Localized Structures Constructed by Multi-Valued Functions in the $(2+1)$-dimensional Generalized Nizhnik–Novikov–Veselov Equation

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A modified mapping method is presented to derive a variable separation solution with two arbitrary functions of the $(2+1)$-dimensional generalized Nizhnik–Novikov–Veselov equation. By selecting appropriate functions in the variable separation solution, we discuss interaction behaviours among special solitons, constructed by multi-valued functions, including the compacton-like dromion, compacton-like peakon, and compacton-like semi-foldon.

Key words: Modified Mapping Method; Generalized Nizhnik–Novikov–Veselov Equation; Variable Separation Solution; Interactions between Special Solitons.
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1. Introduction

In linear wave theory, the Fourier analysis and the variable separation approach (VSA) are two most universal and powerful means for the study of linear partial differential equations (PDEs). As a nonlinear analogue of the Fourier analysis, the celebrated inverse scattering transformation plays an important role to analyze nonlinear wave dynamics [1]. The extension of the VSA to nonlinear field has also been a highlight, and there come out some methods: the formal VSA [2], the multilinear VSA [3, 4], and the VSA based on the mapping method [5, 6], and so on. Moreover, many direct methods based on different mapping equations, including the improved projective approach [7 – 9], the q-deformed hyperbolic functions method [10], and the projective Ricatti equation method (PREM) [11, 12], were chosen to realize the variable separation to nonlinear equations.

Many single-valued localized structures (dromions, peakons, and compactons etc.) have been extensively investigated [3 – 12]. However, in the real natural phenomena, there exist very complicated folded phenomena such as the folded protein [13], folded brain and skin surfaces, and many other kinds of folded biologic systems [14]. Moreover, semifolded structures can also be realized. For example, ocean waves may fold in one direction, say x, and localize in an usual single valued way in another direction, say y. These special localized structures can be constructed by multi-valued functions. Of course, at the present stage, it is impossible to make satisfactory analytic descriptions for such complicated folded natural phenomena. However, it is still worth starting with some simpler cases. For example, the interactions among some semi-structures, such as compacton-like semi-dromion, compacton-like semi-peakon, and compacton-like semi-foldon, were little reported in previous literature.

Naturally, some significant and interesting issues arise: Can other mapping equations be used to obtain variable separation solutions of some $(2+1)$-dimensional nonlinear physics systems? Can we discuss some new dynamical behaviours among semi-structures based on these variable separation solutions? In order to answer these issues, we study the following well-known $(2+1)$-dimensional generalized Nizhnik–Novikov–Veselov (GNNV) equation:

$$\begin{aligned} &u_t + au_{xxx} + bu_{yyy} + cu_x + du_y \\ &- 3a(uv)_x - 3b(uw)_y = 0\,, \\ &u_x = v_y,\ \ u_y = w_x\,, \end{aligned} \qquad (1)$$

where a, b, c, and d are arbitrary constants. For $c = d = 0$, the GNNV system will be degenerated to the usual two-dimensional Nizhnik–Novikov–Veselov (NNV) system, which is an isotropic Lax extension of the classical $(1+1)$-dimensional shallow water-wave Korteweg–de Vries (KdV) model. When $a = 1$, $b = c = d = 0$ in (1), we get the asymmetric NNV equation, which may be considered as a model for an incompressible fluid. Some types of the soliton solutions of the GNNV equation have been studied by many authors. For instance, Boiti et al. [15] solved the GNNV equation via the inverse scattering transformation. Zhang obtained many exact solutions of this system based on an extended homogeneous balance approach [16]. However, the GNNV equation yields many interesting soliton structures that have not yet been found, and the interaction between the solitons is still not clear. In [17] and [18], authors obtained variable separation solutions of (1).

2. The Modified Mapping Method

Consider a given nonlinear PDE with independent variables $x = (x_0 = t, x_1, x_2, x_3, \ldots, x_m)$ and dependent variable u,

$$L(u, u_t, u_{x_i}, u_{x_i x_j}, \ldots) = 0, \tag{2}$$

where L is in general a polynomial function of its argument, and the subscripts denote the partial derivatives.

The basic idea of the mapping method is to seek for an ansatz with positive and negative symmetric form

$$u = \sum_{i=0}^{n} a_i(x)\phi^i[q(x)], \tag{3}$$

where a_i are arbitrary functions of x to be determined, and n is fixed by balancing the linear term of the highest order with the nonlinear term in (2).

