

# A new Analytical Solution Procedure for the Motion of a Spherical Particle in a Plane Couette Flow

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In this article, we applied the variational iteration method along with a Padé approximation (VIM-Padé) to obtain the analytical approximate solution for the motion of a spherical particle in a plane Couette flow. We studied the effects of different flow parameters on the velocity field. It is examined that the present analytical technique is extremely efficient and easy to apply for such a problem.

**Key words:** Spherical Particle; Couette Flow; Variation Iteration Method; Analytical Solution.

## 1. Introduction

The problem of the movement of immersing bodies in fluids exists in a combination of manufacturing development such as chemical engineering and dust system. Several mechanisms could be found in the scientific writing [1–3] which discussed the impact of isolated particles in solid media.

The accelerated motion of a falling sphere contains the measurement of the particle position, velocity, and acceleration at any instant of time in Newtonian fluids. These motions also comprise centrifugal and gravity collection, where it is often essential to understand the paths of particle accelerating in a fluid for the purposes of construction or improved performance. In a further model in Newtonian fluids, for instance raindrop settling velocity measurements and viscosity computations using the falling sphere method, it is also essential to understand the time and space necessary to complete the ultimate velocity for a given sphere–fluid mixture. In the last few years, significant interest has been devoted to the study of the accelerated motion of a sphere in a fluid, and theoretical advancement in this area has been given for Newtonian fluids [4–8].

Though an analytical approach is suitable for engineering computations and is also the clear starting position for a better understanding of the connection between the physical characteristics of the sphere fluid–

mixture and the accelerated motion of the sphere. The aim of the present paper is to find a new analytical solution for the equations that govern the two-dimensional motion of a spherical particle in plane Couette fluid flow and also study the effects of different flow parameters on velocity and acceleration profiles.

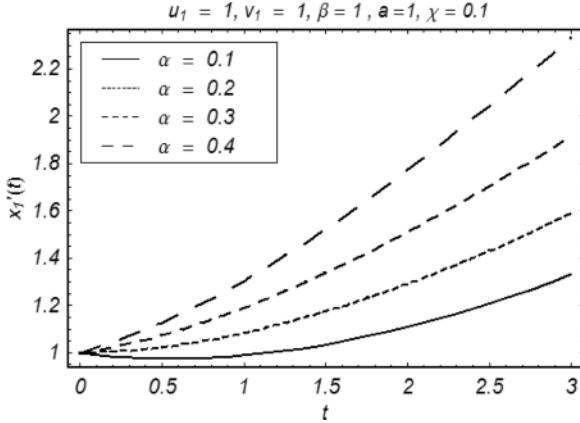
The paper is prepared as follows: In Section 2, the fundamental concept of the variational iteration method is presented. Section 3 includes the basic idea of rational polynomial approximation, Section 4 contains governing equations. Section 5 is devoted to convergence analysis of VIM. In Section 6, we present series solutions obtained with VIM, and results are discussed in Section 7. The concluding remark is given in the last segment.

## 2. Variational Iterative Method

To demonstrate the fundamental idea of the variational iteration method (VIM), we consider the following general differential equation:

$$Lw + Nw + R w = g(x), \quad (1)$$

where  $L$  is a linear operator,  $N$  is a nonlinear operator,  $R$  is the remaining of the linear operator, and  $g(x)$  is the forcing term. According to VIM [9–12], we can construct a correction functional as

Fig. 1. Effects of parameter  $\alpha$  on the velocity field.

$$w_{n+1}(x) = w_n(x) + \int_0^x \lambda(\xi) \left( Lw_n(\xi) + N\tilde{w}_n(\xi) + R w_n(\xi) - g(\xi) \right) d\xi, \quad (2)$$

where  $\lambda$  is a Lagrange multiplier which can be obtained optimally by means of the variational iteration method. The subscripts  $n$  represent the  $n$ th approximation,  $\tilde{w}_n$  is measured as a restricted variation, that is,  $\delta\tilde{u}_n = 0$ . Relational (2) is called a correctional functional. The principles of VIM and its applicability for different kinds of differential equations are given in [13, 14]. In this method, it is required first to determine the Lagrange multiplier  $\lambda$  optimally. The successive approximations  $w_{n+1}$ ,  $n \geq 0$ , of the solution  $w$  will be readily obtained upon using the determined Lagrange multiplier and any selective function. Therefore, the solution is given by

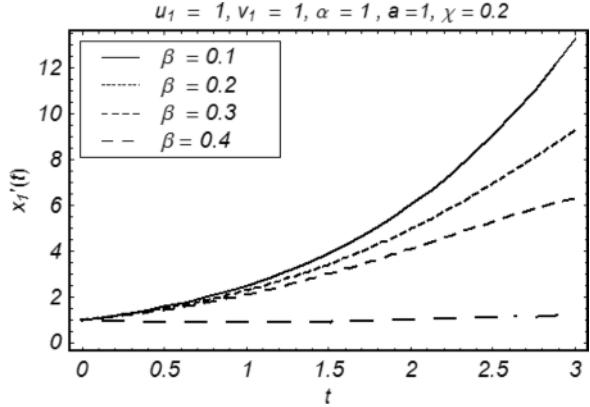
$$w = \lim_{n \rightarrow \infty} w_n. \quad (3)$$

### 3. Rational Approximation

A rational approximation for the function  $w(t)$  is the quotient of two polynomials  $H_K(t)$  and  $S_N(t)$  of the degrees  $K$  and  $N$ , respectively. We make use of the notation  $G_{K,N}(t)$  to indicate this ratio [15, 16]:

$$G_{K,N}(t) = \frac{H_K(t)}{S_N(t)}. \quad (4)$$

The power series of  $w(t)$  in terms of  $t$  is given as

Fig. 2. Effects of parameter  $\beta$  on the velocity field.

