# Solitons and Similaritons of a Generalized Nonlinear Schrödinger Equation with Variable Coefficients in a Power-Law Medium

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We obtain the similarity transformation and construct analytical soliton and similariton solutions for the generalized nonlinear Schrödinger equation with varying dispersion, power-law nonlinearity, and attenuation, which could describe the propagation of optical pulse in inhomogeneous fiber systems. Based on these solutions, we discuss dynamical behaviours of the chirped similariton and the chirp-free soliton in the dispersion decreasing fiber and the periodic distributed system. In the first soliton control system, we can control the compression and stretching by modulating the dispersion parameter $\sigma$. The pulse is compressed for parameter $\sigma > 0$, while the pulse is stretched for parameter $\sigma < 0$. In the second soliton control system, the snake-like propagation behaviour disappears little by little and the period of waves gradually decreases with the increasing of the index of the power-law nonlinearity. Compared with chirped similaritons, chirp-free solitons remain with the certain amplitude and width in two systems.

*Key words:* Variable-Coefficient Nonlinear Schrödinger Equation; Similarity Transformation; Similariton; Soliton; Power-Law Nonlinearity.
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## 1. Introduction

During the last three decades, optical solitons have been witnessed as good information carriers for long distance trans-continental and trans-oceanic communication and all-optical ultrafast switching devices [1 – 3]. The possibility of the propagation of solitons in optical fibers was theoretically predicted by Hasegawa and Tappert [4] and was experimentally demonstrated by Mollenauer et al. [5]. Moreover, the so-called optical similariton [6 – 10], which appears when the coaction of nonlinearity, dispersion, and gain in a high-power fiber amplifier causes the shape of an arbitrary input pulse to converge asymptotically to a pulse with a self-similar shape, recently has flourished into a research area of great importance and interest in many different contexts of nonlinear optics because it can tolerate strong nonlinearity without wave breaking and enhance linearity of chirp [11] compared with a soliton. Meanwhile, recently nonautonomous solitons in Bose–Einstein condensates have been reported [12].

In the picosecond regime, the nonlinear evolution equation that takes into account this balance between the group velocity dispersion (GVD) and the self-phase modulation (SPM) and which describes the dynamics of solitons is the well-known nonlinear Schrödinger equation (NLSE) with Kerr nonlinearity. However, communication grade optical fibers or as a matter of fact any optical transmitting medium does possess a finite attenuation coefficient, thus optical loss is inevitable, and the pulse is often deteriorated by this loss. Due to the nonsaturable nature of fiber, the Kerr nonlinearity is inadequate to describe the soliton dynamics in the ultrahigh bit rate transmission. When the transmission bit rate is very high, the peak power of the incident field accordingly becomes very large for soliton formation. Thus higher-order (power-law) nonlinearities may become significant even at moderate intensities in certain materials such as semiconductor doped glass fibers. Under circumstances, as mentioned above, non-Kerr law nonlinearities come into play changing essentially the physical features of optical soliton propagation [13, 14]. More material realization for non-

Kerr law nonlinearities can be found in [15]. Therefore, when a very high bit rate transmission or transmission through materials with higher nonlinear coefficients are considered, it is necessary to take into account higher-order nonlinearities. This problem can be addressed by incorporating the non-Kerr law nonlinearity in the NLSE. Spatial solitons have been investigated in media that have a power-law dependence on the intensity $I^n$ for continuum values of $n$ (with $I$ being the intensity) [16, 17].

In a real fiber, the core medium is not homogeneous [18]. There will always be some nonuniformity due to many factors, and important among them are:

(i) that which arises from a variation in the lattice parameters of the fibre medium, so that the distance between two neighbouring atoms is not constant throughout the fibre,
(ii) that due to the variation of the fibre geometry (diameter fluctuations, etc).