Note that many mapping equations for ϕ have been used, such as the Riccati equation $\phi' = l_0 + \phi^2$ (l_0 is a constant and the prime denotes differentiation with respect to q) [4–6], $\phi' = \sigma\phi + \phi^2$ (σ is a constant) [7–9], and $\phi' = l_1 + l_2\phi^2$ (l_1 and l_2 are two constants) [19]. Here we seek for a solution of the given nonlinear evolution equation (NLEE) (2) with the new mapping equation [20]

$$\phi' = (A\phi - C)(B\phi - D), \tag{4}$$

which is known to possess the general solution

$$\phi = \frac{D\exp[(BC-AD)q] - C\exp[C_1(AD-BC)]}{B\exp[(BC-AD)q] - A\exp[C_1(AD-BC)]}. \tag{5}$$

Here C_1 is an integration constant, further, A, B, C, and D are arbitrary constants.

To determined u explicitly, we take following three steps:

Step 1: Determine n by balancing the highest nonlinear terms and the highest-order partial differential terms in the given nonlinear PDE (2).

Step 2: Substituting (3) along with (4) into (2) yields a set of polynomials for ϕ^i. Eliminating all the coefficients of the powers of ϕ^i, yields a series of partial differential equations, from which the parameters a_0, a_i $(i = 1, \ldots, n)$, and q are explicitly determined.

Step 3: Substituting a_0, a_i, q, and (5) into (3), one can obtain possible solutions of (2).

3. Variable Separation Solutions for the $(2+1)$-Dimensional GNNV Equation

Along with the modified mapping method in Section 2, by balancing the higher-order derivative terms with the nonlinear terms in (1), we suppose that it has the following formal solutions:

$$\begin{aligned} u(x,y,t) &= a_0(x,y,t) + a_1(x,y,t)\phi(q) \\ &\quad + a_2(x,y,t)\phi(q)^2, \\ v(x,y,t) &= b_0(x,y,t) + b_1(x,y,t)\phi(q) \\ &\quad + b_2(x,y,t)\phi(q)^2, \\ w(x,y,t) &= c_0(x,y,t) + c_1(x,y,t)\phi(q) \\ &\quad + c_2(x,y,t)\phi(q)^2, \end{aligned} \tag{6}$$

where ϕ satisfies (5) and $q \equiv q(x,y,t)$. Inserting (6) into (1), selecting the variable separation ansatz

$$q = \chi(x,t) + \psi(y,t), \tag{7}$$

and eliminating all the coefficients of the powers of ϕ^i, one gets a set of partial differential equations, from which we obtain a solution, namely

$$
\begin{aligned}
&a_0 = 2ABCD\chi_x\psi_y\,,\;\; a_1 = -2AB(AD+BC)\chi_x\psi_y\,,\;\; a_2 = 2A^2B^2\chi_x\psi_y\,,\\
&b_0 = \frac{a\chi_{xxx} + c\chi_x + \chi_t + a\left[(AD+BC)^2 + 2ABCD\right]\chi_x^3 - 3a(AD+BC)\chi_x\chi_{xx}}{3a\chi_x}\,,\\
&b_1 = -2AB\left[(AD+BC)\chi_x^2 - \chi_{xx}\right]\,,\;\; b_2 = 2A^2B^2\chi_x^2\,,\\
&c_0 = \frac{b\psi_{yyy} + d\psi_y + \psi_t + a\left[(AD+BC)^2 + 2ABCD\right]\psi_y^3 - 3a(AD+BC)\psi_y\psi_{yy}}{3b\psi_y}\,,\\
&c_1 = -2AB\left[(AD+BC)\psi_y^2 - \psi_{yy}\right]\,,\;\; c_2 = 2A^2B^2\psi_y^2\,,
\end{aligned} \tag{8}
$$

where χ and ψ are arbitrary functions of x,t and y,t, respectively.

