Integral Methods to Solve the Variable Coefficient Nonlinear Schrödinger Equation

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In this paper, we use two different integral techniques, the first integral and the direct integral method, to study the variable coefficient nonlinear Schrödinger (NLS) equation arising in arterial mechanics. The application of the first integral method yielded periodic and solitary wave solutions. Using the direct integration lead to solitary wave solution and Jacobi elliptic function solutions.

Key words: NLS Equation with Variable Coefficient; First Integral Method; Direct Integral Method; Periodic, Solitary and Jacobi Elliptic Function Solutions.

1. Introduction

Treating the arteries as a thin-walled and prestressed elastic tube with a stenosis and the blood as a Newtonian fluid with negligible viscosity, Demiray in [1 – 3] has studied the amplitude modulation of nonlinear waves in such a composite system by using the reductive perturbation method. The governing evolution equation is obtained as the variable-coefficient NLS equation

$$\mathrm{i}U_t + \mu_1 U_{xx} + \mu_2 |U|^2 U - \mu_3 h(t) U = 0\,, \qquad (1)$$

where the coefficients μ_1, μ_2, and μ_3 are constants, $h(t)$ is an arbitrary real function, and $U = U(x,t)$ is a complex-valued function of two real variables x, t. By seeking a progressive wave type of solution to the NLS equation with variable coefficient in [1 – 3], it is found that the speed of the harmonic wave increases with distance from the center of the stenosis. Such a result is to be expected from physical considerations. When $\mu_3 = 0$, we have the NLS equation with constant coefficients, and when $h(t) = 1$, the dissipative NLS equation is obtained. The NLS equations describe a wide class of physical nonlinear phenomena such as hydrodynamics [4, 5], nonlinear optics [6, 7], self-focusing in laser pulses [8], thermodynamic processes in meso scale systems [9], propagation of heat pulses in crystals, helical motion of very thin vortex filaments, models of protein dynamics [10], magnetic thin films [11], the dynamics of Bose–Einstein condensate at extremely low temperature [12], and models of energy transfer in molecular systems [13] and plasmas [14].

The objectives of this work are twofold. First, we apply the first integral method on the variable coefficient NLS equation to obtain periodic and solitary wave solutions. Second, we aim using the well-known direct integration on the reduced nonlinear ordinary differential equation obtained after using the travelling wave transformation on the NLS equation to get many exact solutions in the form of solitary wave and Jacobi elliptic function solutions.

2. The First Integral Method

Consider the nonlinear partial differential equation in the from [15 – 19]

$$F(U, U_x, U_t, U_{xx}, U_{tt}, U_{xt}, \ldots) = 0\,, \qquad (2)$$

where $U = U(x,t)$ is a solution of the nonlinear partial differential equation (2). We use the transformation

$$U(x,t) = f(\xi)\,, \qquad (3)$$

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where $\xi = x + \lambda t$. This enables us to use the following changes:

$$\frac{\partial}{\partial t}(.) = \lambda \frac{\partial}{\partial \xi}(.), \quad \frac{\partial}{\partial x}(.) = \frac{\partial}{\partial \xi}(.), \quad \frac{\partial^2}{\partial x^2}(.) = \frac{\partial^2}{\partial \xi^2}(.), \quad \cdots \tag{4}$$

by using (4) to transfer the nonlinear partial differential equation (2) to the nonlinear ordinary differential equation

$$G\left(f(\xi), \frac{\partial f(\xi)}{\partial \xi}, \frac{\partial^2 f(\xi)}{\partial \xi^2}, \ldots\right) = 0. \tag{5}$$

Next, we introduce a new independent variable

$$X(\xi) = f(\xi), \quad Y = \frac{\partial f(\xi)}{\partial \xi}, \tag{6}$$

which leads to a system of nonlinear ordinary differential equations

$$\frac{\partial X(\xi)}{\partial \xi} = Y(\xi), \quad \frac{\partial Y(\xi)}{\partial \xi} = F_1(X(\xi), Y(\xi)). \tag{7}$$