These nonuniformities influence various effects such as attenuation (or gain), dispersion, phase modulation, etc. Thus, in order to model these features in the soliton dynamics, from a practical standpoint, the variable coefficient NLSE with power-law nonlinearity should be considered as follows [19]:

$$\mathrm{i}u_z + \frac{1}{2}\beta(z)u_{tt} + g_m(z)|u|^{2m}u = \mathrm{i}\gamma(z)u\,, \tag{1}$$

where $u(z,t)$ is the complex envelope of the electrical field in the moving frame, $z$ the coordinate along the propagation direction, and $t$ the reduced time. All coordinates are made dimensionless by the choice of coefficients. The function $\beta(z)$ represents the dispersion coefficient and $\gamma(z)$ the gain ($\gamma > 0$) or loss ($\gamma < 0$) coefficient. The functions $g_m(z)$ for $m = 1,2,\ldots,n$ stand for the nonlinearities of orders up to $2n+1$. For $m = 1$, one has the simple Kerr nonlinearity, for $m = 2$ the quintic, for $m = 3$ the septic, and so on. For Kerr nonlinearity, many authors obtained soliton and similariton solutions [8 – 10, 20, 21]. For quintic nonlinearity, Senthilnathan et al. [22] discussed a self-similar Townes soliton. In [18], authors obtained travelling wave and soliton solutions of (1) by means of a direct method. The motivation of this paper is to look for exact similariton solutions of the generalized NLSE with power-law nonlinearity via the similarity transformation method.

The NLS-type equation with variable coefficients has been extensively investigated [20 – 28] as the governing equation for optical soliton control, which is an important development in the application of solitons after the first soliton dispersion management experiment in a fiber with hyperbolically decreasing group velocity dispersion was realized by Dianov's group at the General Physics Institute [29]. Then controlling optical solitons in soliton communication systems and generating soliton train have been effectively realized as early as in 1991 [30]. Therefore, the concept of control of soliton propagation described by the NLSE (1) with power-law nonlinearity is a new and important development in the application of solitons for optical communication systems. However, to our knowledge, the control of soliton and similariton for NLSE (1) in dispersion decreasing fiber (DDF) is hardly studied, which is discussed in this paper.

## 2. Soliton and Similariton Solutions

In order to construct the relation between the variable-coefficient NLSE (1) with power-law nonlinearity and the constant-coefficient one,

$$\mathrm{i}U_Z + \frac{1}{2}BU_{TT} + G_m|U|^{2m}U = 0\,, \tag{2}$$

we construct the similarity transformation

$$u(z,t) = \rho(z)U[T(z,t),Z(z)]\exp[\mathrm{i}\phi(z,t)]\,, \tag{3}$$

where $B$ and $G_m$ are both real constants, describing dispersion and nonlinearity, respectively. Moreover, the amplitude $\rho(z)$ and the phase $\phi(z,t)$ satisfy

$$\begin{aligned}\rho &= \rho_0 C_\mathrm{p}^{1/2}\exp[\Gamma(z)]\,,\\ \phi &= -\frac{c_0 C_\mathrm{p}(z)}{2}t^2 - l_0 C_\mathrm{p}(z)t - \frac{l_0^2}{2}C_\mathrm{p}(z)D(z)\,,\end{aligned} \tag{4}$$

and the effective propagation distance $Z(z)$ and similarity variable $T(z,t)$ are also real functions in the form

$$Z = \frac{C_\mathrm{p}(z)D(z)}{BW_0^2}\,,\quad T = \frac{t - t_c(z)}{W(z)}\,, \tag{5}$$

with the width of pulse $W(z) = W_0/C_\mathrm{p}(z)$, position of pulse $t_c(z) = t_0 - (l_0 + c_0 t_0)D(z)$, the accumulated dispersion and gain/loss $D(z) = \int_0^z \beta(s)\,\mathrm{d}s$ and $\Gamma(z) = \int_0^z \gamma(s)\,\mathrm{d}s$. Here $C_\mathrm{p}(z) = [1 - c_0 D(z)]^{-1}$ is related to the wave front curvature and presents a measure of the phase chirp imposed on the wave. The subscript 0 denotes the initial values of the given functions at distance $z = 0$. Note that the accumulated dispersion $D(z)$ influences the form of the amplitude, the width, the phase, the chirp, and the effective propagation distance. Here free parameters have very clear physical meaning: $c_0$ and $l_0$ are the initial curvature and position of the wavefront, $\rho_0$ denotes the initial amplitude of the pulse center, $W_0$ is related to the initial similariton width. Also note that from (3) with (4) and (5), soliton solutions can be obtained by setting $c_0 = 0$. Similaritons have as essential feature the linear chirp ($c_0 \neq 0$), which leads to an efficient compression or amplification and thus are particularly useful in the design of optical fiber amplifiers and optical pulse compressors [31].