$$w(t) = \sum_{i=0}^{\infty} a_i t^i, \quad (5)$$

$$w(t) = \frac{H_K(t)}{S_N(t)} + O(t^{K+N+1}). \quad (6)$$

We imposed the normalization condition to a polynomial in the denominator which is given below:

$$S_0(t) = s_0 = 1. \quad (7)$$

Expanding polynomials  $H_K(t)$  and  $S_N(t)$  in a power series in terms of  $t$  of order  $K$  and  $N$ , we get

$$\begin{aligned} H_K(t) &= h_0 + h_1 t + h_2 t^2 + \dots + h_K t^K, \\ S_N(t) &= 1 + s_1 t + s_2 t^2 + \dots + s_N t^N. \end{aligned} \quad (8)$$

Utilizing (7)–(8) in (6), we have

$$\begin{aligned} \sum_{i=0}^{\infty} a_i t^i &= \frac{h_0 + h_1 t + h_2 t^2 + \dots + h_K t^K}{1 + s_1 t + s_2 t^2 + \dots + s_N t^N} \\ &\quad + O(t^{K+N+1}). \end{aligned} \quad (9)$$

$$\begin{aligned} (1 + s_1 t + s_2 t^2 + \dots + s_N t^N)(a_0 + a_1 t + a_2 t^2 + \dots) \\ = h_0 + h_1 t + h_2 t^2 + \dots + h_K t^K + O(t^{K+N+1}). \end{aligned} \quad (10)$$

From (10), we arrive at a linear system of equations

$$\begin{aligned} a_0 &= h_0, \\ a_1 + a_0 s_1 &= h_1, \\ a_2 + a_1 s_1 + a_0 s_2 &= h_2, \\ &\vdots \\ a_K + a_{K-1} s_1 + a_0 s_K &= h_K \end{aligned} \quad (11)$$

and

$$\begin{aligned} a_{K+1} + a_K s_1 + \dots + a_{K-N+1} s_K &= 0, \\ a_{K+2} + a_{K+1} s_1 + \dots + a_{K-N+2} s_N &= 0, \\ &\vdots \\ a_{K+N} + a_{K+N-1} s_1 + \dots + a_K s_N &= 0. \end{aligned} \quad (12)$$

From (12), we find  $s_i$ ,  $1 \leq i \leq N$ . The values of  $s_1, s_2, \dots, s_N$  inserted in (11) give the unknown values of the quantities  $h_0, h_1, h_2, \dots, h_K$ , respectively.

#### 4. Mathematical Formulation of the Problem

Consider the two-dimensional spherical motion of a particle in an incompressible Newtonian plane Couette flow. We consider that the particle will rotate with a constant angular velocity. The particle movement is completely traced by the collective effects of drag, inertia, and lift. Buoyancy and gravitational effects are assumed to be negligible [1, 2, 5, 6]. Therefore, the governing mathematical equations are given as follows:

$$\frac{4}{3}\pi r^3 \rho_1 \ddot{x}_1 = \frac{1}{2}\pi r^3 \rho_2 a \dot{x}_2 - 6\pi \mu r (\dot{x}_1 - a x_2), \quad (13)$$

$$\begin{aligned} \frac{4}{3}\pi r^3 \rho_1 \ddot{x}_2 &= \left( \frac{1}{2}\pi r^3 \rho_2 a + 6.46r^2 \rho_2 \sqrt{a} \sqrt{v} \right) \\ &\cdot (a x_2 - \dot{x}_1) - 6\pi \mu r \dot{x}_2, \end{aligned} \quad (14)$$

where  $r$  and  $\rho_1$  indicate the radius and density of the particle, respectively, and  $\mu$  is the fluid viscosity. Furthermore, dots indicate differentiation with respect to time. We consider that the particle and fluid relative velocities are small. To apply VIM, we write (13) and (14) in the simplified form

$$\ddot{x}_1 - a \dot{x}_2 + \beta (\dot{x}_1 - a x_2) = 0, \quad (15)$$

$$\ddot{x}_2 + \beta \dot{x}_2 + (\alpha + \chi) (\dot{x}_1 - a x_2) = 0, \quad (16)$$

where coefficients  $\alpha$  to  $\chi$  are defined as

$$\begin{aligned} \alpha &= \frac{3a\rho_2}{8\rho_1}, \quad \beta = \frac{9v\rho_2}{2r^2\rho_1}, \\ \text{and } \chi &= 1.542 \frac{\sqrt{v}\sqrt{a}\rho_2}{r\rho_1}. \end{aligned} \quad (17)$$

The associated boundary conditions signify insertion of the particle into the fluid:

$$x_1 = 0, \quad \dot{x}_1 = u_1 \quad \text{at } t = 0, \quad (18a)$$

$$x_2 = 0, \quad \dot{x}_2 = v_1 \quad \text{at } t = 0. \quad (18b)$$

To solve (15)–(18) with the use of VIM, we construct a correctional functional as following:

$$\begin{aligned} x_{1,n+1}(t) &= x_{1,n}(t) + \int_0^t \lambda_1(t, \xi) \left( \frac{d^2 x_{1,n}(\xi)}{d\xi^2} \right. \\ &\quad \left. - \alpha \frac{d\tilde{x}_{2,n}(\xi)}{d\xi} + \beta \left( \frac{d\tilde{x}_{1,n}(\xi)}{d\xi} - a x_{2,n}(\xi) \right) \right) d\xi, \end{aligned} \quad (19a)$$

$$\begin{aligned} x_{2,n+1}(t) &= x_{2,n}(t) + \int_0^t \lambda_2(t, \xi) \left( \frac{d^2 x_{2,n}(\xi)}{d\xi^2} \right. \\ &\quad \left. + \beta \frac{d\tilde{x}_{2,n}(\xi)}{d\xi} + (\alpha + \chi) \left( \frac{d\tilde{x}_{1,n}(\xi)}{d\xi} \right. \right. \\ &\quad \left. \left. - a x_{2,n}(\xi) \right) \right) d\xi, \end{aligned} \quad (19b)$$

