Solitons for a Forced Extended Korteweg–de Vries Equation with Variable Coefficients in Atmospheric Dynamics

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Under investigation is a forced extended Korteweg–de Vries equation with variable coefficients, which can describe the atmospheric blocking phenomenon. The nonisospectral Lax pair for this equation is constructed via symbolic computation, and new integrable conditions are given. One- and two-soliton solutions are derived explicitly through the binary-Bell-polynomial method under the integrable conditions. Based on the solutions, kink-type and bell-profile-like (BPL) solitons are obtained under certain conditions. The analysis shows that the variable coefficients not only influence the amplitudes and velocities of the kink-type and BPL solitons, but also affect the background and the type of interaction.

Key words: Forced Extended Korteweg–de Vries Equation in Atmospheric Dynamics; Integrability; Soliton Solutions; Symbolic Computation; Binary Bell Polynomial.
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1. Introduction

As a nonlinear model, the Korteweg–de Vries (KdV) equation,

$$u_t + 6uu_x + u_{xxx} = 0\,, \tag{1}$$

has arisen in such physical situations as the internal solitary waves in shallow water [1], ion-acoustic soliton in plasmas [2], and dust acoustic solitary structures in magnetized dusty plasmas [3]. Hereby, $u = u(x,t)$ represents the amplitude of the relevant wave mode and determines the time evolution of the vertical displacement on the isopycnal surface, x is the scaled space variable in the direction of wave propagation, and t is the scaled time. Specially for the internal solitary waves, the quadratic nonlinear term uu_x in (1), defined by the density stratification, may not be stable or even vanish when a buoyancy frequency profile is nearly symmetric about the middepth [4]. Therefore, it is necessary to let the quadratic and cubic nonlinear terms appear at the same order in an asymptotic perturbation, which leads to the extended KdV (eKdV) equation (also called the Gardner equation) [4],

$$u_t + a_1 uu_x + a_2 u^2 u_x + a_3 u_{xxx} = 0\,, \tag{2}$$

which can discribe the internal waves in a stratified ocean [5], propagation of the long wave in an inhomogeneous two-layer shallow liquid [6], and ion-sound waves in plasmas with negative ions [7], where a_j $(j = 1,2)$ are the coefficients of quadratic and cubic nonlinear terms, respectively, while a_3 denotes the effect of dispersion. Different from (1), (2) can present kinds of solutions such as the breather, plateau, and kink-type soliton solutions due to the co-existence of the quadratic (uu_x) and cubic nonlinear (u^2u_x) terms [8].

However, in the geophysical and marine applications, for instance, when the waves are generated by the moving ships or flows over the bottom topography, (2) needs to include an external force [9]. Thus, the forced eKdV (feKdV) equation has been derived and applied to the physical settings dealing with the fluid flow [9] and forced generation of nonlinear waves [10, 11]. Further, with the non-uniformities of depth and width, compressibility of fluid, and presence of vorticity [12], the feKdV equation with time- and space-dependent variable coefficients has also been proposed as [13]

$$\begin{aligned} &u_t + f(t)uu_x + g(t)u^2u_x + h(t)u_{xxx} \\ &+ \big[p(t) + q(t)x\big]u_x + k(t)u + l(t) = 0\,, \end{aligned} \tag{3}$$

with the integrable condition [13]

$$2g(t)l(t) = f'(t) + f(t)k(t) - f(t)\frac{l'(t)}{l(t)}, \qquad (4)$$

where $f(t)$, $g(t)$, $h(t)$, $p(t)$, $q(t)$, $k(t)$, and $l(t)$ are all differentiable functions and $g(t)h(t) \neq 0$. Equation (3) can describe the atmospheric blocking phenomenon [13, 14]. In [13], the group classification problems for (3) have been analyzed with respect to the corresponding equivalence groups.

Generally speaking, a variable-coefficient nonlinear evolution equation (NLEE) is not completely integrable unless the variable coefficients satisfy certain constraint conditions. Once a NLEE is integrable, it may have the properties like the infinite conserved quantities in evolution, N-soliton solutions, and solvability through the inverse scattering method [15]. On the other hand, soliton "management" relying on the variable coefficients can be used to "control" the propagation and interaction of solitons [16]. The concept of "soliton management" has been extended to the study of internal waves in stratified fluids, in which a periodic modulation of the dispersion coefficient is introduced into the KdV equation [17].

In this paper, we will check the integrability of (3) from the viewpoint of Lax pair and obtain new integrable conditions. Moreover, we will carry out the analysis of solitons for the new integrable form of (3).

Based on the above motivations, the structure of this paper will be arranged as follows: By means of the Ablowitz–Kaup–Newell–Segur (AKNS) scheme [18] and symbolic computation [19, 20], we will construct the Lax pair of (3) in Section 2. In this procedure, new integrable conditions will be obtained. Moreover, under the scale transformation, the integrable case of (3) will be transformed into an integrable feKdV equation with minimum number of variable coefficients. Via the binary-Bell-polynomial method [29, 30], we will derive the bilinear form of the integrable feKdV equation in Section 3. In Section 4, one- and two-soliton solutions will be given explicitly. Moreover, soliton dynamics and inhomogeneous effect of the variable coefficients will be analyzed. Conclusions will be listed in Section 5.

2. Lax Pair and Scale Transformation for (3)

The integrability of a NLEE can be judged from the Painlevé property [21], Lax pair [15], and symmetries [22]. The existence of a Lax pair can ensure a series of integrable properties such as the infinite conserved quantities, Hamiltonian structures, and Darboux transformations [23, 24]. Hereby, we will make use of the AKNS scheme [18] to construct the Lax pair of (3) and obtain the integrable conditions.

