On the Nth Iterated Darboux Transformation and Soliton Solutions of a Coherently-Coupled Nonlinear Schrödinger System

Tao Xu[a], Ping-Ping Xin[a], Yi Zhang[a], and Juan Li[b,c]

[a] College of Science, China University of Petroleum, Beijing, 102249, China
[b] State Key Laboratory of Remote Sensing Science, Jointly Sponsored by the Institute of Remote Sensing Applications of Chinese Academy of Sciences, and Beijing Normal University, Beijing 100101, China
[c] Demonstration Centre, Spaceborne Remote Sensing National Space Administration, Beijing 100101, China

Reprint requests to T. X.; E-mail: xutao@cup.edu.cn

Z. Naturforsch. **68a,** 261 – 271 (2013) / DOI: 10.5560/ZNA.2012-0110
Received June 8, 2012/ revised October 7, 2012 / published online February 6, 2013

In this paper, we study an integrable coherently-coupled nonlinear Schrödinger system arising from low birefringent fibers and weakly anisotropic media. We construct the Nth iterated Darboux transformation (DT) in the explicit form and give a complete proof for the gauge equivalence of the associated Lax pair. By the DT-based algorithm, we derive the N-soliton solutions which can be uniformly represented in terms of the four-component Wronskians. We analyze the properties of coherently coupled solitons, revealing the parametric criterion for the non-degenerate solitons to respectively display the one- and double-hump profiles. In addition, we point out that the double-hump solitons may have potential application in realizing the multi-level optical communication.

Key words: Coherently-Coupled Nonlinear Schrödinger System; Darboux Transformation; Soliton Solutions; Four-Component Wronskians.
PACS numbers: 05.45.Yv; 02.30.Ik

1. Introduction

Coupled (or symbiotic) solitons with two or more components (modes) [1] have been experimentally observed in the AlGaAs planar wave guides [2], birefringent optical fibers [3], photorefractive materials [4], and fiber laser resonators [5]. In the Kerr or Kerr-like media, the co-propagation of two optical fields is usually governed by the coupled nonlinear Schrödinger (NLS) system [6],

$$\mathrm{i}\,q_{j,z}+q_{j,tt}+2\left(|q_j|^2+|q_{3-j}|^2\right)q_j=0 \quad (j=1,2), \tag{1}$$

which is also known as the Manakov system [7]. Equation (1) is said to be incoherently because the coupling is phase insensitive [8]. One of the most attractive properties associated with (1) is the Manakov soliton collision [3, 9 – 19]. Due to the intensity-coupling structure in (1), the collisions of the Manakov solitons with the internal degrees of freedom are more complicated than those for the scalar ones [9 – 14]. Depending on the pre-collision soliton parameters, (1) can exhibit the shape-changing collisions along with energy redistribution between two components, as well as shape-preserving collisions without energy transfer between two components [9 – 14]. Such two kinds of collisions are both considered to be elastic in the sense that the total energy of each coupled soliton is conserved [14]. It should be mentioned that the shape-changing collisions have been experimentally observed for the spatial coupled solitons in the Kerr-like media [15] and temporal ones in the linearly birefringent optical fibers [3]. More importantly, the Manakov soliton collisions have brought about the potential applications in implementing the all-optical digital computation [16, 17] and designing the 'solitonets' which are complex networks made up of interacting fields [18, 19].

In low birefringent fibers or weakly anisotropic media, one has to take into account the coherent coupling between two optical fields (that is, the coupling is dependent on relative phases of the interacting fields [8]). In this case, the propagation of two optical fields in a Kerr-type nonlinear medium is described by the following coherently-coupled NLS system [20, 21]:

$$\mathrm{i}\, q_{j,z} + q_{j,tt} + 2\left(|q_j|^2 + 2|q_{3-j}|^2\right) q_j - 2\bar{q}_j q_{3-j}^2 = 0 \quad (j = 1, 2)\,, \tag{2}$$

where the bar means complex conjugate, t represents the retarded time for the temporal case or transverse direction for the spatial case, z the propagation direction, q_j (j=1,2) are slowly varying envelopes of two interacting optical fields, the terms $|q_j|^2 q_j$, $|q_{3-j}|^2 q_j$ and $\bar{q}_j q_{3-j}^2$ are responsible for the self-phase modulation, cross-phase modulation, and coherent coupling for the energy exchange between two fields, respectively.

As an integrable model [21], (2) governs the propagation of optical beams in nonlinear Kerr media with linear optical activity and cubic anisotropy [20], and the trapping of two orthogonally polarized optical pulses in an isotropic medium with the three components $\chi^{(3)}_{xxyy}$, $\chi^{(3)}_{xyxy}$, and $\chi^{(3)}_{xyyx}$ of the third-order susceptibility tensor $\chi^{(3)}$ subject to the relation $\chi^{(3)}_{xxyy} + \chi^{(3)}_{xyxy} = -2\chi^{(3)}_{xyyx}$ [21]. It has been shown that (2) admits degenerate and non-degenerate solitons, where the former is of the usual sech profile [22 – 25], while the latter contains more free parameters and can display both the one- and double-hump profiles [24 – 27]. In the optical communication lines, the binary data '1' and '0' can be respectively represented by the presence and absence of an optical soliton, and thus the proximity of individual solitons determines the bit rate of a fiber communication system [6]. As a kind of complex objects formed by the superposition of two fundamental solitons, the double-hump solitons may be appropriate candidates for the multi-level communication in the birefringent or two-mode fibers [28].

In this paper, we will study (2) from the following three aspects: (i) Authors of [26, 27] have constructed the once-iterated DT and given the general scheme of Nth iterated DT. However, the explicit form of the Nth iterated DT as well as its rigorous proof was absent in [26, 27], and the uniform determinantal representation of N-soliton solutions has also not been obtained. In the way of [29], we will explicitly construct the Nth iterated DT for (2) and represent the general N-soliton solutions in terms of the four-component Wronskians. (ii) To our knowledge, the parametric conditions under which the one- and double-hump solitons can be respectively generated are still uncovered. Via the extreme value analysis, we will study the properties of coherently coupled solitons in (2). (iii) The shape-changing collisions of coupled solitons have potential applications in virtual digital computation [16, 17] and all-optical switching [30]. Authors of [24, 25] have reported the shape-changing collisions between degenerate and non-degenerate solitons. We will explore whether such nontrivial collisions can occur between two degenerate (or non-degenerate) solitons.

2. *N*th Iterated Darboux Transformation

In this section, we will construct the Nth iterated DT in the explicit form for (2) and give a complete proof for the gauge equivalence of the Lax pair associated with (2).

In the frame of the 4×4 Ablowitz–Kaup–Newell–Segur inverse scattering formulation [31], the Lax pair of (2) takes the form [21]

$$\begin{aligned} \Psi_t &= U(\lambda)\Psi = (\lambda\, U^{(\mathrm{I})} + U^{(\mathrm{II})})\Psi, \\ \Psi_z &= V(\lambda)\Psi = (\lambda^2 V^{(\mathrm{I})} + \lambda\, V^{(\mathrm{II})} + V^{(\mathrm{III})})\Psi, \end{aligned} \tag{3}$$

with

$$U^{(\mathrm{I})} = \mathrm{i}\begin{pmatrix} -\mathbf{I} & 0 \\ 0 & \mathbf{I} \end{pmatrix}, \quad U^{(\mathrm{II})} = \begin{pmatrix} 0 & \mathbf{Q} \\ -\mathbf{Q}^\dagger & 0 \end{pmatrix},$$
$$V^{(\mathrm{I})} = 2U^{(\mathrm{I})}, \quad V^{(\mathrm{II})} = 2U^{(\mathrm{II})},$$
$$V^{(\mathrm{III})} = \mathrm{i}\begin{pmatrix} \mathbf{Q}\mathbf{Q}^\dagger & \mathbf{Q}_t \\ \mathbf{Q}_t^\dagger & -\mathbf{Q}^\dagger\mathbf{Q} \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} q_1 & q_2 \\ -q_2 & q_1 \end{pmatrix},$$

where $\mathbf{I}$ is the 2×2 unit matrix, $\Psi = (\psi_1, \psi_2, \psi_3, \psi_4)^{\mathrm{T}}$ (the superscript T signifies the vector transpose) is the four-dimensional vector eigenfunction, λ is the spectral parameter, and the compatibility condition $U_z(\lambda) - V_t(\lambda) + [U(\lambda), V(\lambda)] = 0$ is exactly equivalent to (2).