where  $\lambda_1$  and  $\lambda_2$  are Lagrange multipliers which can be determined optimally.  $\tilde{x}_{1,n}(\xi)$ ,  $\tilde{x}_{2,n}(\xi)$  are considered as restricted variations, that is  $\delta\tilde{x}_{1,n}(\xi) = 0$  and  $\delta\tilde{x}_{2,n}(\xi) = 0$ . To find the optimal values of  $\lambda$ , we have

$$\begin{aligned} \delta x_{1,n+1}(t) &= \delta x_{1,n}(t) + \delta \int_0^t \lambda_1(t, \xi) \left( \frac{d^2 x_{1,n}(\xi)}{d\xi^2} \right. \\ &\quad \left. - \alpha \frac{d\tilde{x}_{2,n}(\xi)}{d\xi} + \beta \left( \frac{d\tilde{x}_{1,n}(\xi)}{d\xi} \right. \right. \\ &\quad \left. \left. - a x_{2,n}(\xi) \right) \right) d\xi, \end{aligned} \quad (20a)$$

$$\begin{aligned} \delta x_{2,n+1}(t) &= \delta x_{2,n}(t) + \delta \int_0^t \lambda_2(t, \xi) \left( \frac{d^2 x_{2,n}(\xi)}{d\xi^2} \right. \\ &\quad \left. + \beta \frac{d\tilde{x}_{2,n}(\xi)}{d\xi} + (\alpha + \chi) \left( \frac{d\tilde{x}_{1,n}(\xi)}{d\xi} \right. \right. \\ &\quad \left. \left. - a x_{2,n}(\xi) \right) \right) d\xi, \end{aligned} \quad (20b)$$

$$\begin{aligned} \delta x_{1,n+1}(t) &= \delta x_{1,n}(t) \\ &\quad + \delta \int_0^t \lambda_1(t, \xi) \left( \frac{d^2 x_{1,n}(\xi)}{d\xi^2} \right) d\xi, \end{aligned} \quad (21a)$$

$$\begin{aligned} \delta x_{2,n+1}(t) &= \delta x_{2,n}(t) \\ &\quad + \delta \int_0^t \lambda_2(t, \xi) \left( \frac{d^2 x_{2,n}(\xi)}{d\xi^2} \right) d\xi. \end{aligned} \quad (21b)$$

Following [10], the stationary conditions are given by

$$1 - \lambda'_1(\xi)|_{\xi=t} = 0, \quad \lambda''_1(\xi)|_{\xi=t} = 0, \quad (22a)$$

$$\lambda_1(\xi)|_{\xi=t} = 0,$$

$$1 - \lambda'_2(\xi)|_{\xi=t} = 0, \quad \lambda''_2(\xi)|_{\xi=t} = 0, \quad (22b)$$

$$\lambda_2(\xi)|_{\xi=t} = 0.$$

On solving (22a) and (22b), we get

$$\lambda_1(\xi, t) = \xi - t \text{ and } \lambda_2(\xi, t) = \xi - t. \quad (23)$$

Substituting these values of the Lagrange multipliers in (19a) and (19b), we get an iterative formula of the form

$$x_{1,n+1}(t) = x_{1,n}(t) + \int_0^t (\xi - t) G_1(x_{1,n}(\xi), x'_{1,n}(\xi), x''_{1,n}(\xi), x_{2,n}(\xi), x'_{2,n}(\xi), x''_{2,n}(\xi)) d\xi, \quad (24a)$$

$$G_1 = \frac{d^2 x_{1,n}(\xi)}{d\xi^2} - \alpha \frac{d\tilde{x}_{2,n}(\xi)}{d\xi} + \beta \left( \frac{d\tilde{x}_{1,n}(\xi)}{d\xi} - ax_{2,n}(\xi) \right), \quad (24b)$$

$$x_{2,n+1}(t) = x_{2,n}(t) + \int_0^t (\xi - t) G_2(x_{1,n}(\xi), x'_{1,n}(\xi), x''_{1,n}(\xi), x_{2,n}(\xi), x'_{2,n}(\xi), x''_{2,n}(\xi)) d\xi, \quad (24c)$$

$$G_2 = \frac{d^2 x_{2,n}(\xi)}{d\xi^2} + \beta \frac{dx_{2,n}(\xi)}{d\xi} + (\alpha + \chi) \left[ \frac{dx_{1,n}(\xi)}{d\xi} - ax_{2,n}(\xi) \right]. \quad (24d)$$

## 5. Convergence Theorem

In this section, we described the convergence criteria of our analytical approximate series solution obtained with the help of the variational iterative method [13].

### 5.1. Functions of Class $C^k$

A function is said to be of class  $C^k$  if the first  $k$  derivatives  $x(t), \dot{x}(t), \ddot{x}(t), \dots, x^k(t)$  all exist and are continuous.

**Theorem 1.** If for any  $j$ ,  $x_j(t) \in C^2$  over  $[0, T]$  satisfies the correctional functional of variational iterative, it is equivalent to the following iterative relation:

$$L[x_{j+1}(t) - x_j(t)] = G(x_j, \dot{x}_j, \ddot{x}_j), \quad (25)$$

where  $L = \frac{d^2}{dt^2}$  is a second-order linear operator.