With the AKNS scheme, the linear nonisospectral eigenvalue problems (or Lax pair) of (3) are given as

$$\Psi_x = M\Psi, \quad \Psi_t = N\Psi, \qquad (5)$$

with

$$M = \begin{pmatrix} \lambda(t) & \mathrm{i}u(x,t) + \mathrm{i}\rho(t) \\ -\mathrm{i}u(x,t) - \mathrm{i}\rho(t) & -\lambda(t) \end{pmatrix}, \quad N = \begin{pmatrix} A(x,t) & B(x,t) \\ C(x,t) & -A(x,t) \end{pmatrix}, \qquad (6)$$

where $\Psi = (\psi_1, \psi_2)^{\mathrm{T}}$ is the vector eigenfunction, ψ_j $(j=1,2)$ are the eigenfunctions, T represents the transpose of the vector, $\lambda(t)$ is the nonisospectral parameter, and $\rho(t)$, $A(x,t)$, $B(x,t)$, and $C(x,t)$ are the functions to be determined. Considering that the spectral parameter $\lambda(t)$ is varying with t, we assume that $\lambda'(t) = \lambda_0(t)\lambda(t)$. It can be checked that the zero curve equation $U_t - V_x + [U,V] = 0$ is exactly equivalent to (3) with the following functions:

$$\lambda_0(t) = -k(t), \quad \rho(t) = \mathrm{e}^{-\int k(t)\,\mathrm{d}t} \int \mathrm{e}^{\int k(t)\,\mathrm{d}t} l(t)\,\mathrm{d}t,$$

$$A(x,t) = -4h(t)\lambda^3(t) - \Big[4\rho(t)^2 h(t) + p(t) + xk(t) - 4\rho(t)h(t)u - 2h(t)u^2\Big],$$

$$B(x,t) = -\mathrm{i}\Big[4\rho(t)h(t) + 4h(t)u\Big]\lambda^2(t) - 2\mathrm{i}h(t)u_x\lambda(t) - \mathrm{i}\Big[4\rho(t)^3 h(t) + xk(t)\rho(t) + \rho(t)p(t) + xk(t)u + p(t)u - 2h(t)u^3 - 6h(t)u^3\rho(t) + h(t)u_{xx}\Big],$$

$$C(x,t) = \mathrm{i}\Big[4\rho(t)h(t) + 4h(t)u\Big]\lambda^2(t) + 2\mathrm{i}h(t)u_x\lambda(t) + \mathrm{i}\Big[4\rho(t)^3 h(t) + xk(t)\rho(t) + \rho(t)p(t) + xk(t)u + p(t)u - 2h(t)u^3 - 6h(t)u^3\rho(t) + h(t)u_{xx}\Big],$$

under the conditions

$$q(t) = k(t), \quad f(t) = -12\rho(t)h(t), \quad g(t) = -6h(t). \qquad (7)$$

As we know, the generation of the inhomogeneous solitons relies on the balance among nonlinearity, dispersion and variable coefficients. With the balance relations, i. e., conditions (7), (3) presents a new integrable form:

$$u_t - 12h(t)\,\mathrm{e}^{-\int k(t)\,\mathrm{d}t}\int \mathrm{e}^{\int k(t)\,\mathrm{d}t} l(t)\,\mathrm{d}t\, u u_x - 6h(t)u^2 u_x + h(t)u_{xxx} + \Big[p(t) + xk(t)\Big]u_x + k(t)u + l(t) = 0\,, \tag{8}$$

which is different from that studied in [13] since the coefficients do not satisfy condition (4). To minimize the number of the variable coefficients for (8), we take the scale transformation as the following general form:

$$u = A(t)U\Big[X(x,t), T(t)\Big]\,. \tag{9}$$

Our calculation shows that transformation (9) with

$$\begin{aligned} A(t) &= 2\,\mathrm{e}^{-\int k(t)\,\mathrm{d}t}\int \mathrm{e}^{\int k(t)\,\mathrm{d}t} l(t)\,\mathrm{d}t\,,\\ T(t) &= 16\sqrt{2}\int \mathrm{e}^{-3\int k(t)\,\mathrm{d}t} h(t)\left[\int \mathrm{e}^{\int k(t)\,\mathrm{d}t} l(t)\,\mathrm{d}t\right]^3 \mathrm{d}t\,,\\ X(x,t) &= 2\sqrt{2}x\,\mathrm{e}^{-\int k(t)\,\mathrm{d}t}\int \mathrm{e}^{\int k(t)\,\mathrm{d}t} l(t)\,\mathrm{d}t\\ &\quad + 2\sqrt{2}\int \mathrm{e}^{\int k(t)\,\mathrm{d}t} l(t)\,\mathrm{d}t \times \int \mathrm{e}^{-3\int k(t)\,\mathrm{d}t}\\ &\quad \cdot \left\{8h(t)\left[\int \mathrm{e}^{\int k(t)\,\mathrm{d}t} l(t)\,\mathrm{d}t\right]^2 - \mathrm{e}^{2\int k(t)\,\mathrm{d}t} p(t)\right\}\mathrm{d}t \end{aligned}$$

converts (8) into a simplified form with only one variable coefficient:

$$U_T - 3UU_X - 3U^2U_X + U_{XXX} + \big[1 + X\alpha(T)\big]U_X + \alpha(T)U + \frac{\alpha(T)}{2} = 0 \tag{10}$$

with

$$\alpha(T) = \frac{\mathrm{e}^{4\int k(t)\,\mathrm{d}t} l(t)}{16\sqrt{2}h(t)\left[\int \mathrm{e}^{\int k(t)\,\mathrm{d}t} l(t)\,\mathrm{d}t\right]^4}\,.$$

