Transient Oscillatory Flows of a Generalized Burgers' Fluid in a Rotating Frame

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In this paper, the unsteady oscillatory flows of a generalized Burgers' fluid in a rotating frame are investigated. The constitutive equations of the generalized Burgers' fluid are used in the mathematical formulation of the problem. The solutions are obtained by using the Laplace transform method. The graphical results are displayed and discussed for various parameters of interest. It is found that the velocity profiles reflect some interesting results for the rotation parameter and rheological fluid parameters.

Key words: Oscillatory Flow; Generalized Burgers' Fluid; Rotating Frame.

1. Introduction

The rotating flows of non-Newtonian fluids have paramount importance in meteorology, geophysics, cosmical fluid dynamics, turbomachinery etc., and currently it is an area of research undergoing rapid growth in the modern fluid mechanics. Specifically, the rotating flows are significant in the processing engineering and geofluid mechanics where the earth Coriolis force is considered and high velocity flows are required. The Coriolis force in the basic field equations is considered to be more significant in comparison to inertial forces. The literature on the rotating flows of viscous fluids is substantial ([1 – 7] and the references therein). However, such investigations are narrowed down when the rotating flows of non-Newtonian fluids are considered. The reason that the rotating flows have not been well studied for non-Newtonian fluids is the difficulty of the resultant problems. In these fluids, the governing equations are of higher order and more complicated in comparison to Navier–Stokes fluids. The constitutive relationships between stress and rate of strain in these fluids are complex in nature and give rise to extra terms in the arising equations [8 – 17]. Therefore in this study, we have chosen the generalized Burgers' fluid as a non-Newtonian fluid to investigate the unsteady rotating flows due to oscillatory motion of the boundary.

The layout of the paper is organized as follows. The mathematical formulation of the problem is given in Section 2. Section 3 comprises the solution expression in the transformed plane. The results and discussion are given in Section 4, and the influence of the emerging parameters on the velocity field is analyzed using graphs. The concluding remarks are given in Section 5.

2. Problem Formulation

Here, we assume that the rotating unsteady flow of an incompressible generalized Burgers' fluid occupies the semi-infinite non-porous space $z > 0$, with an infinite plate coinciding with the plane $z = 0$. The z-axis is taken normal to the plate. The fluid and the plate are in state of rigid body rotation with a constant angular velocity $\Omega_1 = \Omega_1 \hat{k}$ ($\hat{k}$ is a unit vector parallel to the z-axis). The flow in the fluid is caused by the oscillatory motion of the plate. Under the above assumptions, the continuity equation is identically satisfied and the momentum equations is given by [17]

$$\rho\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial F}{\partial t}+2\mathrm{i}\Omega_1 F\right) = \mu\left(1+\lambda_3\frac{\partial}{\partial t}+\lambda_4\frac{\partial^2}{\partial t^2}\right)\frac{\partial^2 F}{\partial z^2}, \quad (1)$$

where ρ is the fluid density and μ is the dynamic viscosity; λ_1 and λ_3 $(< \lambda_1)$, respectively, are the relaxation and retardation times and λ_2 and λ_4 are material constants having dimensions as the square of time and $F = u + \mathrm{i}v$.

The corresponding boundary and initial conditions are

$$F(0,t) = U_0 H(t)\cos(\omega_0 t) \ \text{or} \ U_0 \sin(\omega_0 t), \ t > 0, \tag{2}$$

$$F(\infty,t) = 0, \ t > 0, \tag{3}$$

$$F(z,0) = \frac{\partial F(z,0)}{\partial t} = \frac{\partial^2 F(z,0)}{\partial t^2} = 0, \ z > 0, \tag{4}$$

in which U_0 designates the constant plate velocity, ω_0 is the frequency of oscillation of the plate, and $H(t)$ is the Heaviside function.