By the qualitative theory of ordinary differential equations [20]: if we can find the integrals to (7) under the same conditions, then the general solutions to (7) can be solved directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way for telling us what these first integrals are. We will apply the division theorem to obtain one first integral to (7) which reduces (5) to a first order integrable ordinary differential equation. An exact solution to (2) is then obtained by solving this equation. Now, let us recall the division theorem:

Division Theorem. *Suppose that $P(\omega,z)$ and $Q(\omega,z)$ are polynomials in $C[\omega,z]$; and $P(\omega,z)$ is irreducible in $C[\omega,z]$. If $Q(\omega,z)$ vanishes at all zero points of $P(\omega,z)$, then there exists a polynomial $G(\omega,z)$ in $C[\omega,z]$ such that*

$$Q(\omega,z) = P(\omega,z)\, G(\omega,z).$$

3. Solutions by Using the First Integral Method

In this section, we study the NLS equation with variable coefficients by using the transformation

$$U(x,t) = \exp(\mathrm{i}\theta) f(\xi), \quad \theta = \alpha x + \beta(t), \quad \xi = x - \lambda t, \tag{8}$$

where α is a constant and $\beta(t)$, $f(\xi)$ are real functions

Substituting (8) into (1), we obtain an ordinary differential equation:

$$(-\beta'(t) - \mu_3 h(t)) f(\xi) + \mathrm{i}(-\lambda + 2\alpha\mu_1)\frac{\partial f(\xi)}{\partial \xi} - \alpha^2 \mu_1 f(\xi) + \mu_1 \frac{\partial^2 f(\xi)}{\partial \xi^2} + \mu_2 (f(\xi))^3 = 0. \tag{9}$$

By setting the coefficient of $f(\xi)$ and the complex coefficient of $\frac{\partial f(\xi)}{\partial \xi}$ to be zero, we get

$$\lambda = 2\alpha\mu_1, \quad \beta(t) = -\mu_3 \int h(t)\,\mathrm{d}t + c, \tag{10}$$

where c is an arbitrary integration constant.

Substituting (10) into (8), we obtain the transformation

$$U(x,t) = \exp(\mathrm{i}\theta) f(\xi), \quad \theta = \alpha x - \mu_3 \int h(t)\,\mathrm{d}t + c, \quad \xi = x - 2\alpha\mu_1 t, \tag{11}$$

where α is a constant and $f(\xi)$ is a real function.

Substituting (11) into (1), we obtain an ordinary differential equation:

$$-\alpha^2 \mu_1 f(\xi) + \mu_1 \frac{\partial^2 f(\xi)}{\partial \xi^2} + \mu_2 (f(\xi))^3 = 0. \tag{12}$$

Using (6) and (12) in (7), we get

$$\dot{X}(\xi) = Y(\xi), \tag{13}$$

$$\dot{Y}(\xi) = \alpha^2 X(\xi) - \frac{\mu_2}{\mu_1}(X(\xi))^3. \tag{14}$$

According to the first integral method, we suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (13) and (14), and

$$Q(X,Y) = \sum_{i=0}^{m} a_i(X) Y^i = 0$$

is an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$Q(X(\xi),Y(\xi)) = \sum_{i=0}^{m} a_i(X(\xi))Y^i(\xi) = 0, \quad (15)$$

where $a_i(X)$ $(i = 0,1,\ldots,m)$, are polynomials of X and $a_m(X) \neq 0$. Equation (15) is called the first integral to (13) and (14). Due to the division theorem, there exists a polynomial $g(X)+h(X)Y$, in the complex domain $C[X,Y]$ such that

$$\begin{aligned}\frac{dQ}{d\xi} &= \frac{dQ}{dX}\frac{dX}{d\xi} + \frac{dQ}{dY}\frac{dY}{d\xi} \\ &= (g(X)+h(X)Y)\sum_{i=0}^{m} a_i(X)Y^i.\end{aligned} \quad (16)$$

In this paper, we take two different cases, assuming that $m = 1$ and $m = 2$ in (15).