In this case, the inhomogeneous effect of the variable coefficients has been mainly led by the terms of phase speed, damping, and external force. In the following, we will focus our studies on the soliton solutions and inhomogeneous effect for (10). Note that (10) cannot be transformed into a constant-coefficient one unless the condition

$$h(t) = \xi\,\frac{\mathrm{e}^{4\int k(t)\,\mathrm{d}t} l(t)}{\left[\int \mathrm{e}^{\int k(t)\,\mathrm{d}t} l(t)\,\mathrm{d}t\right]^4} \tag{11}$$

is satisfied, where ξ is an arbitrary constant. Our studies in the following will take no account of condition (11).

3. Bilinear Form via the Binary Bell Polynomials

Among the methods to obtain the soliton solutions, the Hirota method is a direct analytic tool for certain NLEEs [25, 26]. Once the bilinear presentation is given, multi-soliton solutions will be derived through the truncated formal perturbation expansion at different levels [27, 28]. Moreover, the Bell-polynomial scheme has been developed to deal with the NLEEs, to directly derive the bilinear form of a given NLEE through some properties of the binary Bell polynomials, rather than the dependent variable transformations [29, 30]. Hereby, we will make use of the muti-dimensional binary Bell polynomials to construct the bilinear representation of (10) and then obtain the soliton solutions via the Hirota method.

3.1. Multi-Dimensional Binary Bell Polynomials

Let $f = f(x_1, \ldots, x_n)$ (x_j are the variables, $j = 1, \ldots n$) be a C^∞ multi-variable function, then the multi-dimensional Bell polynomials can be defined as [29, 30]

$$\begin{aligned} Y_{n_1x_1,\ldots,n_lx_l}(f) &\equiv Y_{n_1,\ldots,n_l}(f_{r_1x_1,\ldots,r_lx_l})\\ &= \mathrm{e}^{-f}\partial_{x_1}^{n_1}\cdots\partial_{x_l}^{n_l}\mathrm{e}^{f}\,, \end{aligned} \tag{12}$$

where n_j ($j = 1, \ldots, l$) are the nonzero integers, $f_{r_1x_1,\ldots,r_lx_l} = \partial_{x_1}^{r_1}\cdots\partial_{x_l}^{r_l} f (r_k = 0, \ldots, n_k, k = 1, \ldots, l)$ and $Y_{n_1x_1,\ldots,n_lx_l}(f)$ denotes the multivariable polynomial with respect to $f_{r_1x_1,\ldots,r_lx_l}$. Specially, for $f = f(x,t)$, the associated two-dimensional Bell polynomials are

$$\begin{aligned} Y_{2x}(f) &= f_{2x} + f_x^2\,,\\ Y_{3x}(f) &= f_{3x} + 3f_{2x}f_x + f_x^3\,, \end{aligned} \tag{13}$$

$$\begin{aligned} Y_{x,t}(f) &= f_{x,t} + f_xf_t\,,\\ Y_{2x,t}(f) &= f_{2x,t} + f_{2x}f_t + 2f_{x,t}f_x + f_x^2f_t\,,\\ &\;\;\vdots \end{aligned} \tag{14}$$

In order to differ the odd and even order derivatives of $f_{r_1x_1,\ldots,r_lx_l}$, the multi-dimensional binary Bell polynomial ($\mathcal{Y}$-polynomial) is introduced as [30]

$$\mathcal{Y}_{n_1x_1,\ldots,n_lx_l}(v,\omega) = Y_{n_1x_1,\ldots,n_lx_l}(f)\Bigg|_{f_{r_1x_1,\ldots,r_lx_l}=\begin{cases} v_{r_1x_1,\ldots,r_lx_l}\,, & r_1+\cdots+r_l \text{ is odd.} \\ \omega_{r_1x_1,\ldots,r_lx_l}\,, & r_1+\cdots+r_l \text{ is even.}\end{cases}} \tag{15}$$

Then (14) can be rewritten in the form of binary Bell polynomials as

$$\begin{aligned} &\mathcal{Y}_{2x}(v,\omega) = \omega_{2x} + v_x^2\,, \\ &\mathcal{Y}_{3x}(v,\omega) = v_{3x} + 3\omega_{2x}v_x + v_x^3\,, \end{aligned} \tag{16}$$

$$\begin{aligned} &\mathcal{Y}_{x,t}(v,\omega) = \omega_{x,t} + v_x v_t\,, \\ &\mathcal{Y}_{2x,t}(v,\omega) = v_{2x,t} + \omega_{2x}v_t + 2\omega_{x,t}v_x + v_x^2 v_t\,, \\ &\quad\vdots \end{aligned} \tag{17}$$

Note that the $\mathcal{Y}$-polynomial and Hirota expression $D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}F\cdot G$ can be linked via the identity [30]

$$\begin{aligned} &\mathcal{Y}_{n_1x_1,\ldots,n_lx_l}(v=\ln F/G, \omega=\ln FG) \\ &= (FG)^{-1}D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}F\cdot G\,, \end{aligned} \tag{18}$$

where the Hirota bilinear operators are defined as [27]

$$\begin{aligned} D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}(F\cdot G) = &\left(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_1'}\right)^{n_1}\cdots \\ &\cdot\left(\frac{\partial}{\partial x_l}-\frac{\partial}{\partial x_l'}\right)^{n_l}F(x_1,\ldots,x_l) \\ &\times G(x_1',\ldots,x_l')\Big|_{x_1'=x_1,x_l'=x_l}\,. \end{aligned}$$

Therefore, a nonlinear equation can be transformed into the corresponding bilinear equation via (18), once this nonlinear equation is expressible as a linear combination of $\mathcal{Y}$-polynomials.