Therefore, the variable separation solution of the (2+1)-dimensional GNNV equation reads

$$
\begin{aligned}
u = {} & 2ABCD\chi_x\psi_y - 2AB(AD+BC)\chi_x\psi_y \cdot \frac{D\exp[(BC-AD)(\chi+\psi)] - C\exp[C_1(AD-BC)]}{B\exp[(BC-AD)(\chi+\psi)] - A\exp[C_1(AD-BC)]}\\
& + 2A^2B^2\chi_x\psi_y \cdot \left\{\frac{D\exp[(BC-AD)(\chi+\psi)] - C\exp[C_1(AD-BC)]}{B\exp[(BC-AD)(\chi+\psi)] - A\exp[C_1(AD-BC)]}\right\}^2,
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
v = {} & \frac{a\chi_{xxx} + c\chi_x + \chi_t + a[(AD+BC)^2 + 2ABCD]\chi_x^3 - 3a(AD+BC)\chi_x\chi_{xx}}{3a\chi_x}\\
& - 2AB[(AD+BC)\chi_x^2 - \chi_{xx}] \cdot \frac{D\exp[(BC-AD)(\chi+\psi)] - C\exp[C_1(AD-BC)]}{B\exp[(BC-AD)(\chi+\psi)] - A\exp[C_1(AD-BC)]}\\
& + 2A^2B^2\chi_x^2 \cdot \left\{\frac{D\exp[(BC-AD)(\chi+\psi)] - C\exp[C_1(AD-BC)]}{B\exp[(BC-AD)(\chi+\psi)] - A\exp[C_1(AD-BC)]}\right\}^2,
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
w = {} & \frac{b\psi_{yyy} + d\psi_y + \psi_t + a[(AD+BC)^2 + 2ABCD]\psi_y^3 - 3a(AD+BC)\psi_y\psi_{yy}}{3b\psi_y}\\
& - 2AB[(AD+BC)\psi_y^2 - \psi_{yy}] \cdot \frac{D\exp[(BC-AD)(\chi+\psi)] - C\exp[C_1(AD-BC)]}{B\exp[(BC-AD)(\chi+\psi)] - A\exp[C_1(AD-BC)]}\\
& + 2A^2B^2\psi_y^2 \cdot \left\{\frac{D\exp[(BC-AD)(\chi+\psi)] - C\exp[C_1(AD-BC)]}{B\exp[(BC-AD)(\chi+\psi)] - A\exp[C_1(AD-BC)]}\right\}^2,
\end{aligned} \tag{11}
$$

where $\chi(x,t)$ and $\psi(y,t)$ are two arbitrary variable separation functions.

4. Localized Structures in the GNNV Equation

Based on the solutions (9)–(11), we can obtain many rich coherent localized structures such as non-propagating solitons, dromions, peakons, compactons, foldons, instantons, and ring solitons [3–12]. Here we will pay attention to interaction behaviours between special solitons for the physical quantity u expressed by (9). The interaction behaviours between solitons in (2+1)-dimensional nonlinear models are usually considered to be completely elastic, which means that the amplitude, velocity, and shape of a soliton do not undergo any change after the nonlinear interaction. Otherwise, the interaction between solitons is non-elastic (non-completely elastic and completely non-elastic). Like the collisions between two classical particles, a collision in which the solitons stick together is sometimes called completely non-elastic.

4.1. Localized Structures Constructed by Multi-Valued Functions

We discuss the three special combined soliton structures, i. e. compacton-like dromion, compacton-like peakon, and compacton-like semi-foldon, by introducing multi-valued function as

$$\psi_y = \sum_{i=1}^{N} \kappa_i(\zeta - d_i t)\,, \quad y = \zeta + \sum_{i=1}^{N} \eta_i(\zeta - d_i t)\,, \tag{12}$$

where d_i $(i = 1, 2, \ldots, N)$ are arbitrary constants, κ_i and η_i are localized excitations with the properties $\kappa_i(\pm\infty) = 0$, $\eta_i(\pm\infty) = \text{const}$. From (12), one can know that ζ may be a multi-valued function in some suitable regions of y by choosing the functions η_i appropriately. Therefore, the function p_x, which is obviously an interaction solution of N localized excitations due to the property $\zeta|_{x\to\infty} \to \infty$, may be a multi-valued function of x in these areas, though it is a single-valued function of ζ. Actually, most of the known multi-loop solutions are special cases of (12).