Case 1. Suppose that $m = 1$. By comparing the coefficients of Y^i $(i = 2,1,0)$ on both sides of (16), we have

$$\dot{a}_1(X) = h(X)a_1(X), \quad (17)$$

$$\dot{a}_0(X) = g(X)a_1(X) + h(X)a_0(X), \quad (18)$$

$$a_1(X)\left[\alpha^2 X(\xi) - \frac{\mu_2}{\mu_1}(X(\xi))^3\right] = g(X)a_0(X). \quad (19)$$

Since $a_i(X)$ $(i = 0,1)$ are polynomials, then from (17) we deduce that $a_1(X)$ is constant and $h(X) = 0$. For simplicity, we take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that $g(X) = A_1X + B_0$, then from (18) we obtain

$$a_0(X) = B_1 + B_0X + \frac{1}{2}A_1X^2, \quad (20)$$

where A_1, B_0 are arbitrary constants, and B_1 is an arbitrary integration constant to be determined.

Substituting $a_0(X)$ and $g(X)$ into (19) and setting all coefficients of powers X to be zero, we obtain a system of nonlinear algebraic equations, and by solving it, we get

$$B_0 = 0,\ A_1 = \sqrt{-2\frac{\mu_2}{\mu_1}},\ B_1 = \frac{\alpha^2}{\sqrt{-2\frac{\mu_2}{\mu_1}}}, \quad (21)$$

$$B_0 = 0,\ A_1 = -\sqrt{-2\frac{\mu_2}{\mu_1}},\ B_1 = -\frac{\alpha^2}{\sqrt{-2\frac{\mu_2}{\mu_1}}}. \quad (22)$$

Using the conditions (21) and (22) in (15), we obtain

$$Y(\xi) = \pm\left(\frac{\alpha^2}{\sqrt{-2\frac{\mu_2}{\mu_1}}} + \frac{1}{2}\sqrt{-2\frac{\mu_2}{\mu_1}}X^2(\xi)\right). \quad (23)$$

Combining (23) with (13), we obtain the exact solution of (13) and (14). So that the exact solution of the NLS equation with variable coefficient can be written as

$$\begin{aligned}U_1(x,t) = &\pm\alpha\sqrt{-\frac{\mu_1}{\mu_2}}\, e^{i(\alpha x - \mu_3\int h(t)\,dt + c)} \\ &\cdot \tan\left(\frac{1}{\sqrt{2}}\alpha(x - 2\alpha\mu_1 t + \xi_0)\right),\end{aligned} \quad (24)$$

where ξ_0 is an arbitrary integration constant.

Case 2. Suppose that $m = 2$. By comparing the coefficients of Y^i $(i = 3,2,1,0)$ on both sides of (16), we have

$$\dot{a}_2(X) = h(X)a_2(X), \quad (25)$$

$$\dot{a}_1(X) = g(X)a_2(X) + h(X)a_1(X), \quad (26)$$

$$\begin{aligned}\dot{a}_0(X) = &-2a_2(X)\left[\alpha^2 X(\xi) - \frac{\mu_2}{\mu_1}(X(\xi))^3\right] \\ &+ g(X)a_1(X) + h(X)a_0(X),\end{aligned} \quad (27)$$

$$a_1(X)\left[\alpha^2 X(\xi) - \frac{\mu_2}{\mu_1}(X(\xi))^3\right] = g(X)a_0(X). \quad (28)$$