Introducing the following dimensionless variables:

$$\begin{aligned} &\xi = \frac{ZU_0}{\nu}, \ G = \frac{F}{U_0} = \frac{u+\mathrm{i}v}{U_0}, \\ &\tau = \frac{U_0^2 t}{\nu}, \ \Omega = \frac{\Omega_1 \nu}{U_0^2}, \ \omega = \frac{\omega_0 \nu}{U_0^2}, \\ &\beta_i = \frac{\lambda_i U_0^2}{\nu} \ (i = 1,3), \\ &\gamma_i = \frac{\lambda_i U_0^4}{\nu^2} \ (i = 2,4), \end{aligned} \tag{5}$$

into (1)–(4), we obtain the following dimensionless system:

$$\left(1 + \beta_1 \frac{\partial}{\partial \tau} + \gamma_2 \frac{\partial^2}{\partial \tau^2}\right)\left(\frac{\partial G}{\partial \tau} + 2\mathrm{i}\Omega G\right)$$

$$= \left(1 + \beta_3 \frac{\partial}{\partial \tau} + \gamma_4 \frac{\partial^2}{\partial \tau^2}\right)\frac{\partial^2 G}{\partial \xi^2}, \tag{6}$$

$$G(0,\tau) = H(\tau)\cos(\omega\tau) \ \text{or} \ \sin(\omega\tau), \ \tau > 0, \tag{7}$$

$$G(\infty,\tau) = 0, \ \tau > 0, \tag{8}$$

$$G(\xi,0) = \frac{\partial G(\xi,0)}{\partial \tau} = \frac{\partial^2 G(\xi,0)}{\partial \tau^2} = 0, \ \xi > 0. \tag{9}$$

3. Solution of the Problem

Applying the Laplace transform to (6)–(8) and using the initial conditions (9), the boundary value problem in the transformed (ξ,q)-plane is given by

$$\frac{\mathrm{d}^2 \overline{G}(\xi,q)}{\mathrm{d}\xi^2} - \left(\gamma_2 q^3 + q^2(\beta_1 + 2\mathrm{i}\Omega\gamma_2) + q(1 + 2\mathrm{i}\Omega\beta_1) + 2\mathrm{i}\Omega\right)\left[\gamma_4 q^2 + \beta_3 q + 1\right]^{-1} \overline{G}(\xi,q) = 0, \tag{10}$$

$$\overline{G}(0,q) = \frac{q}{q^2+\omega^2} \ \text{or} \ \overline{G}(0,q) = \frac{\omega}{q^2+\omega^2}, \tag{11}$$

$$\overline{G}(\xi,q) \to 0 \ \text{as} \ \xi \to \infty, \tag{12}$$

where the Laplace transform of $G(\xi,\tau)$ is

$$\overline{G}(\xi,q) = L\{G(\xi,\tau)\} = \int_0^\infty \mathrm{e}^{-q\tau} G(\xi,\tau)\,\mathrm{d}\tau. \tag{13}$$

The general solution of (10) is given by

$$\overline{G}(\xi,q) = c_1 \mathrm{e}^{-\alpha\xi} + c_2 \mathrm{e}^{\alpha\xi}, \tag{14}$$

where

$$\alpha = \sqrt{\frac{\gamma_2 q^3 + q^2(\beta_1 + 2\mathrm{i}\Omega\gamma_2) + q(1 + 2\mathrm{i}\Omega\beta_1) + 2\mathrm{i}\Omega}{\gamma_4 q^2 + \beta_3 q + 1}}; \quad \operatorname{Re}\alpha > 0. \tag{15}$$

In view of (11) and (12), from (14), we find that

$$\overline{G}_{\mathrm{c}}(\xi,q) = \frac{q}{q^2+\omega^2}\exp\left[-\xi\sqrt{\frac{\gamma_2 q^3 + q^2(\beta_1 + 2\mathrm{i}\Omega\gamma_2) + q(1 + 2\mathrm{i}\Omega\beta_1) + 2\mathrm{i}\Omega}{\gamma_4 q^2 + \beta_3 q + 1}}\right], \tag{16}$$

$$\overline{G}_{\mathrm{s}}(\xi,q) = \frac{\omega}{q^2+\omega^2}\exp\left[-\xi\sqrt{\frac{\gamma_2 q^3 + q^2(\beta_1 + 2\mathrm{i}\Omega\gamma_2) + q(1 + 2\mathrm{i}\Omega\beta_1) + 2\mathrm{i}\Omega}{\gamma_4 q^2 + \beta_3 q + 1}}\right], \tag{17}$$