Specifically, χ and ψ are chosen as

$$\chi = \begin{cases} 0 & x \le -\dfrac{\pi}{4}\,, \\ 4\sin(2x) + 1 & -\dfrac{\pi}{4} < x \le \dfrac{\pi}{4}\,, \\ 5 & x > \dfrac{\pi}{4}\,, \end{cases} \tag{13}$$

$$\psi_y = 0.5\,\mathrm{sech}^2(\zeta - 0.5t)\,, \quad y = \zeta - E\tanh(\zeta - 0.5t)\,, \tag{14}$$

where E is a characteristic parameter, which determines the localized structure. Figure 1 describes these special localized structures, i. e. compacton-like dromion, compacton-like peakon, and compacton-like semi-foldon, with $E = 0.1$, 0.95, and 1.5, respectively. They are localized as compacton in the y-direction and bell-like soliton, peakon, and loop soliton in the x-direction, respectively.

4.2. Completely Elastic Interaction Among Solitons

Let us study interaction behaviours among these special solitons produced by multi-valued functions above. If we take the specific choice $N = 3$, $d_1 = 0$, $d_2 = 0.5$, and $d_3 = -0.5$ in (12), one has

$$\begin{aligned} \psi_y &= 0.6\,\mathrm{sech}^2(\zeta) + 0.5\,\mathrm{sech}^2(\zeta - 0.5t) \\ &\quad + 0.7\,\mathrm{sech}^2(\zeta + 0.5t)\,, \\ y &= \zeta - E\tanh(\zeta) - F\tanh(\zeta - 0.5t) \\ &\quad - G\tanh(\zeta + 0.5t)\,, \end{aligned} \tag{15}$$

where C, D, and E are characteristic parameters, which determine the types of interaction. Moreover, χ is given by (13). From the expression u with (15) and (13), one can obtain three solitons, one is static, another is moving along positive y-direction, and the last one is moving along negative y-direction.

If we take the specific values $E = G = 0.95$, $F = 1.5$ in (15), then we can successfully construct the interaction among two compacton-like peakons and one compacton-like semifoldon, which possess a phase shift for the physical quantity u depicted in Figure 2. From Figure 2, one can find that the interaction may exhibit a completely elastic behaviour since solitons' shapes and amplitudes are completely maintained after the interaction.

The phase shift can also be observed. Prior to interaction, the velocities of the smallest compacton-like semifoldon, middle and the largest compacton-like peakons have set to be $v_{02x} = d_2 = 0.5$,, $v_{01x} = d_1 = 0$, and $v_{03x} = d_3 = -0.5$, respectively. The middle compacton-like peakon site changes from $y = -0.6$ to $y = 0.6$, then resides at $x = 0.6$ and maintains its

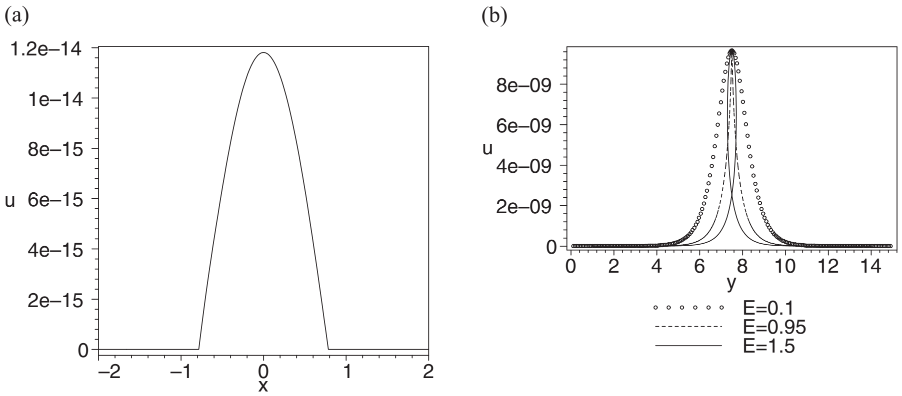

Fig. 1. Sectional views of special solitons at (a) $x = 0$ and (b) $y = 0$ for parameters $A = 2$, $B = 1$, $C = 0.5$, $D = 3$, $C_1 = 0.1$ at time $t = 15$.