*Proof.* Let us suppose that  $x_j(t)$  and  $x_{j+1}(t)$  satisfy the correctional functional defined in (19), i.e.

$$x_{j+1}(t) = x_j(t) + \int_0^t \lambda_{(t,\xi)} G(x_j, \dot{x}_j, \ddot{x}_j) d\xi. \quad (26)$$

Equivalently, (26) can be written as follows:

$$x_{j+1}(t) - x_j(t) = \int_0^t \lambda_{(t,\xi)} G(x_j, \dot{x}_j, \ddot{x}_j) d\xi. \quad (27)$$

Now we apply the linear operator to (27) that yields

$$\frac{d^2}{dt^2} [x_{j+1}(t) - x_j(t)] = \int_0^t \frac{\partial^2 \lambda_{(t,\xi)}}{\partial t^2} G d\xi + \frac{\partial \lambda_{(t,\xi)}}{\partial t} \Big|_{\xi=t} \left[ G + \frac{d}{dt} \left[ \lambda_{(t,\xi)} \right] \Big|_{\xi=t} G \right]. \quad (28)$$

Utilizing the conditions (22a)–(22b), we have

$$\frac{d^2}{dt^2} [x_{j+1}(t) - x_j(t)] = G(x_j, \dot{x}_j, \ddot{x}_j). \quad (29)$$

Equation (29) can be written in operator form by the use of the definition of the linear operator:

$$L[x_{j+1}(t) - x_j(t)] = G(x_j, \dot{x}_j, \ddot{x}_j). \quad (30)$$

Conversely, suppose that  $x_j(t)$  and  $x_{j+1}(t)$  satisfy (25). Applying the definition of the linear operator first, we have

$$\frac{d^2}{dt^2} [x_{j+1}(t) - x_j(t)] = G(x_j, \dot{x}_j, \ddot{x}_j) \quad (31)$$

or

$$[\ddot{x}_{j+1}(t) - \ddot{x}_j(t)] = G(x_j, \dot{x}_j, \ddot{x}_j). \quad (32)$$

We multiply nonzero Lagrange multipliers  $\lambda_{(t,\xi)}$  and apply integration on both sides from 0 to  $t$ , which gives

$$\int_0^t \lambda_{(t,\xi)} [\ddot{x}_{j+1}(\xi) - \ddot{x}_j(\xi)] d\xi = \int_0^t \lambda_{(t,\xi)} G d\xi. \quad (33)$$

Employing integration by parts on the left hand side of (33), yields

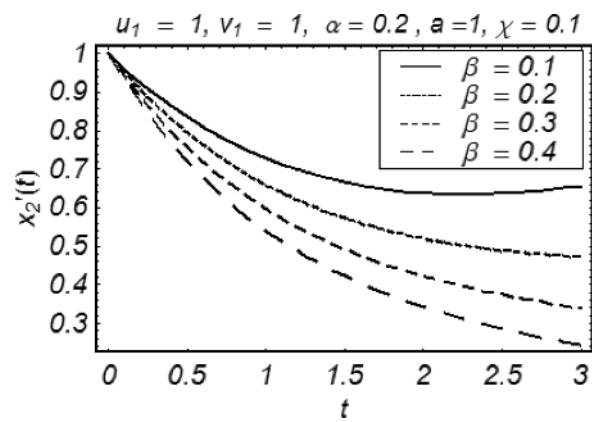
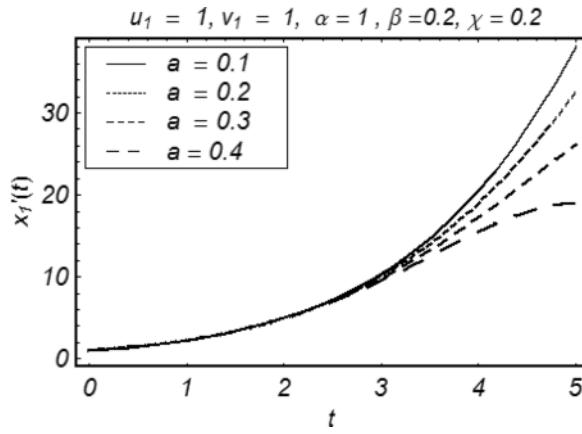
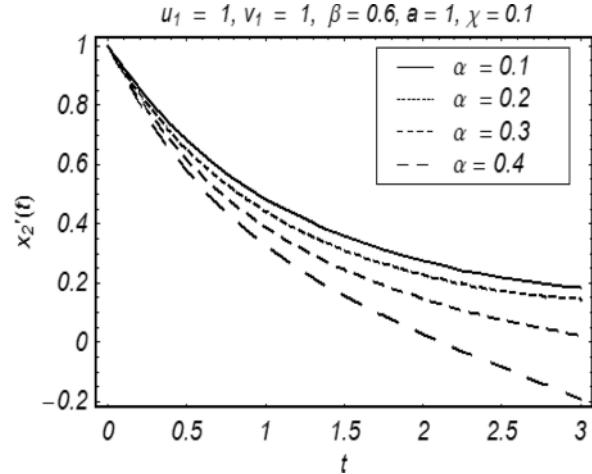
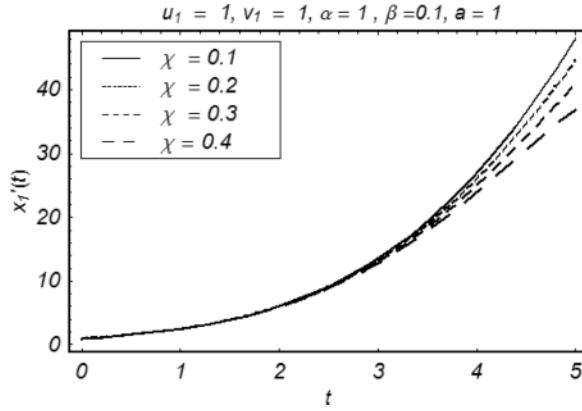
$$\begin{aligned} & \lambda_{(t,\xi)} \Big|_{\xi=t} [\ddot{x}_{j+1}(t) - \ddot{x}_j(t)] - \frac{\partial \lambda_{(t,\xi)}}{\partial \xi} \Big|_{\xi=t} [x_{j+1}(t) - x_j(t)] \\ & + \int_0^t \frac{\partial^2 \lambda_{(t,\xi)}}{\partial \xi^2} [x_{j+1}(\xi) - x_j(\xi)] d\xi \\ & = \int_0^t \lambda_{(t,\xi)} G d\xi. \end{aligned} \quad (34)$$