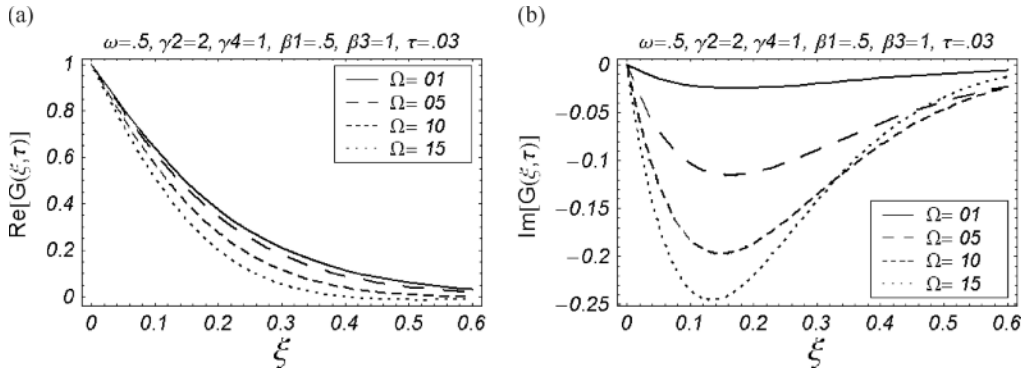

Fig. 1. Variation of velocity parts for various values of Ω.

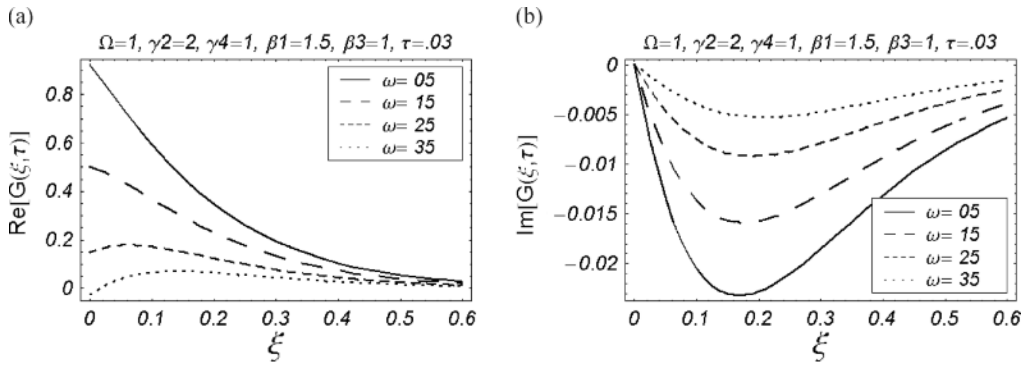

Fig. 2. Variation of velocity parts for various values of ω.

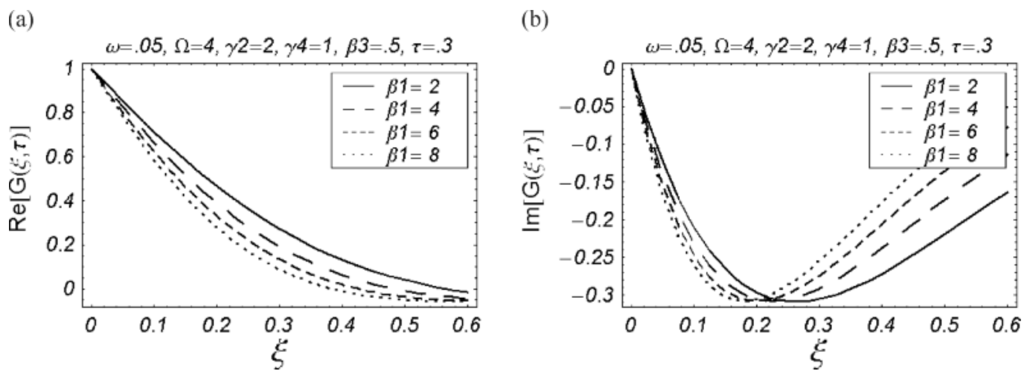

Fig. 3. Variation of velocity parts for various values of β_1.