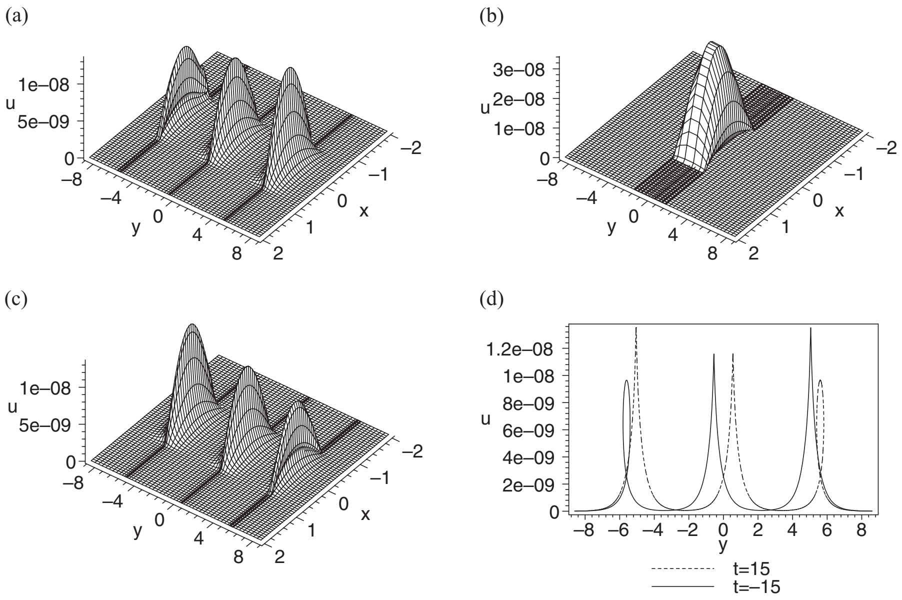

Fig. 2. Completely elastic interaction among two compacton-like peakons and one compacton-like semifoldon at time (a) $t=-15$, (b) $t=-0.1$, and (c) $t=15$. (d) Sectional view of (a) and (c) at $x=0$. The parameters are chosen as $A=2$, $B=1$, $C=0.5$, $D=3$, $C_1=0.1$, $E=G=0.95$, $F=1.5$.

initial velocity $v_{1x}=v_{01x}=0$ (i. e. static) after interaction. Therefore, the magnitude of the phase shift of the static middle compacton-like peakon is 1.2. The final velocities v_{2x} and v_{3x} of the other moving compacton-like peakons also completely maintain their initial values $v_{2x}=v_{02x}=0.5$ and $v_{3x}=v_{03x}=-0.5$. The phase shifts of them can also been observed in Figure 2d. From Figure 2, the smallest compacton-like semifoldon, middle and the largest compacton-like peakons preserve their amplitudes $0.96\cdot10^{-8}$, $1.2\cdot10^{-8}$, and $1.32\cdot10^{-8}$, respectively, before and after interaction. The amplitudes, velocities, and shapes of the solitons do not undergo any change after the nonlinear interaction, and thus this interaction is completely elastic.

Similarly, if we choose the specific values $E=F=G=1.5$ in (15), then we can successfully obtain interaction among three compacton-like semifoldons. This interaction has also a completely elastic behaviour since solitons' shapes and amplitudes are not completely maintained any more after the interaction (c.f. Fig. 3). However, different from the interaction among two compacton-like peakons and one compacton-like semifoldon in Figure 2, here no phase shift is observed in Figure 3d. Before and after interaction, the static smallest semifoldon is both located at $x=0$ and the other two semifoldons exactly exchange the corresponding position. The middle, smallest, and largest compacton-like semifoldons maintain their initial velocities $v_{2x}=v_{02x}=0.5$ (positive y-direction moving), $v_{1x}=v_{01x}=0$ (i. e. static), and $v_{3x}=v_{03x}=-0.5$ (negative y-direction moving), respectively. From Figure 3, the three compacton-like semifoldons preserve their amplitudes $7.3\cdot10^{-9}$, $6.4\cdot10^{-9}$, and $5.2\cdot10^{-9}$, respectively, before and after interaction. This interaction is completely elastic because the amplitudes, velocities, and shapes of the solitons maintain unchanged after the nonlinear interaction.