Using again the variational conditions (22a)–(22b), we get

$$x_{j+1}(t) = x_j(t) \quad (35)$$

$$+ \int_0^t \lambda_{(t,\xi)} G [x_j(\xi), \dot{x}_j(\xi), \ddot{x}_j(\xi)] d\xi,$$

which is the required proof.  $\square$



## 6. Series Solutions of Variational Iterative Method

In this section, we used the correctional functional in order to calculate few components of the series solution of the problem formulated in Section 4. The correctional functional forms (19a)–(19b) are

$$x_{1n+1}(t) = x_{1n}(t) + \int_0^t \lambda_1(t, \xi) \left( \frac{d^2 x_{1n}(\xi)}{d\xi^2} - \alpha \frac{d\tilde{x}_{2n}(\xi)}{d\xi} + \beta \left( \frac{d\tilde{x}_{1n}(\xi)}{d\xi} - \alpha x_{2n}(\xi) \right) \right) d\xi, \quad (36)$$

$$x_{2n+1}(t) = x_{2n}(t) + \int_0^t \lambda_2(t, \xi) \left( \frac{d^2 x_{2n}(\xi)}{d\xi^2} + \beta \frac{d\tilde{x}_{2n}(\xi)}{d\xi} + (\alpha + \chi) \left( \frac{d\tilde{x}_{1n}(\xi)}{d\xi} - \alpha x_{2n}(\xi) \right) \right) d\xi. \quad (37)$$

Now the components of the series solution are

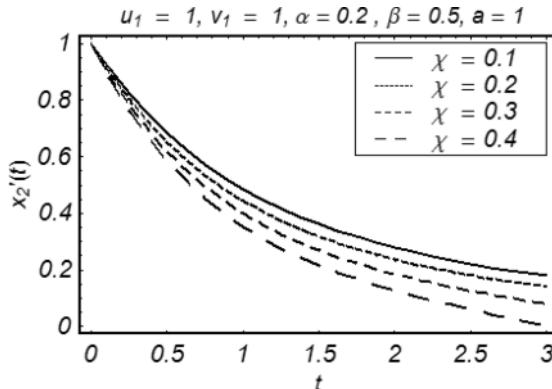
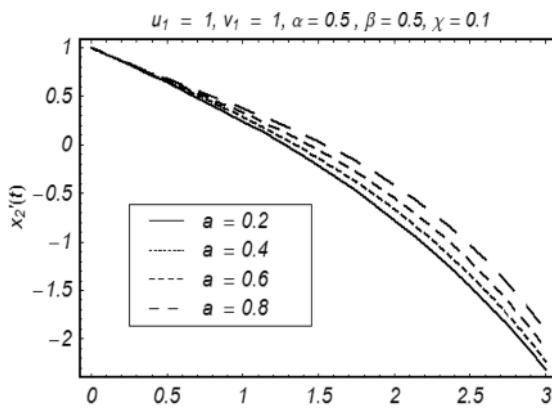
$$\begin{aligned} x_{11}(t) &= tu_1 - \frac{1}{2}t^2\beta u_1 + \frac{1}{2}t^2\alpha v_1 + \frac{1}{6}at^3\beta v_1, \\ x_{21}(t) &= tv_1 - \frac{1}{2}t^2\alpha u_1 - \frac{1}{2}t^2\chi u_1 + \frac{1}{6}at^3\alpha v_1 - \frac{1}{2}t^2\beta v_1 \\ &\quad + \frac{1}{6}at^3\chi v_1, \\ x_{12}(t) &= tu_1 - \frac{1}{6}t^3\alpha^2 u_1 - \frac{1}{2}t^2\beta u_1 - \frac{1}{24}at^4\alpha\beta u_1 \\ &\quad + \frac{1}{6}t^3\beta^2 u_1 - \frac{1}{6}t^3\alpha\chi u_1 - \frac{1}{24}at^4\beta\chi u_1 \\ &\quad + \frac{1}{2}t^2\alpha v_1 + \frac{1}{24}at^4\alpha^2 v_1 + \frac{1}{6}at^3\beta v_1 \\ &\quad - \frac{1}{3}t^3\alpha\beta v_1 + \frac{1}{120}a^2t^5\alpha\beta v_1 - \frac{1}{12}at^4\beta^2 v_1 \\ &\quad + \frac{1}{24}at^4\alpha\chi v_1 + \frac{1}{120}a^2t^5\beta\chi v_1, \end{aligned}$$