3.2. Binary Bell-Polynomial Form of (10)

First of all, (10) can be rewritten in the form

$$\begin{aligned} &W_T - 6\beta\,\mathrm{e}^{-\int\alpha(T)\,\mathrm{d}T}WW_X - 3W^2W_X \\ &+W_{XXX} + \Big[7/4 - 3\beta^2\,\mathrm{e}^{-2\int\alpha(T)\,\mathrm{d}T} \\ &+X\alpha(T)\Big]W_X + \alpha(T)W = 0 \end{aligned} \tag{19}$$

through the dependent variable transformation

$$U = W + \beta\,\mathrm{e}^{-\int\alpha(T)\,\mathrm{d}T} - \frac{1}{2}\,, \tag{20}$$

where W is a differentiable function of X and T, while β is a constant. In order to transform (19) into the binary Bell-polynomial form, we introduce a potential field P by

$$W = \mu(T)P_X\,, \tag{21}$$

where $\mu(T)$ is a real function to be determined, P is a differentiable function of X and T. Substituting (21) into (19) and integrating with respect to X yields

$$\begin{aligned} &P_T - 3\beta\mu(T)\,\mathrm{e}^{-\int\alpha(T)\,\mathrm{d}T}P_X^2 - \mu(T)^2P_X^3 \\ &+P_{XXX} + \Big[7/4 - 3\beta^2\,\mathrm{e}^{-2\int\alpha(T)\,\mathrm{d}T} \\ &+X\alpha(T)\Big]P_X + \frac{\mu'(T)}{\mu(T)}P = 0\,. \end{aligned} \tag{22}$$

By means of (15), we replace the terms P_T, P_X, and P_{XXX} with $\mathcal{Y}_T(P,Q)$, $\mathcal{Y}_X(P,Q)$, and $\mathcal{Y}_{3X}(P,Q) - 3Q_{2X}P_X - P_X^3$, respectively, in (22) and obtain

$$\begin{aligned} &\mathcal{Y}_T(P,Q) + \mathcal{Y}_{3X}(P,Q) + \Big[7/4 \\ &-3\beta^2\,\mathrm{e}^{-2\int\alpha(T)\,\mathrm{d}T} + X\alpha(T)\Big]\mathcal{Y}_X(P,Q) \\ &-\mathcal{Y}_X(P,Q)\Big\{3\beta\mu(T)\,\mathrm{e}^{-\int\alpha(T)\,\mathrm{d}T}\mathcal{Y}_X(P,Q) \\ &+\left[\mu^2(T)+1\right]P_X^2 + 3Q_{XX}\Big\} + \frac{\mu'(T)}{\mu(T)}P = 0 \end{aligned} \tag{23}$$

with $Q = Q(X,T)$ being an arbitrary function. Comparing the last two terms above with (15) and (16), we set

$$\mu'(T) = 0\,,\;\; 1+\mu(T)^2 = 3 \tag{24}$$

and obtain the linear binary Bell-polynomial equations of (19):

$$\begin{aligned} &\mathcal{Y}_T(P,Q) + \mathcal{Y}_{3X}(P,Q) + \Big[7/4 - 3\beta^2\,\mathrm{e}^{-2\int\alpha(T)\,\mathrm{d}T} \\ &+X\alpha(T)\Big]\mathcal{Y}_X(P,Q) = 0, \end{aligned} \tag{25a}$$

$$\beta\mu(T)\,\mathrm{e}^{-\int\alpha(T)\,\mathrm{d}T}\mathcal{Y}_X(P,Q) + \mathcal{Y}_{2X}(P,Q) = 0\,. \tag{25b}$$

Through the dependent variable transformations $P = \ln F/G$, $Q = \ln FG$, and (18), we obtain the bi-