Moreover, we can analyze asymptotic behaviours of the localized excitations to discuss the type of interaction. In general, if the function χ and ψ (considering (12)) are selected as multi-localized solitonic excitations with ($z_i\equiv\zeta-d_it$)

$$\chi|_{t\to\mp\infty}=\sum_{j=1}^{M}\chi_j^{\mp}\,,\quad \chi_j^{\mp}\equiv\chi_j(x-c_jt+\Delta_j^{\mp})\,, \tag{16}$$

$$\begin{aligned}\psi|_{t\to\mp\infty}&=\sum_{i=1}^{N}\psi_i^{\mp}\,,\\ \psi_i^{\mp}(z_i)&\equiv\psi_i(\zeta-d_it)\equiv\int\kappa_i\,\mathrm{d}y|_{z_i\to\mp\infty}\,,\end{aligned} \tag{17}$$

where $\{\chi_j,\psi_i\}\forall j$ and i are localized functions, then the physical quantity expressed by (9) delivers $M\times N$ $(2+1)$-dimensional localized excitations with the asymptotic behaviour

$$u|_{t\to\mp\infty} \to \sum_{i=1}^{N}\sum_{j=1}^{M} \frac{2\chi_{jx}^{\mp}\psi_{iz_i}^{\mp}}{1+\eta_{iz_i}^{\mp}} \Bigg\{ ABCD - AB(AD+BC) \\ \cdot \frac{D\exp\left[(BC-AD)(\chi_j^{\mp}+\tilde{\chi}_j^{\mp}+\psi_i^{\mp}+\tilde{\psi}_i^{\mp})\right] - C\exp[C_1(AD-BC)]}{B\exp\left[(BC-AD)(\chi_j^{\mp}+\tilde{\chi}_j^{\mp}+\psi_i^{\mp}+\tilde{\psi}_i^{\mp})\right] - A\exp[C_1(AD-BC)]} \\ + A^2B^2\left(\frac{D\exp\left[(BC-AD)(\chi_j^{\mp}+\tilde{\chi}_j^{\mp}+\psi_i^{\mp}+\tilde{\psi}_i^{\mp})\right] - C\exp[C_1(AD-BC)]}{B\exp\left[(BC-AD)(\chi_j^{\mp}+\tilde{\chi}_j^{\mp}+\psi_i^{\mp}+\tilde{\psi}_i^{\mp})\right] - A\exp[C_1(AD-BC)]}\right)^2 \Bigg\} \equiv \sum_{i=1}^{N}\sum_{j=1}^{M} u_{ij}^{\mp}, \tag{18}$$