We assume the Nth iterated eigenfunction transformation for Lax pair (3) be of the form

$$\Psi_N = T_N(\lambda)\Psi\,, \tag{4}$$

in which $\Psi_N = (\psi_{1N}, \psi_{2N}, \psi_{3N}, \psi_{4N})^{\mathrm{T}}$ is the Nth iterated eigenfunction that satisfies $\Psi_{N,t} = U_N(\lambda)\ \Psi_N$ and $\Psi_{N,z} = V_N(\lambda)\Psi_N$ with $U_N(\lambda)$ and $V_N(\lambda)$ being the same as $U(\lambda)$ and $V(\lambda)$ except that q_1 and q_2 are respectively replaced by the Nth iterated potentials q_{1N} and q_{2N}, and $T_N(\lambda)$ is the undetermined Nth iterated

Darboux matrix

$$T_N(\lambda)=\begin{pmatrix} A_{11}(\lambda) & A_{12}(\lambda) & B_{11}(\lambda) & B_{12}(\lambda)\\ A_{21}(\lambda) & A_{22}(\lambda) & B_{21}(\lambda) & B_{22}(\lambda)\\ C_{11}(\lambda) & C_{12}(\lambda) & D_{11}(\lambda) & D_{12}(\lambda)\\ C_{21}(\lambda) & C_{22}(\lambda) & C_{21}(\lambda) & D_{22}(\lambda)\end{pmatrix} \quad (5)$$

with

$$A_{ij}(\lambda)=(-\mathrm{i}\lambda)^N\delta_{ij}-\sum_{n=1}^{N}a_{ij}^{(n)}(-\mathrm{i}\lambda)^{n-1} \quad (1\le i,j\le 2)\,, \quad (6)$$

$$B_{ij}(\lambda)=-\sum_{n=1}^{N}b_{ij}^{(n)}(\mathrm{i}\lambda)^{n-1} \quad (1\le i,j\le 2)\,, \quad (7)$$

$$C_{ij}(\lambda)=-\sum_{n=1}^{N}c_{ij}^{(n)}(-\mathrm{i}\lambda)^{n-1} \quad (1\le i,j\le 2)\,, \quad (8)$$

$$D_{ij}(\lambda)=(-\mathrm{i}\lambda)^N\delta_{ij}-\sum_{n=1}^{N}d_{ij}^{(n)}(\mathrm{i}\lambda)^{n-1} \quad (1\le i,j\le 2)\,, \quad (9)$$

where δ_{ij} is the Kronecker delta function, $a_{ij}^{(n)}$, $b_{ij}^{(n)}$, $c_{ij}^{(n)}$, and $d_{ij}^{(n)}$ $(1\le i,j\le 2;\ 1\le n\le N)$ are the functions to be determined.

Note that if $\Psi_k=\big(e_k,f_k,g_k,h_k\big)^{\mathrm{T}}$ satisfies Lax pair (3) with $\lambda=\lambda_k$ $(1\le k\le N)$, then $\Psi_k'=\big(f_k,-e_k,h_k,-g_k\big)^{\mathrm{T}}$ is also a solution of Lax pair (3) with $\lambda=\lambda_k$, and $\Phi_k=\big(-\bar g_k,\bar h_k,\bar e_k,-\bar f_k\big)^{\mathrm{T}}$ and $\Phi_k'=\big(\bar h_k,\bar g_k,-\bar f_k,\ -\bar e_k\big)^{\mathrm{T}}$ are the solutions of Lax pair (3) with $\lambda=\bar\lambda_k$ [26, 27]. On the other hand, $\{\Psi_k\}_{k=1}^N$, $\{\Psi_k'\}_{k=1}^N$, $\{\Phi_k\}_{k=1}^N$, and $\{\Phi_k'\}_{k=1}^N$ are four sets of linearly-independent solutions. Hence, the functions $a_{ij}^{(n)}, b_{ij}^{(n)}, c_{ij}^{(n)}$, and $d_{ij}^{(n)}$ $(1\le i,j\le 2;\ 1\le n\le N)$ can be uniquely determined by requiring that

$$T_N(\lambda_k)\Psi_k=\mathbf{0}\quad(1\le k\le N)\,, \quad (10a)$$
$$T_N(\lambda_k)\Psi_k'=\mathbf{0}\quad(1\le k\le N)\,, \quad (10b)$$
$$T_N(\bar\lambda_k)\Phi_k=\mathbf{0}\quad(1\le k\le N)\,, \quad (10c)$$
$$T_N(\bar\lambda_k)\Phi_k'=\mathbf{0}\quad(1\le k\le N)\,. \quad (10d)$$

For convenience of calculating $a_{ij}^{(n)}$, $b_{ij}^{(n)}$, $c_{ij}^{(n)}$, and $d_{ij}^{(n)}$ $(1\le i,j\le 2;\ 1\le n\le N)$ from (10a)–(10d) via Cramer's rule, we introduce the following matrices and vectors:

$$E_M=\begin{pmatrix} e_1 & \cdots & (-\mathrm{i}\lambda_1)^{M-1}e_1\\ \vdots & \ddots & \vdots\\ e_N & \cdots & (-\mathrm{i}\lambda_N)^{M-1}e_N\end{pmatrix},\quad F_M=\begin{pmatrix} f_1 & \cdots & (-\mathrm{i}\lambda_1)^{M-1}f_1\\ \vdots & \ddots & \vdots\\ f_N & \cdots & (-\mathrm{i}\lambda_N)^{M-1}f_N\end{pmatrix}, \quad (11)$$

$$G_M=\begin{pmatrix} g_1 & \cdots & (\mathrm{i}\lambda_1)^{M-1}g_1\\ \vdots & \ddots & \vdots\\ g_N & \cdots & (\mathrm{i}\lambda_N)^{M-1}g_N\end{pmatrix},\quad H_M=\begin{pmatrix} h_1 & \cdots & (\mathrm{i}\lambda_1)^{M-1}h_1\\ \vdots & \ddots & \vdots\\ h_N & \cdots & (\mathrm{i}\lambda_N)^{M-1}h_N\end{pmatrix}, \quad (12)$$

$$\mathbf{e}=\big[(-\mathrm{i}\lambda_1)^Ne_1,\ldots,(-\mathrm{i}\lambda_N)^Ne_N\big],\quad \mathbf{f}=\big[(-\mathrm{i}\lambda_1)^Nf_1,\ldots,(-\mathrm{i}\lambda_N)^Nf_N\big]\,, \quad (13)$$

$$\mathbf{g}=\big[(\mathrm{i}\lambda_1)^Ng_1,\ldots,(\mathrm{i}\lambda_N)^Ng_N\big],\quad \mathbf{h}=\big[(\mathrm{i}\lambda_1)^Nh_1,\ldots,(\mathrm{i}\lambda_N)^Nh_N\big]\,, \quad (14)$$

$$\mathbf{a}_{ij}=\big(a_{ij}^{(1)},\ldots,a_{ij}^{(N)}\big),\quad \mathbf{b}_{ij}=\big(b_{ij}^{(1)},\ldots,b_{ij}^{(N)}\big)\,, \quad (15)$$

$$\mathbf{c}_{ij}=\big(c_{ij}^{(1)},\ldots,c_{ij}^{(N)}\big),\quad \mathbf{d}_{ij}=\big(d_{ij}^{(1)},\ldots,d_{ij}^{(N)}\big)\,. \quad (16)$$

Hereby, (10a)–(10d), with the change of the order of equations, can be written in the matrix form

$$A_\tau(\mathbf{a}_{11},\mathbf{a}_{12},\mathbf{b}_{11},\mathbf{b}_{12})^{\mathrm{T}}=(\mathbf{e},\mathbf{f},-\bar{\mathbf{g}},\bar{\mathbf{h}})^{\mathrm{T}}\,, \quad (17a)$$
$$A_\tau(\mathbf{a}_{21},\mathbf{a}_{22},\mathbf{b}_{21},\mathbf{b}_{22})^{\mathrm{T}}=(\mathbf{f},-\mathbf{e},\bar{\mathbf{h}},\bar{\mathbf{g}})^{\mathrm{T}}\,, \quad (17b)$$
$$A_\tau(\mathbf{c}_{11},\mathbf{c}_{12},\mathbf{d}_{11},\mathbf{d}_{12})^{\mathrm{T}}=(-1)^N(\mathbf{g},\mathbf{h},\bar{\mathbf{e}},-\bar{\mathbf{f}})^{\mathrm{T}}\,, \quad (17c)$$
$$A_\tau(\mathbf{c}_{21},\mathbf{c}_{22},\mathbf{d}_{21},\mathbf{d}_{22})^{\mathrm{T}}=(-1)^N(\mathbf{h},-\mathbf{g},-\bar{\mathbf{f}},-\bar{\mathbf{e}})^{\mathrm{T}}\,, \quad (17d)$$

with

$$A_\tau=\begin{pmatrix} E_N & F_N & G_N & H_N\\ F_N & -E_N & H_N & -G_N\\ -\bar G_N & \bar H_N & \bar E_N & -\bar F_N\\ \bar H_N & \bar G_N & -\bar F_N & -\bar E_N\end{pmatrix}. \quad (18)$$