Since $a_i(X)$ $(i = 0,1,2)$ are polynomials, then from (25) we deduce that $a_2(X)$ is a constant and $h(X) = 0$. For simplicity, we take $a_2(X) = 1$. Balancing the degrees of $g(X)$, $a_1(X)$, and $a_2(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that $g(X) = A_1X + B_0$, then from (26) we find $a_1(X)$ and $a_0(X)$ as follows:

$$a_1(X) = B_1 + B_0X + \frac{1}{2}A_1X^2, \quad (29)$$

$$\begin{aligned}a_0(X) = &\ d + B_0B_1X + \frac{1}{2}\left(-2\alpha^2 + A_1B_1 + B_0^2\right)X^2 \\ &+ \frac{1}{2}A_1B_0X^3 + \frac{1}{4}\left(\frac{2\mu_2}{\mu_1} + \frac{1}{2}A_1^2\right)X^4,\end{aligned} \quad (30)$$

where A_1, B_0 are arbitrary constants, and B_1, d are arbitrary integration constants.

Substituting $a_0(X)$, $a_1(X)$, and $g(X)$ in (28) and setting the coefficients of all powers X to be zero, we obtain a system of nonlinear algebraic equations, and by solving it with the aid of symbolic computation program Maple, we obtain

$$B_0=0\,,\;\; B_1=-\frac{\sqrt{-2\mu_1\mu_2}}{\mu_2}\alpha^2\,,\quad A_1=2\frac{\sqrt{-2\mu_1\mu_2}}{\mu_1}\,,\;\; d=-\frac{1}{2}\frac{\mu_1}{\mu_2}\alpha^4\,, \tag{31}$$

$$B_0=0\,,\;\; B_1=\frac{\sqrt{-2\mu_1\mu_2}}{\mu_2}\alpha^2\,,\quad A_1=-2\frac{\sqrt{-2\mu_1\mu_2}}{\mu_1}\,,\;\; d=-\frac{1}{2}\frac{\mu_1}{\mu_2}\alpha^4\,. \tag{32}$$

Using the conditions (31) and (32) into (15), we get

$$Y(\xi)=\pm\frac{\sqrt{-2\mu_1\mu_2}(-\alpha^2\mu_1+\mu_2X^2(\xi))}{2\mu_1\mu_2}\,. \tag{33}$$

By using (33) with (13), we obtain the exact solution of (13) and (14). Also, by back substitution, we get the following exact solution of the NLS equation with variable coefficient:

$$U_2(x,t)=\pm\sqrt{\frac{\alpha^2\mu_1}{\mu_2}}\,\mathrm{e}^{\mathrm{i}(\alpha x-\mu_3\int h(t)\,\mathrm{d}t+c)}\cdot\tanh\left(\sqrt{-\frac{1}{2}\alpha^2}\,(x-2\alpha\mu_1 t+\xi_0)\right)\,, \tag{34}$$

where ξ_0 is an arbitrary integration constant. This solitary wave solution is similar to that solution obtained by Demiray in [1].

4. Solutions by Using the Direct Integration Method

In this section, we multiply (12) by $\frac{\partial f(\xi)}{\partial\xi}$ and get

$$-\alpha^2\mu_1 f(\xi)\frac{\mathrm{d}f(\xi)}{\mathrm{d}\xi}+\mu_1\frac{\mathrm{d}f(\xi)}{\mathrm{d}\xi}\frac{\mathrm{d}^2f(\xi)}{\mathrm{d}\xi^2}+\mu_2(f(\xi))^3\frac{\mathrm{d}f(\xi)}{\mathrm{d}\xi}=0\,. \tag{35}$$

Case 3. Integrating (35) once and considering the constant of integration to be zero, we obtain

$$\left(\frac{\mathrm{d}f(\xi)}{\mathrm{d}\xi}\right)^2=-\frac{\mu_2}{2\mu_1}(f(\xi))^4+\alpha^2(f(\xi))^2-\frac{2c_0}{\mu_1}\,, \tag{36}$$

where c_0 is an arbitrary integration constant. In that case consider $c_0=0$, so we obtain

$$\frac{\mathrm{d}f(\xi)}{f(\xi)\sqrt{\alpha^2-\frac{\mu_2}{2\mu_1}(f(\xi))^2}}=\mathrm{d}\xi\,. \tag{37}$$

By integrating both sides of (37) it can be proved that

$$f(\xi)=\sqrt{\frac{2\mu_1\alpha^2}{\mu_2}}\operatorname{sech}[\alpha(\xi+\xi_0)]\,, \tag{38}$$

where ξ_0 is an arbitrary integration constant.