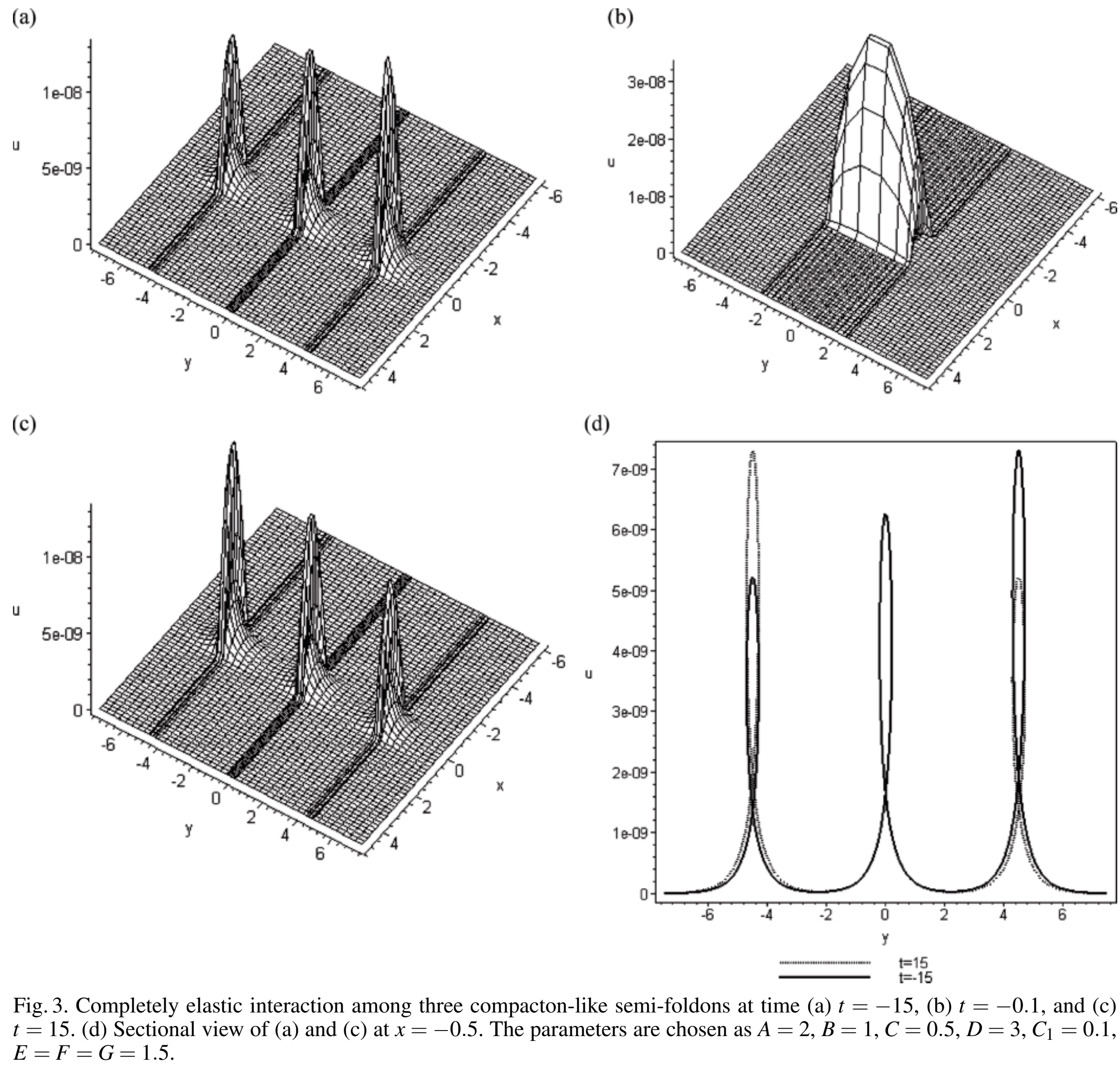

Fig. 3. Completely elastic interaction among three compacton-like semi-foldons at time (a) $t = -15$, (b) $t = -0.1$, and (c) $t = 15$. (d) Sectional view of (a) and (c) at $x = -0.5$. The parameters are chosen as $A = 2$, $B = 1$, $C = 0.5$, $D = 3$, $C_1 = 0.1$, $E = F = G = 1.5$.

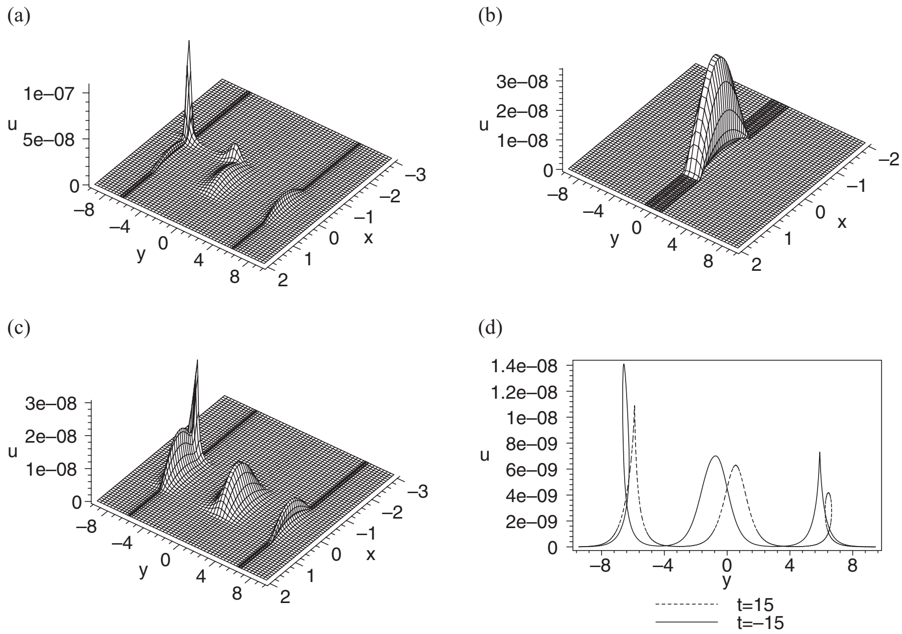

Fig. 4. Completely elastic interaction among compacton-like dromion, compacton-like peakon, and compacton-like semifoldon at time (a) $t=-15$, (b) $t=-0.1$, and (c) $t=15$. (d) Sectional view of (a) and (c) at $x=-0.5$. The parameters are chosen as $A=2$, $B=1$, $C=0.5$, $D=3$, $C_1=0.1$, $E=0.1$, $F=1.5$, $G=0.95$.