By Cramer's rule, one can obtain from (17a) – (17d) that

$$b_{11}^{(N)} = \frac{-\chi_{11}}{\tau}, \qquad b_{12}^{(N)} = \frac{(-1)^{N-1}\chi_{12}}{\tau}, \tag{19}$$

$$b_{21}^{(N)} = \frac{(-1)^{N-1}\chi_{21}}{\tau}, \qquad b_{22}^{(N)} = \frac{-\chi_{22}}{\tau}, \tag{20}$$

$$c_{11}^{(N)} = \frac{(-1)^{N}\chi'_{11}}{\tau}, \qquad c_{12}^{(N)} = \frac{\chi'_{12}}{\tau}, \tag{21}$$

$$c_{21}^{(N)} = \frac{\chi'_{21}}{\tau}, \qquad c_{22}^{(N)} = \frac{(-1)^{N}\chi'_{22}}{\tau}, \tag{22}$$

where $\tau = |A_\tau|$; χ_{ij} and χ'_{ij} $(1 \le i, j \le 2)$ are the following determinants:

$$\chi_{11} = \begin{vmatrix} E_{N+1} & F_N & G_{N-1} & H_N \\ F_{N+1} & -E_N & H_{N-1} & -G_N \\ -\bar{G}_{N+1} & \bar{H}_N & \bar{E}_{N-1} & -\bar{F}_N \\ \bar{H}_{N+1} & \bar{G}_N & -\bar{F}_{N-1} & -\bar{E}_N \end{vmatrix}, \quad \chi_{12} = \begin{vmatrix} E_{N+1} & F_N & G_N & H_{N-1} \\ F_{N+1} & -E_N & H_N & -G_{N-1} \\ -\bar{G}_{N+1} & \bar{H}_N & \bar{E}_N & -\bar{F}_{N-1} \\ \bar{H}_{N+1} & \bar{G}_N & -\bar{F}_N & -\bar{E}_{N-1} \end{vmatrix}, \tag{23}$$

$$\chi_{21} = \begin{vmatrix} E_N & F_{N+1} & G_{N-1} & H_N \\ F_N & -E_{N+1} & H_{N-1} & -G_N \\ -\bar{G}_N & \bar{H}_{N+1} & \bar{E}_{N-1} & -\bar{F}_N \\ \bar{H}_N & \bar{G}_{N+1} & -\bar{F}_{N-1} & -\bar{E}_N \end{vmatrix}, \quad \chi_{22} = \begin{vmatrix} E_N & F_{N+1} & G_N & H_{N-1} \\ F_N & -E_{N+1} & H_N & -G_{N-1} \\ -\bar{G}_N & \bar{H}_{N+1} & \bar{E}_N & -\bar{F}_{N-1} \\ \bar{H}_N & \bar{G}_{N+1} & -\bar{F}_N & -\bar{E}_{N-1} \end{vmatrix}, \tag{24}$$

$$\chi'_{11} = \begin{vmatrix} E_{N-1} & F_N & G_{N+1} & H_N \\ F_{N-1} & -E_N & H_{N+1} & -G_N \\ -\bar{G}_{N-1} & \bar{H}_N & \bar{E}_{N+1} & -\bar{F}_N \\ \bar{H}_{N-1} & \bar{G}_N & -\bar{F}_{N+1} & -\bar{E}_N \end{vmatrix}, \quad \chi'_{12} = \begin{vmatrix} E_N & F_{N-1} & G_{N+1} & H_N \\ F_N & -E_{N-1} & H_{N+1} & -G_N \\ -\bar{G}_N & \bar{H}_{N-1} & \bar{E}_{N+1} & -\bar{F}_N \\ \bar{H}_N & \bar{G}_{N-1} & -\bar{F}_{N+1} & -\bar{E}_N \end{vmatrix}, \tag{25}$$

$$\chi'_{21} = \begin{vmatrix} E_{N-1} & F_N & G_N & H_{N+1} \\ F_{N-1} & -E_N & H_N & -G_{N+1} \\ -\bar{G}_{N-1} & \bar{H}_N & \bar{E}_N & -\bar{F}_{N+1} \\ \bar{H}_{N-1} & \bar{G}_N & -\bar{F}_N & -\bar{E}_{N+1} \end{vmatrix}, \quad \chi'_{22} = \begin{vmatrix} E_N & F_{N-1} & G_N & H_{N+1} \\ F_N & -E_{N-1} & H_N & -G_{N+1} \\ -\bar{G}_N & \bar{H}_{N-1} & \bar{E}_N & -\bar{F}_{N+1} \\ \bar{H}_N & \bar{G}_{N-1} & -\bar{F}_N & -\bar{E}_{N+1} \end{vmatrix}. \tag{26}$$

To verify the form-invariance of Lax pair (3), we need to prove that

$$T_{N,t}(\lambda) + T_N(\lambda)U(\lambda) = U_N(\lambda)T_N(\lambda), \tag{27a}$$

$$T_{N,z}(\lambda) + T_N(\lambda)V(\lambda) = V_N(\lambda)T_N(\lambda) \tag{27b}$$

are satisfied with the Darboux matrix $T_N(\lambda)$ given by (5), in which $a_{ij}^{(n)}$, $b_{ij}^{(n)}$, $c_{ij}^{(n)}$, and $d_{ij}^{(n)}$ $(1 \le i, j \le 2;\ 1 \le n \le N)$ are determined by (17a) – (17d). Based on Lemmas 1 – 3 (see Appendix), we can arrive at the following proposition:

Proposition 1. Suppose that $(e_k, f_k, g_k, h_k)^{\mathrm{T}}$ is the solution of Lax pair (3) with $\lambda = \lambda_k$ $(1 \le k \le N)$. Then, the Darboux matrix $T_N(\lambda)$ in (5) obeys the conditions in (27a) and (27b), provided that $a_{ij}^{(n)}$, $b_{ij}^{(n)}$, $c_{ij}^{(n)}$, and $d_{ij}^{(n)}$ $(1 \le i, j \le 2;\ 1 \le n \le N)$ are determined by (17a)–(17d), and the Nth iterated potential transformations are given by

$$q_{1N} = q_1 + 2(-1)^{N-1} b_{11}^{(N)}, \quad q_{2N} = q_2 + 2(-1)^{N-1} b_{12}^{(N)}. \tag{28}$$

Proof. We equivalently prove that the Darboux matrix $T_N(\lambda)$ in (5) obeys

$$U_N(\lambda)\det T_N(\lambda) = [T_{N,t}(\lambda) + T_N(\lambda)U(\lambda)]T_N^*(\lambda), \tag{29a}$$

$$V_N(\lambda)\det T_N(\lambda) = [T_{N,z}(\lambda) + T_N(\lambda)V(\lambda)]T_N^*(\lambda), \tag{29b}$$

where $T_N^*(\lambda)$ is the adjoint matrix of $T_N(\lambda)$. Let us define that $[u_{hl}(\lambda)]_{4\times4} = [T_{N,t}(\lambda) + T_N(\lambda)U(\lambda)]T_N^*(\lambda)$ and $[v_{hl}(\lambda)]_{4\times4} = [T_{N,z}(\lambda) + T_N(\lambda)V(\lambda)]T_N^*(\lambda)$.