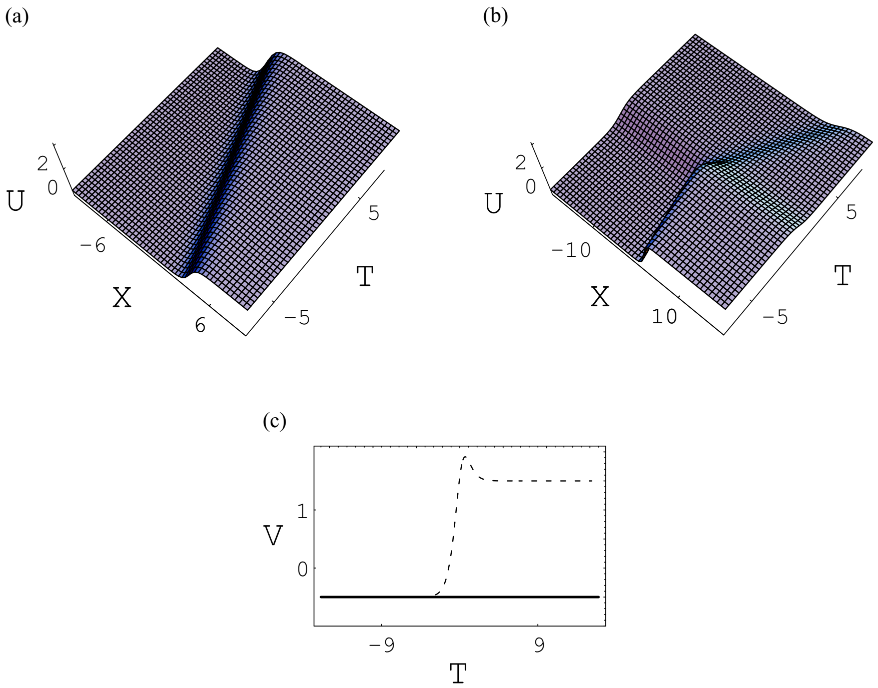

Fig. 1 (colour online). Kink-type soliton via solution (30). (a) $\delta_1 = 0$, $\beta = 1.5$, and $\alpha(T) = 0$. (b) The same as (a) except for $\alpha(T) = \{\ln[2+\tanh(T)]\}_T$. (c) Corresponding variation of the velocities of (a) (solid curve) and (b) (dashed curve).

linear form of (25) as

$$\left\{D_T + D_X^3 + \left[7/4 - 3\beta^2\, \mathrm{e}^{-2\int\alpha(T)\,\mathrm{d}T} + X\alpha(T)\right] D_X\right\} \cdot (F\cdot G) = 0\,, \tag{26a}$$

$$\left[\beta\mu(T)\,\mathrm{e}^{-\int\alpha(T)\,\mathrm{d}T} D_X + D_X^2\right](F\cdot G) = 0\,, \tag{26b}$$

where F and G are both functions of X and T.

To obtain the soliton solutions of (19), we expand G and F with respect to a formal expansion parameter ε as

$$F = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots\,, \tag{27a}$$

$$G = 1 + \varepsilon g_1 + \varepsilon^2 g_2 + \cdots\,, \tag{27b}$$

where f_m ($m = 1,2,3\ldots$) and g_l ($l = 1,2,3,\ldots$) are all real functions to be determined. Substituting (27) into (26) and truncating the perturbation expansion at different levels, we can derive the one- and multi-soliton solutions of (10) via (20).

4. Soliton Solutions and Inhomogeneous Effect of Variable Coefficient

4.1. One-Soliton Solution

To obtain the one-soliton solution of (10), we truncate the power series expansions of F and G to the order of ε and make the assumption

$$f_1 = \mathrm{e}^{\theta_1}\,, \quad g_1 = \zeta_1(T)\,\mathrm{e}^{\theta_1} \tag{28}$$

with $\theta_1 = k_1(T)X + w_1(T)$. Then substituting (27) into (26), we have

$$k_1(T) = \sigma_1\,\mathrm{e}^{-\int\alpha(T)\,\mathrm{d}T}\,, \quad \zeta_1(T) = \frac{\beta\mu(T)+\sigma_1}{\beta\mu(T)-\sigma_1}\,,$$

$$w_1(T) = \delta_1 - \sigma_1 \int \left\{ \mathrm{e}^{-3\int\alpha(T)\,\mathrm{d}T} \left[7/4\,\mathrm{e}^{2\int\alpha(T)\,\mathrm{d}T} - 3\beta^2 + \sigma_1^2\right]\right\} \mathrm{d}T\,,$$

where σ_1 and δ_1 are both constants. By means of (20) and (21), the explicit one-soliton solution of (10) can

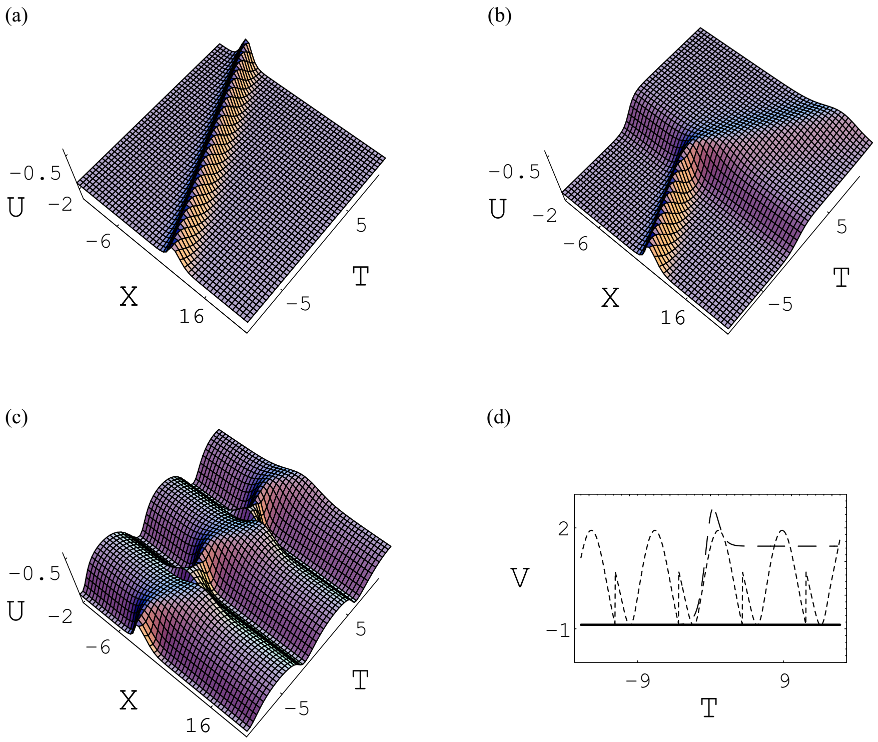

Fig. 2 (colour online). Elevation soliton via solution (33) with $\delta_1 = 0$, $\beta = 1.5$, and (a) $\alpha(T) = 0$ (b) $\alpha(T) = \{\ln[2+\tanh(T)]\}_T$ (c) $\alpha(T) = \{\ln[2+\sin(T)]\}_T$. (d) Corresponding variation of the velocities of (a) (solid curve), (b) (long dashed curve), and (c) (short dashed curve).