$$y|_{t\to\mp\infty} \to \zeta + \delta_i^{\mp} + \eta_i^{\mp}(z_i)\,, \tag{19}$$

with

$$\tilde{\chi}_i^{\mp} = \sum_{j<i} \chi_j(\mp\infty) + \sum_{j>i} \chi_j(\pm\infty)\,, \tag{20}$$

$$\tilde{\psi}_i^{\mp} = \sum_{j<i} \psi_j(\mp\infty) + \sum_{j>i} \psi_j(\pm\infty)\,, \tag{21}$$

$$\delta_i^{\mp} = \sum_{j<i} \eta_j(\mp\infty) + \sum_{j>i} \eta_j(\pm\infty)\,. \tag{22}$$

In the above discussion, the shape of the ij-th localized excitation u_{ij} will be changed (non-completely elastic interaction) if $\tilde{\chi}_j^+ \neq \tilde{\chi}_j^-$, and (or) $\tilde{\psi}_i^+ \neq \tilde{\psi}_i^-$, following the interaction. On the contrary, it will preserve its shape (completely elastic interaction) during the interaction if $\tilde{\chi}_j^+ = \tilde{\chi}_j^-$, and (or) $\tilde{\psi}_i^+ = \tilde{\psi}_i^-$.

Now we take the interaction among three compacton-like semifoldons as an example to illustrate the asymptotic analysis. Here $c_j = 0$ in (16), and thus we only consider whether $\tilde{\psi}_i^+$ is equal to $\tilde{\psi}_i^-$. Analytically, from (15) and (21), we have $\tilde{\psi}_1^+ - \tilde{\psi}_1^- = 0$, $\tilde{\psi}_2^+ - \tilde{\psi}_2^- = 0$, and $\tilde{\psi}_3^+ - \tilde{\psi}_3^- = 0$. That is to say, the completely elastic interaction condition (21) is really satisfied. This result agrees with the qualitative analysis above. Other cases can be analyzed similarly. Here we omit them due to the limit of length.

4.3. Non-Completely Elastic Interaction Among Solitons

It is interesting to note that although the above selections are all completely elastic interaction behaviours, we can also construct localized coherent structures with non-completely elastic interaction behaviours by appropriately selecting the values of E, F, and G in (15).

If we select the specific values $E = 0.1$, $F = 1.5$, and $G = 0.95$ in (15), then we can successfully construct the interaction among compacton-like dromion, compacton-like peakon, and compacton-like semi-foldon for the physical quantity u depicted in Figure 4. From Figure 4, one can find that the interaction among them may exhibit a non-completely elastic behaviour since solitons' shapes and amplitudes are not completely maintained although the final velocities of the solitons preserve the original velocities after interaction. The phase shift is also observed. Before and after interaction, the static smallest compacton-like dromionsite changes from $y = -0.8$ to $y = 0.8$, and the other two solitons also do not exchange the corresponding position and shift some distances.

Similarly, we can discuss the interaction among two compacton-like dromion and one compacton-like semi-foldon by setting the specific values $E = F = 0.1$, and $G = 1.5$ in (15). This case is also a non-completely elastic interaction. For the limit of length, we omit the detailed discussion about it.

5. Summary

In this paper, our interest has been focused on two issues proposed in the introduction. Here we review the main points offered in this paper:

- A new mapping equation is used.
  Besides mapping equations in [5 – 12], a new mapping equation is utilized to obtain variable separation solutions of some $(2+1)$-dimensional nonlinear physics systems. As an example, we apply it to the $(2+1)$-dimensional GNNV equation, and derive variable separation solution with two arbitrary functions.
- Non-completely elastic and completely elastic interactions among solitons are investigated.
  By selecting appropriate functions in the variable separation solution, we discuss interaction behaviours among special solitons, constructed by multi-valued functions, including the compacton-like dromion, compacton-like peakon, and compacton-like semi-foldon. The analysis results exhibit that the interaction behaviours among two compacton-like peakons and one compacton-like semifoldon, and among three compacton-like semifoldons are both completely elastic, while the interaction behaviours among compacton-like dromion, compacton-like peakon, and compacton-like semi-foldon, and among two compacton-like dromion and one compacton-like semi-foldon are both non-completely elastic.

Of course, there are some pending issues to be further studied. How to quantify the notion of complete or non-complete elasticity more suitably? What is the general equation for the distribution of the energy and momentum for these interactions?

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