By calculation, we have

$$\deg[u_{hh}(\lambda)] = 4N + 1,$$
$$\deg[v_{hh}(\lambda)] = 4N + 2 \quad (1 \le h \le 4),$$
$$\deg[u_{hl}(\lambda)] = 4N,$$
$$\deg[v_{hl}(\lambda)] = 4N + 1 \quad (1 \le h, l \le 4;\ h \ne l),$$

where $\deg[f(\lambda)]$ represents the degree of the polynomial $f(\lambda)$. On the other hand, Lemmas 1 and 2 imply that $u_{hl}(\lambda)$ and $v_{hl}(\lambda)$ $(1 \le h, l \le 4)$ can be exactly divided by $\det T_N(\lambda)$. That is to say, the matrices $U_N(\lambda)$ and $V_N(\lambda)$ are of the form

$$U_N(\lambda) = \lambda\, U_N^{(\mathrm{I})} + U_N^{(\mathrm{II})}, \quad V_N(\lambda) = \lambda^2 V_N^{(\mathrm{I})} + \lambda\, V_N^{(\mathrm{II})} + V_N^{(\mathrm{III})}. \tag{30}$$

Substituting (30) into (29a) and (29b), and comparing the coefficients of λ^{4N+1} and λ^{4N} in (29a) and those of λ^{4N+2}, λ^{4N+1} and λ^{4N} in (29b), we find that $U_N^{(\mathrm{I})}=U^{(\mathrm{I})}$, $V_N^{(\mathrm{I})}=V^{(\mathrm{I})}$, and $U_N^{(\mathrm{II})}$, $V_N^{(\mathrm{II})}$, and $V_N^{(\mathrm{III})}$ have the same form as $U^{(\mathrm{II})}$, $V^{(\mathrm{II})}$, and $V^{(\mathrm{III})}$ under the following conditions:

$$q_{1N}=q_1+2\,(-1)^{N-1}b_{11}^{(N)},\quad q_{2N}=q_2+2\,(-1)^{N-1}b_{12}^{(N)},\tag{31}$$

$$q_{1N}=q_1+2\,(-1)^{N-1}b_{22}^{(N)},\quad q_{2N}=q_2-2\,(-1)^{N-1}b_{21}^{(N)},\tag{32}$$

$$\bar q_{1N}=\bar q_1+2\,c_{11}^{(N)},\ \ \bar q_{2N}=\bar q_2-2\,c_{12}^{(N)},\tag{33}$$

$$\bar q_{1N}=\bar q_1+2\,c_{22}^{(N)},\ \ \bar q_{2N}=\bar q_2+2\,c_{21}^{(N)},\tag{34}$$

which can be reduced to (28) by virtue of the relations (A.2a)–(A.2d) in Lemma 3. □

As suggested by Proposition 1, the Darboux matrix $T_N(\lambda)$ assures that the new eigenfunction $\Psi_N=T_N(\lambda)\Psi$ also satisfies Lax pair (3) for the new potentials q_{1N} and q_{2N} in (28). That is to say, the compatibility condition $\Psi_{N,tz}=\Psi_{N,zt}$ yields the same (2) except for q_{1N} and q_{2N} instead of q_1 and q_2, respectively. Therefore, for a given set of linearly-independent solutions $\{\Psi_k,\Psi_k',\Phi_k,\Phi_k'\}_{k=1}^N$ of Lax pair (3), the eigenfunction transformation (4) and potential transformations (28) constitute the Nth iterated DT: $(\Psi,q_1,q_2)\to(\Psi_N,q_{1N},q_{2N})$ of (2), where the Darboux matrix $T_N(\lambda)$ is determined by (17a)–(17d).

3. Soliton Solutions in Terms of the Four-Component Wronskians

In this section, we will derive the four-component Wronskian representation of the N-soliton solutions to (2) by the above Nth iterated DT algorithm starting from $q_1=q_2=0$. On this basis, we will find the parametric conditions for the generation of one- and double-hump solitons, and analyze the collisions of degenerate and non-degenerate coupled solitons in (2). For convenience, we use the subscripts R and I to represent the real and imaginary parts, respectively.

For a given set of complex parameters $\{\lambda_k\}_{k=1}^N$ ($\lambda_k\neq\lambda_l$), we solve Lax pair (3) with $q_1=q_2=0$ and $\lambda=\lambda_k$, obtaining

$$(e_k,f_k,g_k,h_k)^{\mathrm T}=\left(\alpha_k\mathrm e^{\theta_k},\beta_k\mathrm e^{\theta_k},\gamma_k\,\mathrm e^{-\theta_k},\delta_k\,\mathrm e^{-\theta_k}\right)^{\mathrm T}\quad(1\le k\le N),\tag{35}$$

where the phase $\theta_k=-\mathrm i\lambda_k t-2\,\mathrm i\lambda_k^2 z$; α_k, β_k, γ_k, and δ_k are complex constants. Substitution of (35) into (28) gives the four-component Wronskian solutions to (2) as follows:

$$q_{1N}=2\,(-1)^N\frac{\chi_{11}}{\tau},\ \ q_{2N}=2\,\frac{\chi_{12}}{\tau},\tag{36}$$

with

$$\chi_{11}=\begin{vmatrix}\Lambda_1\Theta_+K_{+,N+1} & \Lambda_2\Theta_+K_{+,N} & \Lambda_3\Theta_-K_{-,N-1} & \Lambda_4\Theta_-K_{-,N}\\ \Lambda_2\Theta_+K_{+,N+1} & -\Lambda_1\Theta_+K_{+,N} & \Lambda_4\Theta_-K_{-,N-1} & -\Lambda_3\Theta_-K_{-,N}\\ -\Lambda_3^*\Theta_-^*K_{-,N+1}^* & \Lambda_4^*\Theta_-^*K_{-,N}^* & \Lambda_1^*\Theta_+^*K_{+,N-1}^* & -\Lambda_2^*\Theta_+^*K_{+,N}^*\\ \Lambda_4^*\Theta_-^*K_{-,N+1}^* & -\Lambda_3^*\Theta_-^*K_{-,N}^* & -\Lambda_2^*\Theta_+^*K_{+,N-1}^* & -\Lambda_1^*\Theta_+^*K_{+,N}^*\end{vmatrix},\tag{37}$$

$$\chi_{12}=\begin{vmatrix}\Lambda_1\Theta_+K_{+,N+1} & \Lambda_2\Theta_+K_{+,N} & \Lambda_3\Theta_-K_{-,N} & \Lambda_4\Theta_-K_{-,N-1}\\ \Lambda_2\Theta_+K_{+,N+1} & -\Lambda_1\Theta_+K_{+,N} & \Lambda_4\Theta_-K_{-,N} & -\Lambda_3\Theta_-K_{-,N-1}\\ -\Lambda_3^*\Theta_-^*K_{-,N+1}^* & \Lambda_4^*\Theta_-^*K_{-,N}^* & \Lambda_1^*\Theta_+^*K_{+,N}^* & -\Lambda_2^*\Theta_+^*K_{+,N-1}^*\\ \Lambda_4^*\Theta_-^*K_{-,N+1}^* & -\Lambda_3^*\Theta_-^*K_{-,N}^* & -\Lambda_2^*\Theta_+^*K_{+,N}^* & -\Lambda_1^*\Theta_+^*K_{+,N-1}^*\end{vmatrix},\tag{38}$$

$$\tau=\begin{vmatrix}\Lambda_1\Theta_+K_{+,N} & \Lambda_2\Theta_+K_{+,N} & \Lambda_3\Theta_-K_{-,N} & \Lambda_4\Theta_-K_{-,N}\\ \Lambda_2\Theta_+K_{+,N} & -\Lambda_1\Theta_+K_{+,N} & \Lambda_4\Theta_-K_{-,N} & -\Lambda_3\Theta_-K_{-,N}\\ -\Lambda_3^*\Theta_-^*K_{-,N}^* & \Lambda_4^*\Theta_-^*K_{-,N}^* & \Lambda_1^*\Theta_+^*K_{+,N}^* & -\Lambda_2^*\Theta_+^*K_{+,N}^*\\ \Lambda_4^*\Theta_-^*K_{-,N}^* & -\Lambda_3^*\Theta_-^*K_{-,N}^* & -\Lambda_2^*\Theta_+^*K_{+,N}^* & -\Lambda_1^*\Theta_+^*K_{+,N}^*\end{vmatrix},\tag{39}$$

where $\Lambda_1=\mathrm{diag}(\alpha_1,\ldots,\alpha_N)$, $\Lambda_2=\mathrm{diag}(\beta_1,\ldots,\beta_N)$, $\Lambda_3=\mathrm{diag}(\gamma_1,\ldots,\gamma_N)$, $\Lambda_4=\mathrm{diag}(\delta_1,\ldots,\delta_N)$, $\Theta_+=\mathrm{diag}\left(\mathrm e^{\theta_1},\ldots,\mathrm e^{\theta_N}\right)$, $\Theta_-=\mathrm{diag}\left(\mathrm e^{-\theta_1},\ldots,\mathrm e^{-\theta_N}\right)$, $K_{+,M}=\left[(-\mathrm i\lambda_n)^{m-1}\right]_{N\times M}$, $K_{-,M}=\left[(\mathrm i\lambda_n)^{m-1}\right]_{N\times M}$ ($M=N-1,N,N+1$). Note from Lemma 3 that τ is a real function, and χ_{ij} and χ_{ij}' ($1\le i,j\le 2$) obey the relations:

$$\bar{\chi}_{11} = \chi'_{11}\,, \ \ \bar{\chi}_{12} = \chi'_{21}\,, \ \ \bar{\chi}_{21} = \chi'_{12}\,, \ \ \bar{\chi}_{22} = \chi'_{22}\,. \quad (40)$$