be derived as

$$U = \mu(T)\left[\ln\frac{1+\mathrm{e}^{\theta_1}}{1+\zeta_1(T)\,\mathrm{e}^{\theta_1}}\right]_X + \beta\,\mathrm{e}^{-\int\alpha(T)\,\mathrm{d}T} - \frac{1}{2}$$
$$= \frac{2\mu\sigma_1^2\,\mathrm{e}^{-\int\alpha(T)\,\mathrm{d}T}\,\mathrm{e}^{\theta_1}}{\sigma_1-\beta\mu-2\beta\mu\,\mathrm{e}^{\theta_1}-(\beta\mu+\sigma_1)\,\mathrm{e}^{2\theta_1}} \quad (29)$$
$$+\beta\,\mathrm{e}^{-\int\alpha(T)\,\mathrm{d}T} - \frac{1}{2},$$

where $\mu(T) \equiv \mu = \pm\sqrt{2}$ satisfying condition (24). In the following, we will take $\mu = \sqrt{2}$ as an example for the analysis. It is noted that the solution above include two forms, i. e., the kink-type and bell-profile-like (BPL) soliton solutions. Moreover, the last two terms $\beta\,\mathrm{e}^{-\int\alpha(T)\,\mathrm{d}T} - 1/2$ only influence the background of the wave and don't change the waveform.

Case 1 ($\sigma_1 = -\sqrt{2}\beta$)**.** In this case, the kink-type soliton solution can be derived from (29) as

$$U = -\beta\,\mathrm{e}^{-\int\alpha(T)\,\mathrm{d}T}\tanh\frac{\theta_1}{2} - \frac{1}{2}. \quad (30)$$

Note that the background of the kink-type soliton is a plane wave $U = -1/2$ since the term $\beta\,\mathrm{e}^{-\int\alpha(T)\,\mathrm{d}T}$ is just canceled after the calculation. As seen in (10), the variable coefficient $\alpha(T)$ mainly determines the existence of the phase speed, damping, and external force. Hereby, we will study the propagation characteristic of the kink-type soliton with and without those effects. Amplitude and velocity of the kink-type soliton can be expressed as

$$A = |-\beta\,\mathrm{e}^{-\int\alpha(T)\,\mathrm{d}T}|, \quad (31)$$
$$V = \frac{7}{4} - \beta^2\,\mathrm{e}^{-2\int\alpha(T)\,\mathrm{d}T} + \frac{1}{4}\alpha(T)\,\mathrm{e}^{\int\alpha(T)\,\mathrm{d}T}$$
$$\cdot\int \mathrm{e}^{-3\int\alpha(T)\,\mathrm{d}T}\left[7\,\mathrm{e}^{2\int\alpha(T)\,\mathrm{d}T} - 4\beta^2\right]\mathrm{d}T \quad (32)$$
$$+\frac{2\sqrt{2}\delta_1}{4\beta}\alpha(T)\,\mathrm{e}^{\int\alpha(T)\,\mathrm{d}T}.$$

Setting $\alpha(T) = 0$, we can have the kink-type soliton in Figure 1a. In this case, both the amplitude and velocity are the constants during the propaga-

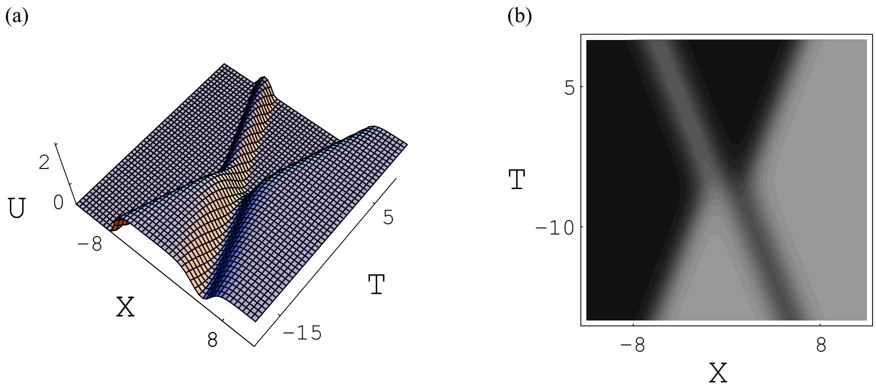

Fig. 3 (colour online). (a) Interaction between a depression soliton and a kink-type soliton via solution (36) with $\beta = 1.2$, $\delta_1 = \delta_2 = 0$, $\sigma_1 = -1.2\sqrt{2}$, $\sigma_2 = 1.5$, and $\alpha(T) = 0$. (b) Contour plot of (a).

tion. When $\alpha(T)$ is selected as a monotonous function $\{\ln[2+\tanh(T)]\}_T$, we can obtain the kink-type soliton propagating with variable amplitude and velocity, as seen in Figure 1b. Meanwhile, its amplitude appears as a tanh function along the T axis due to the effect of the variable coefficient $\alpha(T)$. In contrast, the velocities of the cases in Figures 1a and 1b can be described in Figure 1c, which indicates that $\alpha(T)$ not only influences the magnitude of the velocity but also changes the direction.