where the subscripts c and s respectively indicate the cases for cosine and sine oscillations of the plate.

4. Results and Discussion

The closed form solutions of transient oscillatory flows of a generalized Burgers' fluid in a rotating frame are obtained using the Laplace transform. The integrals involved in the inverse Laplace transform of (16) and (17) are computed by symbolic computation software MATHEMATICA [17 – 19]. The results are plotted for various values of embedded flow parameters. The values of these parameters are clearly pointed out in these graphs. In order to illustrate the role of these parameters on the real and imaginary parts of velocity, Figures 1 – 6 have been displayed. In these figures pan-

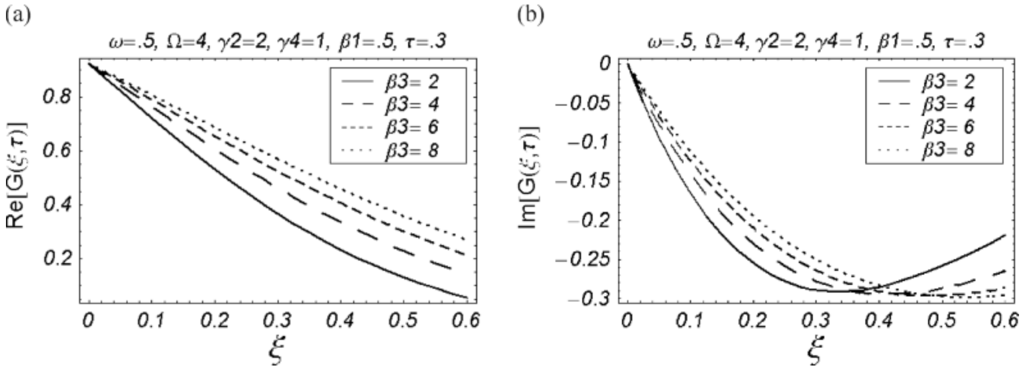

Fig. 4. Variation of velocity parts for various values of β_3.

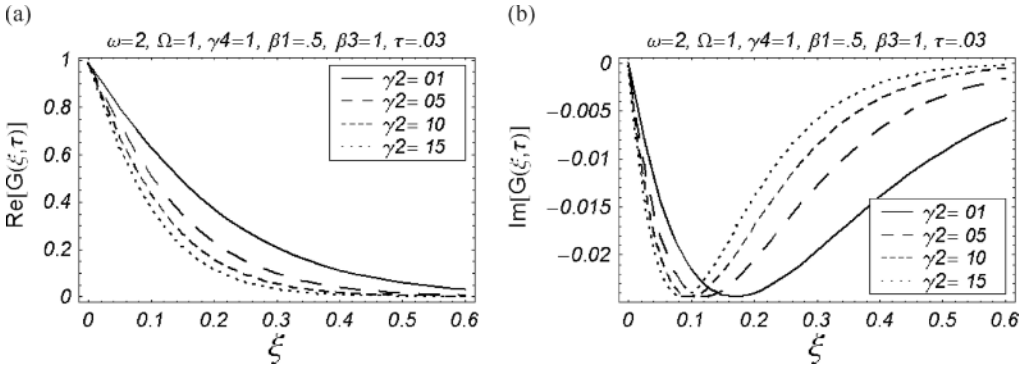

Fig. 5. Variation of velocity parts for various values of γ_2.