On the other hand, via the Laplace expansion technique, we can obtain the following four-component Wronskian identity:

$$\begin{aligned} &\tau\,\tau_{tt} - \tau_t^2 \\ &= 4\,(\chi_{11}\chi'_{11} + \chi_{12}\chi'_{21} + \chi_{21}\chi'_{12} + \chi_{22}\chi'_{22})\,. \end{aligned} \quad (41)$$

Combining (40) and (41), we have

$$\begin{aligned} &2\,(|q_{1N}|^2 + |q_{2N}|^2) \\ &= \frac{4\,(\chi_{11}\bar{\chi}_{11} + \chi_{12}\bar{\chi}_{12} + \chi_{21}\bar{\chi}_{21} + \chi_{22}\bar{\chi}_{22})}{\tau^2} \\ &= \frac{\tau\,\tau_{tt} - \tau_t^2}{\tau^2}\,, \end{aligned} \quad (42)$$

which implies that the function τ has no zeros in the tz-plane unless $\alpha_k = \beta_k = \gamma_k = \delta_k = 0$ for all $1 \le k \le N$. Therefore, we can safely say that (36) together with (37) – (39) represent the N-soliton solutions to (2) if α_k, β_k, γ_k, and δ_k are not all equal to zero. As remarked in [11 – 13], we can without loss of generality take $\alpha_k = 1$ for $1 \le k \le N$. That is to say, the N-soliton solutions to (2) are in general characterized by $4N$ complex parameters $\{\beta_k, \gamma_k, \delta_k, \lambda_k\}_{k=1}^N$, which is greater than the number of those obtained by the non-standard Hirota method in [24, 25].

With $N = 1$, (36) imply the following three families of one-soliton solutions:

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{\lambda_{1I}\,\mu_1 e^{\theta_1 - \bar{\theta}_1}}{|\mu_1\,\kappa_1|}\,\operatorname{sech}\left(\xi_1 + \ln\frac{|\mu_1|}{2\,|\kappa_1|}\right)\begin{pmatrix} \bar{\kappa}_1 \\ \bar{\kappa}_2 \end{pmatrix} \quad (43)$$
$$(\gamma_1 = \pm \mathrm{i}\,\delta_1)\,,$$

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{\lambda_{1I}\,\bar{\nu}_1 e^{\theta_1 - \bar{\theta}_1}}{|\nu_1\,\kappa_1|}\,\operatorname{sech}\left(\xi_1 + \ln\frac{2\,|\kappa_1|}{|\nu_1|}\right)\begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix} \quad (44)$$
$$(\beta_1 = \pm \mathrm{i})\,,$$

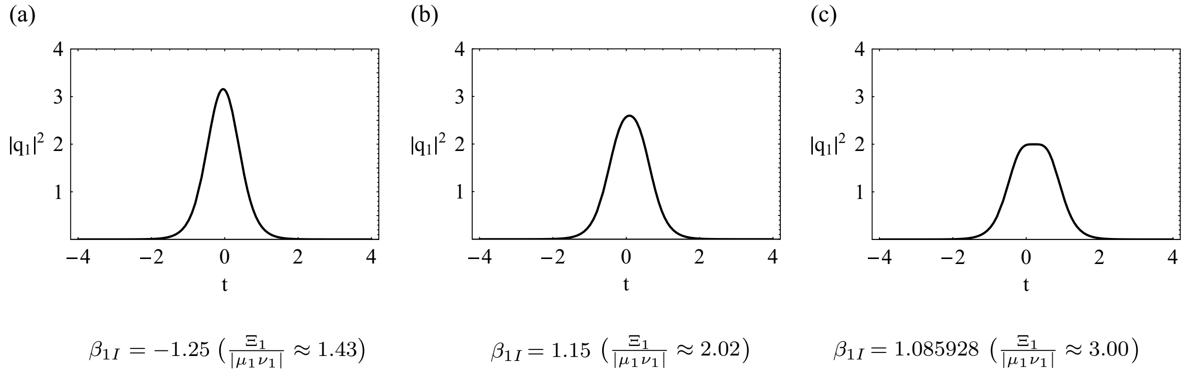

Fig. 1. Non-degenerate one-hump solitons for q_1 via (45) transverse at $z = 0.45$, with the parameters chosen as $\beta_1 = -0.05 + \beta_{1I}\,\mathrm{i}$, $\gamma_1 = 0.01 + 0.01\,\mathrm{i}$, $\delta_1 = 0.01 - 0.02\,\mathrm{i}$, and $\lambda_1 = -1 + \mathrm{i}$.

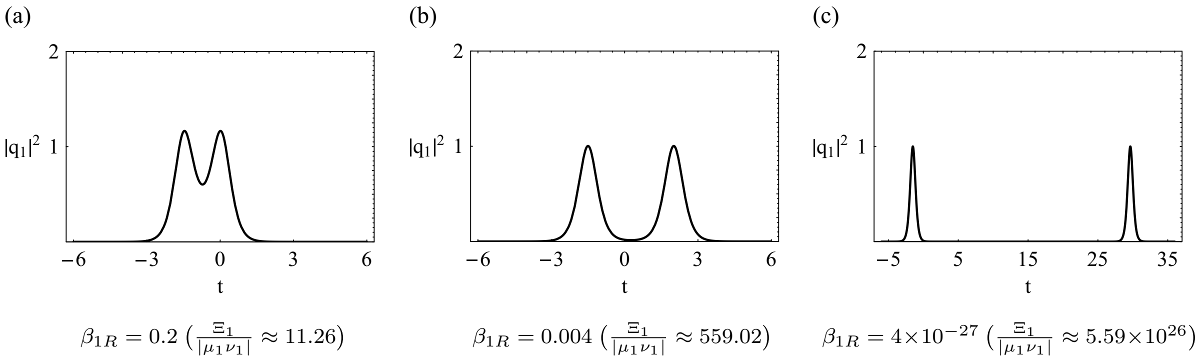

Fig. 2. Non-degenerate double-hump solitons for q_1 via (45) transverse at $z = 0$, with the parameters chosen as $\beta_1 = \beta_{1R} + \mathrm{i}$, $\gamma_1 = -0.1 - 0.1\,\mathrm{i}$, $\delta_1 = -0.1$, and $\lambda_1 = -1 + \mathrm{i}$. The distance between two humps are respectively as follows: (a) $d_1 \approx 1.47$; (b) $d_1 \approx 3.51$; (c) $d_1 \approx 31.14$.

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{\lambda_{1I} e^{\theta_1 - \bar{\theta}_1}}{|\mu_1 \nu_1| \cosh^2\left(\xi_1 + \ln\sqrt{\frac{|\mu_1|}{|\nu_1|}}\right) + \Xi_1 - |\mu_1 \nu_1|} \cdot \left[e^{-\xi_1} \begin{pmatrix} \bar{\nu}_1 \kappa_1 - \frac{|\mu_1 \nu_1|}{\bar{\mu}_1} \bar{\kappa}_1 \\ \bar{\nu}_1 \kappa_2 - \frac{|\mu_1 \nu_1|}{\bar{\mu}_1} \bar{\kappa}_2 \end{pmatrix} + 2\mu_1 \sqrt{\frac{|\nu_1|}{|\mu_1|}} \cosh\left(\xi_1 + \ln\sqrt{\frac{|\mu_1|}{|\nu_1|}}\right) \cdot \begin{pmatrix} \bar{\kappa}_1 \\ \bar{\kappa}_2 \end{pmatrix} \right] \quad (\beta_1 \neq \pm \mathrm{i},\, \gamma_1 \neq \pm \mathrm{i}\delta_1), \tag{45}$$

with

$$\mu_1 = 1 + \beta_1^2,\ \ \nu_1 = \gamma_1^2 + \delta_1^2,\ \ \kappa_1 = \gamma_1 + \beta_1 \delta_1,$$
$$\kappa_2 = \delta_1 - \beta_1 \gamma_1,\ \ \xi_1 = \theta_1 + \bar{\theta}_1,\ \ \Xi_1 = (\beta_1 - \bar{\beta}_1) \cdot (\delta_1 \bar{\gamma}_1 - \gamma_1 \bar{\delta}_1) + (1 + |\beta_1|^2)(|\gamma_1|^2 + |\delta_1|^2).$$

Solutions (43) and (44) are both degenerate in the sense that they represent only the one-hump solitons. In both the two cases, the components q_1 and q_2 have the same intensities. Solution (45) is non-degenerate because it could describe either the one- or double-hump soliton, depending on the choice of parameters.