By back substitution from (38) in (11), we obtain the following exact solution of NLS equation with variable coefficients:

$$U_3(x,t)=\sqrt{\frac{2\mu_1\alpha^2}{\mu_2}}\,\mathrm{e}^{\mathrm{i}(\alpha x-\mu_3\int h(t)\,\mathrm{d}t+c)}\cdot\operatorname{sech}[\alpha(x-2\mu_1\alpha t+\xi_0)]\,. \tag{39}$$

This solution is obtained by Demiray in [1].

Case 4. Consider $c_0\neq 0$ in (36).

This equation has many exact solutions by relations between values of $\left(-\frac{2c_0}{\mu_1},\alpha^2,-\frac{\mu_2}{2\mu_1}\right)$ in the form of Jacobi elliptic functions corresponding to $f(\xi)$ [21]:

$$f_4(\xi)=\operatorname{sn}\left(\xi,\sqrt{\frac{-\mu_2}{2\mu_1}}\right)\,,\quad \alpha=\sqrt{\frac{\mu_2}{2\mu_1}-1}\,,\;\; c_0=\frac{-\mu_1}{2}\,, \tag{40}$$

$$f_5(\xi)=\operatorname{cd}\left(\xi,\sqrt{\frac{-\mu_2}{2\mu_1}}\right)\,,\quad \alpha=\sqrt{\frac{\mu_2}{2\mu_1}-1}\,,\;\; c_0=\frac{-\mu_1}{2}\,, \tag{41}$$

$$f_6(\xi)=\operatorname{cn}\left(\xi,\sqrt{\frac{\mu_2}{2\mu_1}}\right)\,,\quad \alpha=\sqrt{\frac{\mu_2}{\mu_1}-1}\,,\;\; c_0=\frac{\mu_1}{2}\left(\frac{\mu_2}{2\mu_1}-1\right)\,, \tag{42}$$

$$f_7(\xi)=\operatorname{nc}\left(\xi,\sqrt{1+\frac{\mu_2}{2\mu_1}}\right)\,,\quad \alpha=\sqrt{1+\frac{\mu_2}{\mu_1}}\,,\;\; c_0=\frac{\mu_1}{2}\left(1+\frac{\mu_2}{2\mu_1}\right)\,, \tag{43}$$

$$f_8(\xi)=\operatorname{nd}\left(\xi,\sqrt{1-\frac{\mu_2}{2\mu_1}}\right)\,,\quad \alpha=\sqrt{1+\frac{\mu_2}{2\mu_1}}\,,\;\; c_0=\frac{\mu_1}{2}\,, \tag{44}$$

$$f_9(\xi) = \mathrm{sc}\left(\xi, \sqrt{1+\frac{\mu_2}{2\mu_1}}\right), \quad \alpha = \sqrt{1-\frac{\mu_2}{2\mu_1}}, \; c_0 = -\frac{\mu_1}{2}, \tag{45}$$

$$f_{10}(\xi) = \mathrm{sd}\left(\xi, \sqrt{\frac{1}{2}\left(\sqrt{1-\frac{2\mu_2}{\mu_1}}+1\right)}\right), \quad \alpha = \sqrt[4]{1-\frac{2\mu_2}{\mu_1}}, \; c_0 = -\frac{\mu_1}{2}. \tag{46}$$

