, Appl. Anal. **86**, 1293 (2007).
[8] L. Tebou, Math. Control Relat. Fields **2**, 45 (2012).
[9] N. E. Amroun and A. Benaissa, Georgian Math. J. **13**, 397 (2006).
[10] S. A. Messsoudi, J. Math. Anal. Appl. **265**, 296 (2002).
[11] F. Tahamtani and M. Shahrouzi, Bound. Val. Probl. **2012**, 1 (2012).
[12] S. T. Wu and L. Y. Tsai, Taiwan. J. Math. **13**, 545 (2009).
[13] M. M. Cavalcanti, Discrete Contin. Dyn. Syst. **8**, 675 (2002).
[14] M. M. Cavalcanti and V. N. Domingos Cavalcanti, J. Math. Anal. Appl. **291**, 109 (2004).
[15] M. M. Cavalcanti, V. N. Domingos Cavalcanti, and J. A. Soriano, Commun. Contemp. Math. **6**, 705 (2004).
[16] M. M. Cavalcanti, V. N. Domingos Cavalcanti, and P. Martinez, J. Diff. Eqs. **203**, 119 (2004).
[17] E. Vitillaro, J. Diff. Eqs. **186**, 259 (2002).
[18] B. Yamna and B. Benyattou, E. J. Quali. Theory Diff. Eqs. **2012**, 1 (2012).
[19] K. Agre and M. A. Rammaha, Diff. Integ. Eqs. **19**, 1235 (2006).
[20] C. O. Alives, M. M. Cavalcanti, V. N. Domingos Cavalcanti, M. A. Rammaha, and D. Toundykov, Discrete Contin. Dyn. Syst. Ser. S **2**, 583 (2009).
[21] A. Barhoumi, Bol. Soc. Parana. Mat. **27**, 53 (2009).
[22] M. R. Li and L. Y. Tsai, Taiwan. J. Math. **7**, 557 (2003).
[23] M. R. Li and L. Y. Tsai, Nonlin. Anal. **54**, 1397 (2003).
[24] E. Piskin and N. Polat, J. Adv. Res. Appl. Math. **4**, 26 (2012).
[25] S. T. Wu, Electron. J. Diff. Eqs. **105**, 1 (2009).
[26] G. Li, Y. N. Sun, and W. J. Liu, Nonlin. Anal. **74**, 1523 (2011).
[27] W. J. Liu and J. Yu, Stud. Appl. Math. **127**, 315 (2011).