In order to figure out the dependence of the soliton profiles on the parameters in (45), we take the derivatives of $|q_j|^2$ $(j = 1, 2)$ with respect to ξ_1, giving that

$$\frac{\partial |q_j|^2}{\partial \xi_1} = \frac{32 \lambda_{1I}^2 e^{2\xi_1} \left(|\mu_1|^2 e^{4\xi_1} - |\nu_1|^2\right)}{\left(|\mu_1|^2 e^{4\xi_1} + 2\Xi_1 e^{2\xi_1} + |\nu_1|^2\right)^3} \cdot \left[|\kappa_j|^2 |\mu_1|^2 e^{4\xi_1} + |\kappa_j|^2 |\nu_1|^2 + 2\left(\mu_1 \nu_1 \bar{\kappa}_j^2 + \bar{\mu}_1 \bar{\nu}_1 \kappa_j^2 - |\kappa_j|^2 \Xi_1\right) e^{2\xi_1} \right] \ \ (j = 1, 2), \tag{46}$$

which suggests that the maxima of $|q_j|^2$ are related to Δ_j $(j = 1, 2)$ defined by

$$\Delta_1 = \kappa_1^2 \left(\bar{\delta}_1^2 + \bar{\beta}_1^2 \bar{\gamma}_1^2\right) + \bar{\kappa}_1^2 \left(\delta_1^2 + \beta_1^2 \gamma_1^2\right) + \left(\beta_1 \bar{\gamma}_1 \delta_1 - \bar{\beta}_1 \gamma_1 \bar{\delta}_1\right)^2 + \left(|\gamma_1|^2 - |\beta_1|^2 |\delta_1|^2\right)^2, \tag{47}$$

$$\Delta_2 = \kappa_2^2 \left(\bar{\gamma}_1^2 + \bar{\beta}_1^2 \bar{\delta}_1^2\right) + \bar{\kappa}_2^2 \left(\gamma_1^2 + \beta_1^2 \delta_1^2\right) + \left(\beta_1 \bar{\delta}_1 \gamma_1 - \bar{\beta}_1 \delta_1 \bar{\gamma}_1\right)^2 + \left(|\delta_1|^2 - |\beta_1|^2 |\gamma_1|^2\right)^2. \tag{48}$$

For the case $\Delta_j \leq 0$, it is seen from (46) that $|q_j|^2$ has only one maximum

$$|q_j|^2_{\max} = 4\lambda_{1I}^2 \frac{|\nu_1|}{|\mu_1|} \cdot \frac{\left| |\mu_1| \nu_1 |\bar{\kappa}_j + |\mu_1| \bar{\nu}_1 \kappa_j \right|^2}{\left(|\nu_1| \Xi_1 + |\mu_1| |\nu_1|^2\right)^2} \quad (j = 1, 2) \tag{49}$$

along the line $\xi_1 = \frac{1}{2} \ln \frac{|\nu_1|}{|\mu_1|}$ in the tz-plane. Figures 1a–c present three types of one-hump solitons with the parameters satisfying $\Delta_j \leq 0$. One can observe that the top of $|q_j|^2$ tends to be flatter with the increase of $\frac{\Xi_1}{|\mu_1 \nu_1|}$. If $\Delta_j > 0$, the non-degenerate soliton exhibiting the double-hump profile is characterized by the following three features: (i) The two humps in $|q_j|^2$ are symmetric and have the same height, that is,

$$|q_j|^2_{\max} = \frac{16 \lambda_{1I}^2 |\kappa_j|^4 \left| \sqrt{\Delta_j} + |\kappa_j|^2 \Xi_1 - \mu_1 \nu_1 \bar{\kappa}_j^2 \right|^2 \left[\sqrt{\Delta_j} - 2\,\mathrm{Re}(\bar{\mu}_1 \bar{\nu}_1 \kappa_j^2) + |\kappa_j|^2 \Xi_1 \right]}{\left[\Delta_j + 4\sqrt{\Delta_j} |\kappa_j|^2 \Xi_1 + 3|\kappa_j|^4 (\Xi_1^2 + |\mu_1|^2 |\nu_1|^2) + 2\,\mathrm{Re}(\Omega_j) \right]^2} \quad (j = 1, 2), \tag{50}$$

with $\Omega_j = \mu_1^2 \nu_1^2 \bar{\kappa}_j^4 - 4\mu_1 \nu_1 \Xi_1 |\kappa_j|^2 \bar{\kappa}_j^2 - 2\mu_1 \nu_1 \sqrt{\Delta_j} \bar{\kappa}_j^2$, which is reached along two lines in the tz-plane:

$$\xi_1 = \frac{1}{2} \ln \frac{|\kappa_j|^2 \Xi_1 - \mu_1 \nu_1 \bar{\kappa}_j^2 - \bar{\mu}_1 \bar{\nu}_1 \kappa_j^2 \pm \sqrt{\Delta_j}}{|\mu_1|^2 |\kappa_j|^2} \quad (j = 1, 2). \tag{51}$$

(ii) The soliton does not change its shape and remains the separation between the two humps during propagation. (iii) The formulae for the distance between the two humps in $|q_j|^2$ is explicitly given by

$$d_j = \frac{1}{4\lambda_{1I}} \ln \frac{|\kappa_j|^2 \Xi_1 - \mu_1 \nu_1 \bar{\kappa}_j^2 - \bar{\mu}_1 \bar{\nu}_1 \kappa_j^2 + \sqrt{\Delta_j}}{|\kappa_j|^2 \Xi_1 - \mu_1 \nu_1 \bar{\kappa}_j^2 - \bar{\mu}_1 \bar{\nu}_1 \kappa_j^2 - \sqrt{\Delta_j}} \quad (j = 1, 2), \tag{52}$$

which tells us that one hump will be separated with the other one further and even to the infinity as the value of $\frac{\Xi_1}{|\mu_1 \nu_1|}$ increases, as displayed in Figures 2a–c.

For $N \geq 2$, (36) can describe the dynamics of coupled soliton collisions in (2). Here, we make the asymptotic analysis of (36) with $N = 3$, finding the following collision properties: (i) In the collisions among

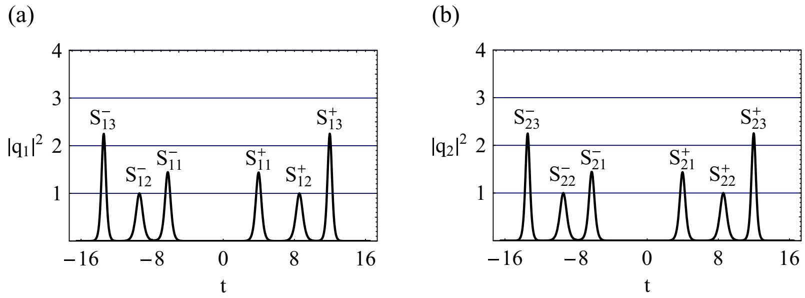

Fig. 3 (colour online). Shape-preserving collisions among three degenerate solitons via (45) transverse respectively at $z = -15$ (for the left solitons) and $z = 15$ (for the right solitons), with $N = 3$, $\beta_1 = \mathrm{i}$, $\beta_2 = 2 - 2\mathrm{i}$, $\beta_3 = \mathrm{i}$, $\gamma_1 = 2$, $\gamma_2 = 2\mathrm{i}$, $\gamma_3 = 3 - \mathrm{i}$, $\delta_1 = 1 - \mathrm{i}$, $\delta_2 = 2$, $\delta_3 = 1$, $\lambda_1 = -0.1 + 1.2\mathrm{i}$, $\lambda_2 = -0.15 - \mathrm{i}$, and $\lambda_3 = -0.2 - 1.5\mathrm{i}$.

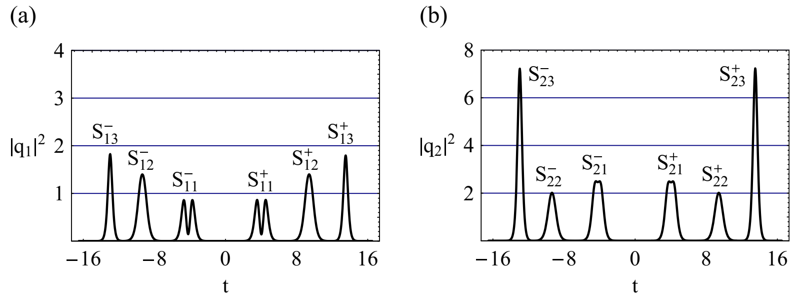

Fig. 4 (colour online). Shape-preserving collisions among three non-degenerate solitons via (45) transverse respectively at $z = -15$ (for the left solitons) and $z = 15$ (for the right solitons), with $N = 3$, $\beta_1 = 1 + 2\mathrm{i}$, $\beta_2 = 2$, $\beta_3 = 2$, $\gamma_1 = 2\mathrm{i}$, $\gamma_2 = 1 + 2\mathrm{i}$, $\gamma_3 = -\mathrm{i}$, $\delta_1 = 1$, $\delta_2 = 1$, $\delta_3 = 0$, $\lambda_1 = -0.1 + 1.2\mathrm{i}$, $\lambda_2 = -0.15 - \mathrm{i}$, and $\lambda_3 = -0.2 - 1.5\mathrm{i}$.