Case 2 ($\sigma_1 \neq -\sqrt{2}\beta$)**.** Solution (29) can be rewritten as

$$U = \frac{-\sqrt{2}\sigma_1^2\,\mathrm{e}^{-\int\alpha(T)\,\mathrm{d}T}}{\sqrt{2}\beta + \sqrt{2\beta^2-\sigma_1^2}\cosh\left(\theta_1 + \ln\sqrt{\frac{\sqrt{2}\beta+\sigma_1}{\sqrt{2}\beta-\sigma_1}}\right)} + \beta\,\mathrm{e}^{-\int\alpha(T)\,\mathrm{d}T} - \frac{1}{2}, \tag{33}$$

where the amplitude A and velocity V for the BPL soliton can be derived as

$$A = \left| \frac{-\sqrt{2}\sigma_1^2\,\mathrm{e}^{-\int\alpha(T)\,\mathrm{d}T}}{\sqrt{2}\beta+\sqrt{2\beta^2-\sigma_1^2}} + \beta\,\mathrm{e}^{-\int\alpha(T)\,\mathrm{d}T} - \frac{1}{2} \right|, \tag{34}$$

$$V = \frac{7}{4} - (3\beta^2-\sigma_1^2)\,\mathrm{e}^{-2\int\alpha(T)\,\mathrm{d}T} + \frac{1}{4}\alpha(T)\,\mathrm{e}^{\int\alpha(T)\,\mathrm{d}T} \cdot \int \mathrm{e}^{-3\int\alpha(T)\,\mathrm{d}T}\left[7\,\mathrm{e}^{2\int\alpha(T)\,\mathrm{d}T} - 12\beta^2 + 4\sigma_1^2\right]\mathrm{d}T - \frac{\alpha(T)\,\mathrm{e}^{\int\alpha(T)\,\mathrm{d}T}}{\sigma_1}\left(\delta_1 + \ln\sqrt{\frac{\sqrt{2}\beta+\sigma_1}{\sqrt{2}\beta-\sigma_1}}\right). \tag{35}$$

From (34) and (35), we find that both the amplitude and velocity vary with the time. Moreover, the variable background of the BPL soliton can be expressed by $U = \beta\,\mathrm{e}^{-\int\alpha(T)\,\mathrm{d}T} - \frac{1}{2}$. Therefore we can investigate how the variable coefficient $\alpha(T)$ affects those physical properties of the BPL soliton. Besides, there exist two families of the BPL solitons, i. e., the elevation and depression ones, which are related to the sign of β. As $\beta < 0$, the elevation BPL soliton will arise, while the depression, when $\beta > 0$. Of those BPL solitons, apart from the ordinary soliton, the plateau ($\beta < 0$) and basin ($\beta > 0$) soliton will appear in the case of $\sigma_1 \sim -\sqrt{2}\beta$. Here, we take the elevation solitons as examples to perform the analysis.

Setting $\alpha(T) = 0$, we find that the amplitude A and velocity V of the elevation soliton both maintain the same during the propagation along the T axis, as seen in Figure 2a. However, the case of $\alpha(T) = \{\ln[2+\tanh(T)]\}_T$ corresponds to the propagation of the elevation soliton on a kink-type background in Figure 2b. Note that the effect of $\alpha(T)$ will lead to the phenomenon of width expansion of the elevation soliton when it travels from the low platform to the high one. Meanwhile, the ordinary soliton becomes the plateau one. If $\alpha(T)$ is chosen as a periodic function, such as $\alpha(T) = \{\ln[2+\sin(T)]\}_T$, the elevation soliton will propagate periodically on a periodic background as seen in Figure 2c. Similar to the case in Figure 2b, the elevation soliton will be widened when it "climbs" onto a high platform. The variation of the velocities in Figures 2a–2c can be seen in Figure 2d. It is found that

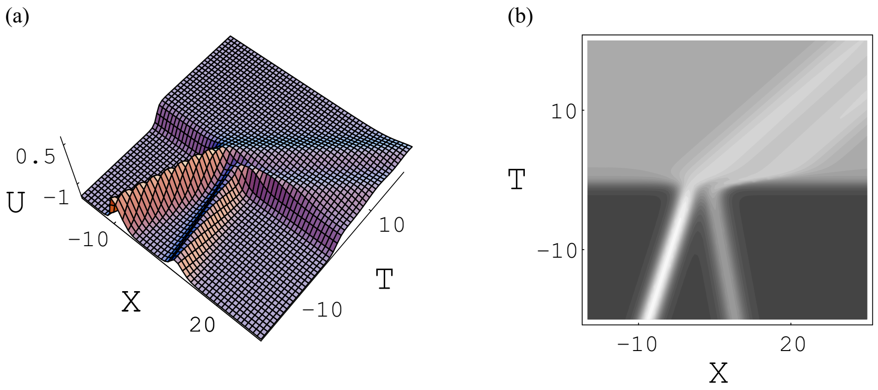

Fig. 4 (colour online). (a) Interaction between two elevation solitons via solution (36) with $\beta=-1$, $\delta_1=\delta_2=0$, $\sigma_1=1.3$, $\sigma_2=1$, and $\alpha(T)=\{\ln[2+\tanh(T)]\}_T$. (b) Contour plot of (a).

the change of the platform affects the direction of the velocity.