$$\begin{aligned}
x_{2_2}(t) = & -\frac{1}{2}t^2\alpha u_1 - \frac{1}{24}at^4\alpha^2 u_1 + \frac{1}{3}t^3\alpha\beta u_1 + \frac{1}{30}at^5\alpha^2\beta u_1 - \frac{1}{2}t^2\chi u_1 - \frac{1}{12}at^4\alpha\chi u_1 + \frac{1}{3}t^3\beta\chi u_1 \\
& - \frac{1}{24}at^4\chi^2 u_1 + tv_1 + \frac{1}{6}at^3\alpha v_1 - \frac{1}{6}t^3a^2 v_1 - \frac{1}{2}t^2\beta v_1 - \frac{1}{8}at^4\alpha\beta v_1 + \frac{1}{6}t^3\beta^2 v_1 - \frac{1}{6}t^3\alpha\chi v_1 + \frac{1}{60}a^2t^5\alpha\chi v_1 \\
& - \frac{1}{8}at^4\beta\chi v_1 + \frac{1}{120}a^2t^5\chi^2 v_1, \\
x_{1_3}(t) = & tu_1 - \frac{1}{6}t^3\alpha^2 u_1 - \frac{1}{120}at^5\alpha^3 u_1 - \frac{1}{2}t^2\beta u_1 - \frac{1}{24}at^4\alpha\beta u_1 + \frac{1}{8}t^4\alpha^2\beta u_1 - \frac{1}{720}a^2t^6\alpha^2\beta u_1 + \frac{1}{6}t^3\beta^2 u_1 \\
& + \frac{1}{40}at^5\alpha\beta^2 u_1 - \frac{1}{24}t^4\beta^3 u_1 - \frac{1}{6}t^3\alpha\chi u_1 - \frac{1}{60}at^5\alpha^2\chi u_1 - \frac{1}{24}at^4\beta\chi u_1 + \frac{1}{8}t^4\alpha\beta\chi u_1 - \frac{1}{360}a^2t^6\alpha\beta\chi u_1 \\
& + \frac{1}{40}at^5\beta^2\chi u_1 - \frac{1}{120}at^5\alpha\chi^2 u_1 - \frac{1}{720}a^2t^6\beta\chi^2 u_1 + \frac{1}{2}t^2\alpha v_1 + \frac{1}{24}at^4\alpha^2 v_1 - \frac{1}{24}t^4\alpha^3 v_1 + \frac{1}{720}a^2t^6\alpha^3 v_1 \\
& + \frac{1}{6}at^3\beta v_1 - \frac{1}{3}t^3\alpha\beta v_1 + \frac{1}{120}a^2t^5\alpha\beta v_1 - \frac{1}{24}at^5\alpha^2\beta v_1 + \frac{1}{5040}a^3t^7\alpha^2\beta v_1 - \frac{1}{12}at^4\beta^2 v_1 + \frac{1}{8}t^4\alpha\beta^2 v_1 \\
& - \frac{1}{180}a^2t^6\alpha\beta^2 v_1 + \frac{1}{40}at^5\beta^3 v_1 + \frac{1}{24}at^4\alpha\chi v_1 - \frac{1}{24}t^4\alpha^2\chi v_1 + \frac{1}{360}a^2t^6\alpha^2\chi v_1 + \frac{1}{120}a^2t^5\beta\chi v_1 \\
& - \frac{1}{24}at^5\alpha\beta\chi v_1 + \frac{1}{2520}a^3t^7\alpha\beta\chi v_1 - \frac{1}{180}a^2t^6\beta^2\chi v_1 + \frac{1}{720}a^2t^6\alpha\chi^2 v_1 + \frac{1}{5040}a^3t^7\beta\chi^2 v_1, \\
x_{2_3}(t) = & -\frac{1}{2}t^2\alpha u_1 - \frac{1}{24}at^4\alpha^2 u_1 + \frac{1}{24}t^4\alpha^3 u_1 - \frac{1}{720}a^2t^6\alpha^3 u_1 + \frac{1}{3}t^3\alpha\beta u_1 + \frac{1}{30}at^5\alpha^2\beta u_1 - \frac{1}{8}t^4\alpha\beta^2 u_1 - \frac{1}{2}t^2\chi u_1 \\
& - \frac{1}{12}at^4\alpha\chi u_1 - \frac{1}{12}t^4\alpha^2\chi u_1 - \frac{1}{240}a^2t^6\alpha^2\chi u_1 + \frac{1}{3}t^3\beta\chi u_1 + \frac{1}{15}at^5\alpha\beta\chi u_1 - \frac{1}{8}t^4\beta^2\chi u_1 - \frac{1}{24}at^4\chi^2 u_1 \\
& + \frac{1}{24}t^4\alpha\chi^2 u_1 - \frac{1}{240}a^2t^6\alpha\chi^2 u_1 + \frac{1}{30}at^5\beta\chi^2 u_1 - \frac{1}{720}a^2t^6\chi^3 u_1 + tv_1 + \frac{1}{6}at^3\alpha v_1 - \frac{1}{6}t^3a^2 v_1 \\
& + \frac{1}{120}a^2t^5\alpha^2 v_1 - \frac{1}{60}at^5\alpha^3 v_1 + \frac{1}{5040}a^3t^7\alpha^3 v_1 - \frac{1}{2}t^2\beta v_1 - \frac{1}{8}at^4\alpha\beta v_1 + \frac{1}{8}t^4\alpha^2\beta v_1 - \frac{1}{144}a^2t^6\alpha^2\beta v_1 \\
& + \frac{1}{6}t^3\beta^2 v_1 + \frac{1}{20}at^5\alpha\beta^2 v_1 - \frac{1}{24}t^4\beta^3 v_1 + \frac{1}{6}at^3\chi v_1 - \frac{1}{6}t^3\alpha\chi v_1 + \frac{1}{60}a^2t^5\alpha\chi v_1 - \frac{1}{30}at^5\alpha^2\chi v_1 \\
& + \frac{1}{1680}a^3t^7\alpha^2\chi v_1 - \frac{1}{8}at^4\beta\chi v_1 + \frac{1}{8}t^4\alpha\beta\chi v_1 - \frac{1}{72}a^2t^6\alpha\beta\chi v_1 + \frac{1}{20}at^5\beta^2\chi v_1 + \frac{1}{120}a^2t^5\chi^2 v_1 \\
& - \frac{1}{60}at^5\alpha\chi^2 v_1 + \frac{1}{1680}a^3t^7\alpha\chi^2 v_1 - \frac{1}{144}a^2t^6\beta\chi^2 v_1 + \frac{1}{5040}a^3t^7\chi^2 v_1, \\
& \vdots
\end{aligned}$$