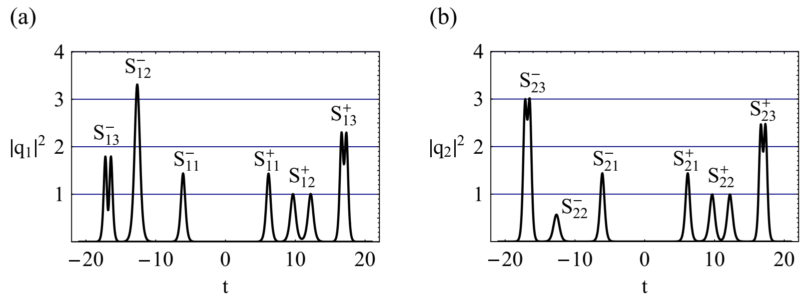

Fig. 5 (colour online). Shape-changing collisions of one degenerate soliton with two non-degenerate solitons via (45) transverse respectively at $z = -20$ (for the left solitons) and $z = 20$ (for the right solitons), where the degenerate soliton (S_{11}, S_{21}) experiences no change upon collision, while the non-degenerate solitons (S_{12}, S_{22}) and (S_{13}, S_{23}) change their amplitudes and profiles after collision. The parameters are chosen as $N = 3$, $\beta_1 = \mathrm{i}$, $\beta_2 = 1$, $\beta_3 = 2$, $\gamma_1 = 2$, $\gamma_2 = 2 + \mathrm{i}$, $\gamma_3 = -\mathrm{i}$, $\delta_1 = 1$, $\delta_2 = 1$, $\delta_3 = 0.8$, $\lambda_1 = -0.1 + 1.2\mathrm{i}$, $\lambda_2 = -0.15 - \mathrm{i}$, and $\lambda_3 = -0.2 - 1.5\mathrm{i}$.

degenerate (or non-degenerate) solitons, both $|q_1|^2$ and $|q_2|^2$ meet the conventional elastic case, that is, each component of interacting solitons maintain individual shape, velocity, and energy after collision. (ii) In the case of degenerate solitons interacting with non-degenerate ones, the degenerate ones in both two components experience always the elastic collisions, while the non-degenerate ones may undergo the change of shapes including the soliton profiles and amplitudes, together with the energy redistribution between two components. The above two properties coincide with those obtained in [24, 25]. Some examples of soliton collisions via (36) with $N = 3$ are illustrated in Figures 3–5, where S_{jn}^- and S_{jn}^+ $(n = 1, 2, 3)$ represent the

nth soliton in the jth component before and after collision, respectively.

4. Concluding Remarks

In this paper, we have studied an integrable coherently-coupled NLS system (2) which can describe the optical pulse evolution in low birefringent fibers and optical beam propagation in weakly anisotropic Kerr-type nonlinear media. Compared with the studies on (2) in [22–27], the new results obtained in this work can be seen as follows: (i) We have completed the rigorous proof for the gauge equivalence of Lax pair (3) under the Nth iterated DT. (ii) We have obtained that each component of the N-soliton solutions can be uniformly expressed as the rational fraction of two four-component Wronskians. (iii) We have found the parametric criterion for the non-degenerate solitons to respectively display the one- and double-hump profiles. In addition, we have analyzed the soliton collisions via (36) with $N = 3$, which confirms the results in [24, 25]. Finally, we would like to address the following two issues:

1. It is conjectured in [12] that the N-soliton solutions to the $(r \times s)$-matrix NLS system can be represented in terms of the $(r+s)$-component Wronskians. This conjecture has been confirmed in this work by noting that (2) can be equivalently written in the (2×2)-matrix NLS system:

$$\mathrm{i}\mathbf{Q}_z + \mathbf{Q}_{tt} + 2\mathbf{Q}\mathbf{Q}^\dagger\mathbf{Q} = \mathbf{0}, \quad \mathbf{Q} = \begin{pmatrix} q_1 & q_2 \\ -q_2 & q_1 \end{pmatrix}, \tag{53}$$

   where the dagger denotes the Hermitian conjugate. In a similar way, one can derive the 2^m-component Wronskian representation of the N-soliton solutions to the general m-coherently-coupled NLS system [23, 25]:

$$\begin{aligned} &\mathrm{i}\,q_{j,z} + q_{j,tt} + 2\left(|q_j|^2 + 2\sum_{k=1,k\neq j}^{m} |q_k|^2\right) q_j \\ &- 2\sum_{k=1,k\neq j}^{m} q_k^2 \bar{q}_j = 0 \quad (1 \le j \le m)\,. \end{aligned} \tag{54}$$

2. The double-hump solitons could allow the multi-level coding of information in each pulse so as to improve the bit rate in the high-speed optical fiber communication systems [28]. In fact, it has been experimentally shown that the double-hump solitons are immune to the time position shifts arising from intrachannel interactions in the dispersion-managed system [32, 33]. Based on this property, the non-degenerate solitons can be used to design the error preventable line-coding scheme in which the binary data are assigned to the one- and double-hump solitons [33]. The shape-changing collisions of degenerate soliton(s) with non-degenerate soliton(s) may find applications in virtual logic and computation [16, 17], and all-optical switching and amplification [30].

Acknowledgements

This work has been supported by the Science Foundation of China University of Petroleum, Beijing (No. BJ-2011-04), by the Special Funds of the National Natural Science Foundation of China (No. 11247267), by the Special Foundation for Young Scientists of Institute of Remote Sensing Applications of Chinese Academy of Sciences (No. Y1S01500CX), and by the National Natural Science Foundations of China under Grants No. 11071257 and No. 11101421.

Appendix: Lemmas for Proving Conditions (27a) and (27b)

Lemma 1. *Let the Darboux matrix $T_N(\lambda)$ be in the form of (5) with the functions $a_{ij}^{(n)}$, $b_{ij}^{(n)}$, $c_{ij}^{(n)}$, and $d_{ij}^{(n)}$ $(1 \le i, j \le 2;\ 1 \le n \le N)$ determined by (17a)–(17d). Then, the determinant of $T_N(\lambda)$ can be expanded as*

$$\det T_N(\lambda) = \prod_{k=1}^{N} (\lambda - \lambda_k)^2 (\lambda - \bar{\lambda}_k)^2\,. \tag{A.1}$$

Proof. It is obvious that $\det T_N(\lambda)$ is a monic polynomial in λ of degree $4N$. By use of (10a) and (10b), one can obtain that λ_k $(1 \le k \le K)$ are the twice-repeated roots of $\det T_N(\lambda)$. Equations (10c) and (10d) imply that $\bar{\lambda}_k$ $(1 \le k \le K)$ are also the twice-repeated roots of $\det T_N(\lambda)$. □

Lemma 2. *Define that $[u_{hl}(\lambda)]_{4\times 4} = [T_{N,t}(\lambda) + T_N(\lambda)\,U(\lambda)]T_N^*(\lambda)$ and $[v_{hl}(\lambda)]_{4\times 4} = [T_{N,z}(\lambda) + T_N(\lambda)V(\lambda)]\,T_N^*(\lambda)$, where $T_N^*(\lambda)$ is the adjoint matrix of $T_N(\lambda)$. Then, λ_k and $\bar{\lambda}_k$ $(1 \le k \le N)$ are both the twice-repeated roots of $u_{hl}(\lambda)$ and $v_{hl}(\lambda)$ $(1 \le h, l \le 4)$.*

Proof. The proof requires a large amount of tedious calculations, but can be achieved with the assistance of Mathematica. The process can be described as follows.