4.2. Two-Soliton Solution

Similarly, we truncate G and F as $G=1+\varepsilon g_1+\varepsilon^2 g_2$ and $F=1+\varepsilon f_1+\varepsilon^2 f_2$, respectively, and substitute them into (26); then the two-soliton solution can be given as

$$U=\mu\left[\ln\frac{1+\mathrm{e}^{\theta_1}+\mathrm{e}^{\theta_2}+\varpi_1(T)\,\mathrm{e}^{\theta_1+\theta_2}}{1+\zeta_1(T)\,\mathrm{e}^{\theta_1}+\zeta_2(T)\,\mathrm{e}^{\theta_2}+\zeta_3(T)\,\mathrm{e}^{\theta_1+\theta_2}}\right]_X +\beta\,\mathrm{e}^{-\int\alpha(T)\,\mathrm{d}T}-\frac{1}{2}, \tag{36}$$

where

$$\theta_j=k_j(T)X+w_j(T)\,,\;\; k_j(T)=\sigma_j\,\mathrm{e}^{\int\alpha(T)\,\mathrm{d}T}\,,$$
$$\zeta_j(T)=\frac{\beta\mu+\sigma_j}{\beta\mu-\sigma_j}\,,\;\; \mu=\pm\sqrt{2}\,,$$
$$w_j(T)=\delta_j-\frac{1}{4}\sigma_j\int \mathrm{e}^{-3\int\alpha(T)\,\mathrm{d}T}\Big[7\,\mathrm{e}^{2\int\alpha(T)\,\mathrm{d}T} -12\beta^2+4\sigma_j^2\Big]\,\mathrm{d}T\,,$$
$$\varpi_1(T)=\frac{(\sigma_1-\sigma_2)^2}{(\sigma_1+\sigma_2)^2}\,,$$
$$\zeta_3(T)=\frac{(\beta\mu+\sigma_1)(\beta\mu+\sigma_2)(\sigma_1-\sigma_2)^2}{(\beta\mu-\sigma_1)(\beta\mu-\sigma_2)(\sigma_1+\sigma_2)^2}\;\;(j=1,2)\,.$$

Then, we will study the interaction of two solitons under the influence of the variable coefficient $\alpha(T)$. In Figure 3, the interaction between a depression soliton and a kink-type soliton is obtained with $\alpha(T)=0$, $\sigma_1=-\sqrt{2}\beta$ and $\sigma_2\ll-\sqrt{2}\beta$. After the interaction, the depression soliton changes its polarity and propagates in the elevation form, as seen in Figure 3b. If we choose $\sigma_j\ll-\sqrt{2}\beta$ $(j=1,2)$ and $\alpha(T)=\{\ln[2+\tanh(T)]\}_T$, interaction of two elevation solitons on a kink-type background will be obtained in Figure 4. When two solitons travel to the high platform of the kink-type background, both of them change the directions of the velocities. But the polarity of the two elevation solitons doesn't change, which is different from that in Figure 3. The interaction is also changed from the head-on form to the overtaking one, as seen in Figure 4b. Therefore, we can conclude that $\alpha(T)$ can not only affect the amplitude and velocity of the solitons, but also change the type of the interaction.

5. Conclusions

In this paper, we have investigated the feKdV equation with time- and space-dependent variable coefficients, as seen in (3), which can describe the atmospheric blocking phenomenon. By means of symbolic computation, we have constructed the nonisospectral Lax Pair (5) of (3). Under the integrable conditions (7), a new integrable form of (3) has been presented, which is different from that studied in [13]. Through scale transformation (9), a simplified equation, as seen in (10), with minimal number of variable coefficients, has been given. Via the binary-Bell-polynomial method, we have given binary Bell polynomial form (25) of (10) and then derived bilinear form (26). One- and two-soliton solutions in (29) and (36) have been given explicitly. Moreover, the soliton dynamics and inhomogeneous effect of variable coefficient $\alpha(T)$ have been analyzed. Attention should be paid to the following aspects:

(i) Via one-soliton solution (29), we have found that two types of solutions can be obtained, i. e., the kink-type and BPL soliton solutions, which depend on the choice of σ_1. That is, for $\sigma_1 = -\sqrt{2}\beta$, the kink-type soliton solution is presented as (30), otherwise we can have BPL soliton solution (33). The analysis also shows that there exist two families of the BPL solitons, i. e., the elevation and depression ones corresponding to the conditions $\beta < 0$ and $\beta > 0$, respectively. Furthermore, from the BPL solitons, we can obtain not only the ordinary soliton but also the plateau and basin solitons in the case of $\sigma_1 \sim -\sqrt{2}\beta$.

(ii) Through the analysis on the effect of $\alpha(T)$, it is found that $\alpha(T)$ mainly influences the amplitudes and velocities of the kink-type and BPL solitons, as seen in Figures 1 and 2. Explicit expressions of the physical quantities have been given respectively in (31), (32), (34), and (35). In addition, the background of the BPL soliton is found to be adjusted through the expression $\beta\,\mathrm{e}^{-\int\alpha(T)\,\mathrm{d}T} - \frac{1}{2}$. By choosing $\alpha(T) = \{\ln[2+\tanh(T)]\}_T$, we have obtained the evolution of a elevation soliton on a kink-type background, which presents the phenomenon of width expansion when travelling from one platform to the other, as seen in Figure 2.

(iii) By choosing $\alpha = 0$, $\sigma_1 = -\sqrt{2}\beta$, and $\sigma_2 \ll -\sqrt{2}\beta$, we have obtained the interaction between a depression soliton and a kink-type soliton, as seen in Figure 3. After the interaction, the depression soliton changes its polarity and propagates in the elevation form. When the effect of $\alpha(T)$ is considered, the interaction of two elevation solitons on a kink-type background has been shown in Figure 4. The amplitudes and velocities of the solitons are changed when they travel from the low platform to the high one of the kink-type background. Moreover, the interaction is changed from the head-on form to the overtaking one, but this background has no influence on the polarity of the solitons.

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