The series solution is given by

$$\begin{aligned}
x_1 &= \lim_{n \rightarrow \infty} x_{1_n}, \quad x_2 = \lim_{n \rightarrow \infty} x_{2_n}. \\
x_1(t) = & tu_1 - \frac{1}{6}t^3\alpha^2 u_1 - \frac{1}{120}at^5\alpha^3 u_1 - \frac{1}{2}t^2\beta u_1 - \frac{1}{24}at^4\alpha\beta u_1 + \frac{1}{8}t^4\alpha^2\beta u_1 - \frac{1}{720}a^2t^6\alpha^2\beta u_1 + \frac{1}{6}t^3\beta^2 u_1 \\
& + \frac{1}{40}at^5\alpha\beta^2 u_1 - \frac{1}{24}t^4\beta^3 u_1 - \frac{1}{6}t^3\alpha\chi u_1 - \frac{1}{60}at^5\alpha^2\chi u_1 - \frac{1}{24}at^4\beta\chi u_1 + \frac{1}{8}t^4\alpha\beta\chi u_1 - \frac{1}{360}a^2t^6\alpha\beta\chi u_1 \\
& + \frac{1}{40}at^5\beta^2\chi u_1 - \frac{1}{120}at^5\alpha\chi^2 u_1 - \frac{1}{720}a^2t^6\beta\chi^2 u_1 + \frac{1}{2}t^2\alpha v_1 + \frac{1}{24}at^4\alpha^2 v_1 - \frac{1}{24}t^4\alpha^3 v_1 + \frac{1}{720}a^2t^6\alpha^3 v_1 \\
& + \frac{1}{6}at^3\beta v_1 - \frac{1}{3}t^3\alpha\beta v_1 + \frac{1}{120}a^2t^5\alpha\beta v_1 - \frac{1}{24}at^5\alpha^2\beta v_1 + \frac{1}{5040}a^3t^7\alpha^2\beta v_1 - \frac{1}{12}at^4\beta^2 v_1 + \frac{1}{8}t^4\alpha\beta^2 v_1 \\
& - \frac{1}{180}a^2t^6\alpha\beta^2 v_1 + \frac{1}{40}at^5\beta^3 v_1 + \frac{1}{24}at^4\alpha\chi v_1 - \frac{1}{24}t^4\alpha^2\chi v_1 + \frac{1}{360}a^2t^6\alpha^2\chi v_1 + \frac{1}{120}a^2t^5\beta\chi v_1 \\
& - \frac{1}{24}at^5\alpha\beta\chi v_1 + \frac{1}{2520}a^3t^7\alpha\beta\chi v_1 - \frac{1}{180}a^2t^6\beta^2\chi v_1 + \frac{1}{720}a^2t^6\alpha\chi^2 v_1 + \frac{1}{5040}a^3t^7\beta\chi^2 v_1 + \dots,
\end{aligned}$$

$$\begin{aligned}
x_2(t) = & -\frac{1}{2}t^2\alpha u_1 - \frac{1}{24}at^4\alpha^2 u_1 + \frac{1}{24}t^4\alpha^3 u_1 \\
& - \frac{1}{720}a^2t^6\alpha^3 u_1 + \frac{1}{3}t^3\alpha\beta u_1 + \frac{1}{30}at^5\alpha^2\beta u_1 \\
& - \frac{1}{8}t^4\alpha\beta^2 u_1 - \frac{1}{2}t^2\chi u_1 - \frac{1}{12}at^4\alpha\chi u_1 - \frac{1}{12}t^4\alpha^2\chi u_1 \\
& - \frac{1}{240}a^2t^6\alpha^2\chi u_1 + \frac{1}{3}t^3\beta\chi u_1 + \frac{1}{15}at^5\alpha\beta\chi u_1 \\
& - \frac{1}{8}t^4\beta^2\chi u_1 - \frac{1}{24}at^4\chi^2 u_1 + \frac{1}{24}t^4\alpha\chi^2 u_1 \\
& - \frac{1}{240}a^2t^6\alpha\chi^2 u_1 + \frac{1}{30}at^5\beta\chi^2 u_1 - \frac{1}{720}a^2t^6\chi^3 u_1 \\
& + tv_1 + \frac{1}{6}at^3\alpha v_1 - \frac{1}{6}t^3a^2v_1 + \frac{1}{120}a^2t^5\alpha^2v_1 \\
& - \frac{1}{60}at^5\alpha^3v_1 + \frac{1}{5040}a^3t^7\alpha^3v_1 - \frac{1}{2}t^2\beta v_1 \\
& - \frac{1}{8}at^4\alpha\beta v_1 + \frac{1}{8}t^4\alpha^2\beta v_1 - \frac{1}{144}a^2t^6\alpha^2\beta v_1 \\
& + \frac{1}{6}t^3\beta^2v_1 + \frac{1}{20}at^5\alpha\beta^2v_1 - \frac{1}{24}t^4\beta^3v_1 + \frac{1}{6}at^3\chi v_1
\end{aligned}$$

Fig. 7. Effects of parameter  $\chi$  on the velocity field.Fig. 8. Effects of parameter  $a$  on the velocity field.

$$\begin{aligned}
& - \frac{1}{6}t^3\alpha\chi v_1 + \frac{1}{60}a^2t^5\alpha\chi v_1 - \frac{1}{30}at^5\alpha^2\chi v_1 \\
& + \frac{1}{1680}a^3t^7\alpha^2\chi v_1 - \frac{1}{8}at^4\beta\chi v_1 + \frac{1}{8}t^4\alpha\beta\chi v_1 \\
& - \frac{1}{72}a^2t^6\alpha\beta\chi v_1 + \frac{1}{20}at^5\beta^2\chi v_1 + \frac{1}{120}a^2t^5\chi^2 v_1 \\
& - \frac{1}{60}at^5\alpha\chi^2 v_1 + \frac{1}{1680}a^3t^7\alpha\chi^2 v_1 - \frac{1}{144}a^2t^6\beta\chi^2 v_1 \\
& + \frac{1}{5040}a^3t^7\chi^3 v_1 + \dots
\end{aligned}$$