First, we replace the first column of $T_N(\lambda_k)$ (or $T_N(\bar{\lambda}_k)$) by virtue of (10a) (or (10c)) and calculate the adjoint matrix $T_N^*(\lambda_k)$ (or $T_N^*(\bar{\lambda}_k)$). Next, we remove the t- and z-derivatives, respectively, for the first columns in $T_{N,t}(\lambda_k)$ and $T_{N,z}(\lambda_k)$ (or $T_{N,t}(\bar{\lambda}_k)$ and $T_{N,z}(\bar{\lambda}_k)$) by taking the t- and z-derivatives of (10a) (or (10c)) and using the fact that $\Psi_{k,t} = U(\lambda_k)\Psi_k$ and $\Psi_{k,z} = V(\lambda_k)\Psi_k$ (or $\Phi_{k,t} = U(\bar{\lambda}_k)\Phi_k$ and $\Phi_{k,z} = V(\bar{\lambda}_k)\Phi_k$). Then, we check $u_{hl}(\lambda_k) = 0$ and $v_{hl}(\lambda_k) = 0$ (or $u_{hl}(\bar{\lambda}_k) = 0$ and $v_{hl}(\bar{\lambda}_k) = 0$) for $1 \le h,l \le 4$. Finally, we follow the above procedure to reach the same results by utilizing (10b) (or (10d)), which suggests that λ_k and $\bar{\lambda}_k$ $(1 \le k \le N)$ are the twice-repeated roots of $u_{hl}(\lambda)$ and $v_{hl}(\lambda)$ $(1 \le h,l \le 4)$. □

Lemma 3. *Let the Darboux matrix $T_N(\lambda)$ in (5) be determined by (17a)–(17d). Then, $b_{11}^{(N)}$, $b_{12}^{(N)}$, $b_{21}^{(N)}$, $b_{22}^{(N)}$, $c_{11}^{(N)}$, $c_{12}^{(N)}$, $c_{21}^{(N)}$, and $c_{22}^{(N)}$ obey the following relations:*

$$b_{11}^{(N)} = b_{22}^{(N)}, \qquad b_{12}^{(N)} = -b_{21}^{(N)}, \tag{A.2a}$$

$$c_{11}^{(N)} = c_{22}^{(N)}, \qquad c_{12}^{(N)} = -c_{21}^{(N)}, \tag{A.2b}$$

$$\bar{b}_{11}^{(N)} = (-1)^{N-1} c_{11}^{(N)}, \quad \bar{b}_{12}^{(N)} = (-1)^{N} c_{12}^{(N)}, \tag{A.2c}$$

$$\bar{b}_{21}^{(N)} = (-1)^{N} c_{21}^{(N)}, \quad \bar{b}_{22}^{(N)} = (-1)^{N-1} c_{22}^{(N)}. \tag{A.2d}$$

Proof. We only need to prove that (A.2a)–(A.2c) are satisfied since (A.2d) is a natural result of (A.2a)–(A.2c). With the change of the order of equations, (17a) and (17c) can be respectively written as

$$A_\tau(-\mathbf{a}_{12}, \mathbf{a}_{11}, -\mathbf{b}_{12}, \mathbf{b}_{11})^{\mathrm{T}} = (\mathbf{f}, -\mathbf{e}, \bar{\mathbf{h}}, \bar{\mathbf{g}})^{\mathrm{T}}, \tag{A.3a}$$

$$A_\tau(-\mathbf{c}_{12}, \mathbf{c}_{11}, -\mathbf{d}_{12}, \mathbf{d}_{11})^{\mathrm{T}} = (-1)^N(\mathbf{h}, -\mathbf{g}, -\bar{\mathbf{f}}, -\bar{\mathbf{e}})^{\mathrm{T}}. \tag{A.3b}$$

Solving (A.3a) and (A.3b) via Cramer's rule, we can obtain that

$$b_{11}^{(N)} = \frac{-\chi_{22}}{\tau}, \qquad b_{12}^{(N)} = \frac{(-1)^N \chi_{21}}{\tau}, \tag{A.4}$$

$$c_{11}^{(N)} = \frac{(-1)^N \chi'_{22}}{\tau}, \qquad c_{12}^{(N)} = \frac{-\chi'_{21}}{\tau}, \tag{A.5}$$

which imply the relations (A.2a) and (A.2b). On the other hand, we take complex conjugate of (17a) and change the order of equations, yielding

$$A_\tau(-\bar{\mathbf{b}}_{11}, \bar{\mathbf{b}}_{12}, \bar{\mathbf{a}}_{11}, -\bar{\mathbf{a}}_{12})^{\mathrm{T}} = (\mathbf{g}, \mathbf{h}, \bar{\mathbf{e}}, -\bar{\mathbf{f}})^{\mathrm{T}}, \tag{A.6}$$

from which, the relation (A.2c) can be derived by means of Cramer's rule. □

[1] G. I. Stegeman and M. Segev, Science **286**, 1518 (1999).
[2] J. U. Kang, G. I. Stegeman, J. S. Aitchison, and N. Akhmediev, Phys. Rev. Lett. **76**, 3699 (1996).
[3] D. Rand, I. Glesk, C. S. Brès, D. A. Nolan, X. Chen, J. Koh, J. W. Fleischer, K. Steiglitz, and P. R. Prucnal, Phys. Rev. Lett. **98**, 053902 (2007).
[4] Z. Chen, M. Segev, T. H. Coskun, D. N. Christodoulides, and Y. S. Kivshar, J. Opt. Soc. Am. B **14**, 3066 (1997).
[5] S. T. Cundiff, B. C. Collings, and W. H. Knox, Opt. Exp. **1**, 12 (1997).
[6] G. P. Agrawal, Nonlinear Fiber Optics, Academic Press, San Diego, California 2002.
[7] S. V. Manakov, Sov. Phys. JETP **38**, 248 (1974).
[8] Yu. S. Kivshar and G. P. Agrawal, Optical Solitons: From Fibers to Photonic Crystals, Academic Press, San Diego California 2003.
[9] R. Radhakrishnan, M. Lakshmanan, and J. Hietarinta, Phys. Rev. E **56**, 2213 (1997).
[10] T. Kanna and M. Lakshmanan, Phys. Rev. Lett. **86**, 5043 (2001).
[11] T. Xu and B. Tian, J. Phys. A **43**, 245205 (2010).
[12] T. Xu and B. Tian, J. Math. Phys. **51**, 033504 (2010).
[13] T. Xu, B. Tian, Y. S. Xue, and F. H. Qi, EPL **92**, 50002 (2010).
[14] M. J. Ablowitz, B. Prinari, and A. D. Trubatch, Inverse Probl. **20**, 1217 (2004).
[15] C. Anastassiou, M. Segev, K. Steiglitz, J. A. Giordmaine, M. Mitchell, M. F. Shih, S. Lan, and J. Martin, Phys. Rev. Lett. **83**, 2332 (1999).
[16] M. H. Jakubowski, K. Steiglitz, and R. Squier, Phys. Rev. E **58**, 6752 (1998).
[17] K. Steiglitz, Phys. Rev. E **63**, 016608 (2000).
[18] I. Kaminer, M. Segev, A. M. Bruckstein, and Y. C. Eldar, Proc. R. Soc. A **465**, 1093 (2009).
[19] I. Kaminer, M. Segev, and A. M. Bruckstein, Phys. Rev. Lett. **105**, 083901 (2010).
[20] N. N. Akhmediev and E. A. Ostrovskaya, Opt. Commun. **132**, 190 (1996).

[21] Q. H. Park and H. J. Shin, Phys. Rev. E **59**, 2373 (1999).
[22] K. Nakkeeran, J. Mod. Opt. **48**, 1863 (2001).
[23] H. Q. Zhang, T. Xu, J. Li, and B. Tian, Phys. Rev. E **77**, 026605 (2008).
[24] X. Lü and B. Tian, Phys. Rev. E **85**, 026117 (2012).
[25] T. Kanna and K. Sakkaravarthi, J. Phys. A **44**, 285211 (2011).
[26] H. Q. Zhang, J. Li, T. Xu, Y. X. Zhang, W. Hu, and B. Tian, Phys. Scr. **76**, 452 (2007).
[27] H. Q. Zhang, T. Xu, J. Li, L. L. Li, C. Zhang, and B. Tian, Z. Naturforsch. **64a**, 300 (2009).
[28] N. Akhmediev and A. Ankiewicz, Chaos **10**, 600 (2000).
[29] T. Xu, B. Tian, and F. H. Qi, Z. Naturforsch. **67a**, 39 (2012).
[30] X. D. Cao and D. D. Meyerhofer, Opt. Lett. **19**, 1711 (1994).
[31] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Phys. Rev. Lett. **31**, 125 (1973).
[32] A. Maruta, Y. Nonaka, and T. Inoue, Electron. Lett. **37**, 1357 (2001).
[33] A. Maruta, T. Inoue, Y. Nonaka, and Y. Yoshika, IEEE J. Sel. Top. Quantum Electron. **8**, 640 (2002).