From inserting (40)–(46) into (11), we obtain the following new exact Jacobi elliptic function solutions of the NLS equation with variable coefficients:

$$U_4(x,t) = \mathrm{e}^{\mathrm{i}\left(\sqrt{\frac{\mu_2}{2\mu_1}-1}x-\mu_3\int h(t)\,\mathrm{d}t\right)+c} \cdot \mathrm{sn}\left(x-2\mu_1\sqrt{\frac{\mu_2}{2\mu_1}-1}\,t, \sqrt{\frac{-\mu_2}{2\mu_1}}\right), \tag{47}$$

$$U_5(x,t) = \mathrm{e}^{\mathrm{i}\left(\sqrt{\frac{\mu_2}{2\mu_1}-1}x-\mu_3\int h(t)\,\mathrm{d}t\right)+c} \cdot \mathrm{cd}\left(x-2\mu_1\sqrt{\frac{\mu_2}{2\mu_1}-1}\,t, \sqrt{\frac{-\mu_2}{2\mu_1}}\right), \tag{48}$$

$$U_6(x,t) = \mathrm{e}^{\mathrm{i}\left(\sqrt{\frac{\mu_2}{\mu_1}-1}x-\mu_3\int h(t)\,\mathrm{d}t\right)+c} \cdot \mathrm{cn}\left(x-2\mu_1\sqrt{\frac{\mu_2}{\mu_1}-1}\,t, \sqrt{\frac{\mu_2}{2\mu_1}}\right), \tag{49}$$

$$U_7(x,t) = \mathrm{e}^{\mathrm{i}\left(\sqrt{1+\frac{\mu_2}{\mu_1}}x-\mu_3\int h(t)\,\mathrm{d}t\right)+c} \cdot \mathrm{nc}\left(x-2\mu_1\sqrt{1+\frac{\mu_2}{\mu_1}}\,t, \sqrt{1+\frac{\mu_2}{2\mu_1}}\right), \tag{50}$$

$$U_8(x,t) = \mathrm{e}^{\mathrm{i}\left(\sqrt{1+\frac{\mu_2}{2\mu_1}}x-\mu_3\int h(t)\,\mathrm{d}t\right)+c} \cdot \mathrm{nd}\left(x-2\mu_1\sqrt{1+\frac{\mu_2}{2\mu_1}}\,t, \sqrt{1-\frac{\mu_2}{2\mu_1}}\right), \tag{51}$$

$$U_9(x,t) = \mathrm{e}^{\mathrm{i}\left(\sqrt{1-\frac{\mu_2}{2\mu_1}}x-\mu_3\int h(t)\,\mathrm{d}t\right)+c} \cdot \mathrm{sc}\left(x-2\mu_1\sqrt{1-\frac{\mu_2}{2\mu_1}}\,t, \sqrt{1+\frac{\mu_2}{2\mu_1}}\right), \tag{52}$$

$$U_{10}(x,t) = \mathrm{e}^{\mathrm{i}\left(\sqrt[4]{1-\frac{2\mu_2}{\mu_1}}x-\mu_3\int h(t)\,\mathrm{d}t\right)+c} \cdot \mathrm{sd}\left(x-2\mu_1\sqrt[4]{1-\frac{2\mu_2}{\mu_1}}\,t, \sqrt{\frac{1}{2}\left(\sqrt{1-\frac{2\mu_2}{\mu_1}}+1\right)}\right). \tag{53}$$

5. Conclusion

In this study, we have applied the first integral and the direct integral method to obtain new exact solutions of the variable coefficient NLS equation model in arterial mechanics. However, the first integral method is a powerful method, and we get by using it two exact periodic and solitary wave solutions. But in general, solving planar systems of ordinary differential equations like (7) directly is a difficult and challenging task by using this method. When we used direct integration, it was very easy to find many new exact solitary and periodic solutions. By comparison of our solutions with that obtained in [1], we have recovered those solutions and obtained many new other solutions.

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