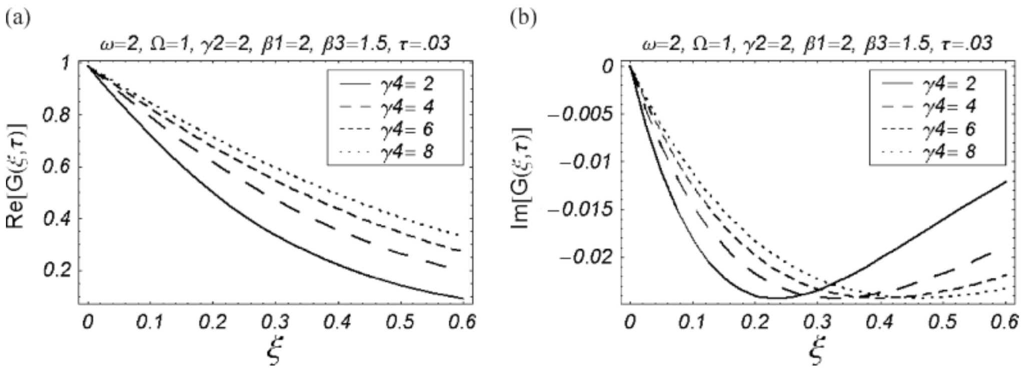

Fig. 6. Variation of velocity parts for various values of γ_4.

els (a) depict the variation of velocity on the real part while panels (b) indicate the variation of velocity on the imaginary part. However, the graphical results are only displayed for the cosine oscillations of the boundary.

It is depicted from Figure 1 that increasing the rotation parameter Ω, the magnitude of the real part of velocity and boundary layer thickness decreases whereas the effect is quite opposite on the imaginary part of the velocity. The magnitude of velocity and boundary layer thickness increases with increasing values of Ω. This is due to the fact that the Coriolis force acts as a constraint in the main fluid flow when the moving plate is suddenly set into oscillation. We can say that the Coriolis force ended fluid flow in the primary flow direction which corresponds to the real part of velocity, to induce cross flow and secondary flow which corresponds to the imaginary part of velocity in the flow field. It is found from Figure 2, that the magnitude of velocity for both real and imaginary parts are decreasing by increasing the frequency of oscillations ω. However, for large values of the independent variable ξ, the fluctuation reduces and the velocity approaches to zero.

Figures 3 and 4 are prepared to show the variation in rotation velocity for different values of non-Newtonian fluid parameters β_1 and β_3 also called the relaxation time and retardation time, respectively, when other parameters are kept fixed. It is noted from Figure 3 that for large values of β_1, the real part of velocity decreases whereas the magnitude of the imaginary part of velocity first increases and then decreases. Physically, it is justified due to the fact that the relaxation time has a reducing effect on the oscillatory flow, and hence the real part of the velocity decreases whereas the imaginary part first decreases and then increases with increasing values of β_1. As expected, the effect of β_3 is quite opposite to β_1 as shown in Figure 4. This is due to the fact that the retardation time enhances the flow field. The variation of γ_2 and γ_4 on the rotation velocity is shown in Figures 5 and 6. We know that λ_2 and λ_4 are material constants having dimensions as the square of time. Therefore, with increasing values of γ_2, we found from Figure 5 that the magnitude of the real part of the velocity decreases while the magnitude of the imaginary part of the velocity first increases and then decrease due to the oscillatory nature of the flow. Moreover, as shown in Figure 6, γ_4 enhances the fluid motion and its effect on the velocity is quite opposite to γ_2. The real part of the velocity increases whereas the magnitude of the imaginary part of the velocity first decreases and then increases with increasing values of γ_4.

5. Concluding Remarks

In this paper, we have obtained the exact solutions for the transient oscillatory flow of a generalized Burgers' fluid in a rotating frame using the Laplace transform technique. Graphical results have been displayed for the real and imaginary parts of the velocity for both cosine and sine oscillations of the plate. It has been observed that the involved parameters have strong influence on the fluid motion. The results for the impulsive motion of the plate (Hayat et al. [17]) can be obtained as a limiting case from the present solutions.

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