In order to see the variations of different flow parameters and the numerical accuracy of VIM-Padé, we plotted velocity curves in Figures 1–8 and tabulated val-

Table 1. Solutions of VIM for  $u_1 = 1$ ,  $v_1 = 1$ ,  $\alpha = 0.1$ ,  $\beta = 0.1$ ,  $\chi = 0.1$ , and  $a = 0.1$ .

t	VIM-Padé solution of $x_1$	VIM-Padé solution of $x_2$
0	0.000000000000000	0.000000000000000
1	0.996828436052782	0.858049015474598
2	1.975846055206008	1.462230376303233
3	2.922348275153172	1.853087957668129
4	3.824538062069779	2.065728885888554
5	4.673001019855675	2.130437364405463
6	5.460160104690702	2.073145772992831
7	6.179688878497031	1.915733735382303
8	6.825869060768159	1.676120182221185

Table 2. Solutions of VIM for  $u_1 = 1$ ,  $v_1 = 1$ ,  $\alpha = 0.1$ ,  $\beta = 0.1$ ,  $\chi = 0.2$ , and  $a = 0.1$ .

t	VIM-Padé solution of $x_1$	VIM-Padé solution of $x_2$
0	0.000000000000000	0.000000000000000
1	0.995242651513581	0.811260601366052
2	1.963768793971209	1.286997767219024
3	2.883518147996572	1.483696022110341
4	3.736780120785085	1.449989345557795
5	4.509396420653593	1.227329635069655
6	5.189946887013818	0.850254347990026
7	5.768913069710821	0.346155377989198
8	6.237834468867694	-0.265544346377235

Table 3. Solutions of VIM for  $u_1 = 1$ ,  $v_1 = 1$ ,  $\alpha = 0.1$ ,  $\beta = 0.1$ ,  $\chi = 0.3$ , and  $a = 0.1$ .

t	VIM-Padé solution of $x_1$	VIM-Padé solution of $x_2$
0	0.000000000000000	0.000000000000000
1	0.99365686729836	0.764472171472630
2	1.95169161478452	1.111763156060844
3	2.84469010031789	1.114270309371584
4	3.64904271295788	0.834000811981103
5	4.34591276417448	0.323057678721741
6	4.92024736300550	-0.376713293577974
7	5.35987821398719	-1.235098471259883
8	5.65480093721809	-2.236049277802759

ues in Tables 1–3 for both components of the velocity fields.

## 7. Results and Discussions

In this section, we mainly discuss the effects of different flow parameters on the velocity field.

From Figures 1–8, we can draw the following results:

- Increasing the values of the parameter  $\alpha$  increases the velocity profile in  $x$ -direction.
- Increasing the values of the parameter  $\beta$  decreases the velocity profiles in  $x$ -direction.
- The velocity profiles decrease as the values of the parameter  $\chi$  increase.
- Increasing the parameter ‘ $a$ ’ increases the velocity profile.

- An increase of the parameter values  $\alpha$ ,  $\beta$ , and  $\chi$  reveals the same effects on the velocity profiles, that is decreasing, by increasing these parameters values in the case of  $y$ -direction.
- Increasing the parameter ‘ $a$ ’ increases the velocity field also in  $y$ -direction.

## 8. Concluding Remark

Our primary purpose here is to examine the effects of physical flow parameters on velocity profiles and offer a new approximate solution scheme for spherical motion of a particle in a plane Couette flow by using VIM-Padé. The technique overcomes the complexity as it arises in other methods for linearizing the original problem. We derived fast convergent results by combining the series obtained by VIM with the diagonal Padé approximants.

- [1] M. Jalaal and D. D. Ganji, *Powder Tech.* **198**, 82 (2010).
- [2] B. Khorshidi, M. Jalaal, E. Esmaeilzadeh, and F. Mohammadi, *J. Coll. Interf. Sci.* **352**, 211 (2010).
- [3] M. Jalaal, B. Khorshidi, and E. Esmaeilzadeh, *Exp. Therm. Fluid Sci.* **34**, 1498 (2010).
- [4] R. Clift, J. Grace, and M. E. Weber, *Bubbles, Drops and Particles*, Academic Press, New York 1978.
- [5] T. J. Vander Werff, *Z. Angew. Math. Phys.* **21**, 825 (1970).
- [6] M. Jalaal, D. D. Ganji, and G. Ahmadi, *Adv. Pow. Tech.* **21**, 298 (2010).
- [7] M. Torabi and H. Yaghoobi, *Adv. Pow. Tech.* **22**, 674 (2011).
- [8] M. Jalaal, M. G. Nejad, P. Jalili, M. Esmaeilpour, H. Bararnia, E. Ghasemi, S. Soleimani, D. D. Ganji, and S. M. Moghimi, *Comp. Math. Appl.* **61**, 2267 (2011).
- [9] J. H. He, *Comp. Meth. Appl. Mech. Eng.* **167**, 57 (1998).
- [10] J. H. He and X. H. Wu, *Comp. Math. Appl.* **54**, 881 (2007).
- [11] J. H. He, *Comp. Math. Appl.* **54**, 879 (2007).
- [12] J. H. He and X. H. Wu, *Comp. Math. Appl.* **54**, 881 (2007).
- [13] A. Ghorbani and J. Saberi-Nadjafi, *Nonlin. Sci. Lett.* **1**, 379 (2010).
- [14] M. Hussain and M. Khan, *Appl. Math. Sci.* **4**, 1931 (2010).
- [15] M. Khan, M. A. Gondal, and S. Kumar, *J. Math. Comp. Sci.* **3**, 135 (2011).
- [16] G. A. Baker, *Essentials of Padé Approximants*, Academic Press, London 1975.