Especially, if the parameters of dispersion, nonlinearity, and gain/loss satisfy the constraints

$$\gamma(z) = \frac{1}{2m}\left[\frac{W\{g_m(z),\beta(z)\}}{g_m(z)\beta(z)} + \frac{(m-2)c_0\beta(z)}{1-c_0D(z)}\right], \quad m = 1,2,\ldots,n, \tag{6}$$

with the notation for the Wronskian $W\{g_m(z),\beta(z)\} = g_m\beta_z - \beta g_{m,z}$ and by solving easier constant-coefficient NLSE (2), we can obtain abundant solutions of (1) via the one-to-one correspondence (3) with (4) and (5).

Equation (6) can be conveniently understood as the integrability conditions on (1). This condition indicates that the connection of the coefficient $\beta(z)$, nonlinear coefficients $g_m(z)$, and gain/loss coefficient $\gamma(z)$ strongly affects the form and the behaviour of solitons and similaritons. Hence, the solution found can exist only under certain conditions and the system parameter functions cannot be all chosen independently. For example, if $\beta(z)$ and $g_m(z)$ are chosen to be free parameters, then $\gamma(z)$ will be determined from (6). Note that the condition (6) includes many special cases in [22, 32]. For $m = 1$, to obtain exact solutions in a lossy medium, the nonlinearity coefficient must grow exponentially. This condition is the one expressed in [32, (8)]. For $m = 2$, this condition includes the one expressed in [22, (5)]. Thus we can choose the equation parameters suitably to investigate the dynamic behaviours for solutions of (1). This choice of real system can be found as systems (8) and (10) in Section 3.

For Kerr nonlinearity ($m = 1$), the reduced equation (3) is the standard NLSE. From the one-to-one correspondence (3) with $c_0 = 0$ in (4) and (5), and based on the corresponding bright and dark soliton solutions of the standard NLSE, we can obtain bright and dark soliton solutions expressed in [32, (11) and (12)] of (1) with $m = 1$. When $c_0 \neq 0$, our result (3) with (4) and (5) here is the corresponding solution expressed in [33, (2) with (6)–(9)]. For quintic nonlinearity ($m = 2$), the reduced equation (3) is the constant-coefficient quintic NLSE, from whose corresponding bright soliton solution and transformation (3), we can obtain the chirped Townes soliton expressed in [22, (28)].

Here we focus on soliton and similariton solutions of the general power-law nonlinearity. Employing the corresponding solutions of (2) and the transformation (3) with (4) and (5), and considering the constraint condition (6), we have bright soliton and similariton solutions of (1) as follows:

$$u = \frac{\rho_0}{[1-c_0D(z)]^{1/2}}\cosh^{-\frac{1}{m}}(\xi) \cdot \exp\left[\Gamma(z) - \mathrm{i}\Phi(z,t)\right], \tag{7}$$

where

$$\xi = \sqrt{\frac{2m^2G_m}{(m+1)B}}\left\{\frac{C_\mathrm{p}(z)\left[t+(l_0+c_0t_0)D(z)-t_0\right]}{W_0} - \frac{\kappa}{W_0^2}\cdot C_\mathrm{p}(z)D(z)\right\},$$

$$\Phi(z,t) = \frac{c_0C_\mathrm{p}(z)}{2}t^2 + \left(l_0 - \frac{\kappa}{W_0}\right)C_\mathrm{p}(z)t + \left\{\frac{l_0^2}{2} - \frac{l_0+c_0t_0}{W_0} - \frac{\kappa^2-2G_m/[(m+1)B]}{2W_0^2}\right\} \cdot C_\mathrm{p}(z)D(z) - \frac{\kappa t_0}{W_0}C_\mathrm{p}(z)$$

with constant $\kappa$. Here solution (7) with $c_0 = 0$ and $c_0 \neq 0$ represents soliton and similariton solutions, respectively.

## 3. Dynamical Behaviours

In this section, we will analyze the mechanism on how the relevant properties of soliton and similariton solutions are affected in two different soliton control systems. Firstly, let us consider the compression problem of the laser pulse in a DDF with the dispersion and nonlinearity parameter [32] according to

$$\beta(z) = \beta_0\exp(-\sigma z), \quad g_m = g_{m0}\exp(\alpha z), \tag{8}$$

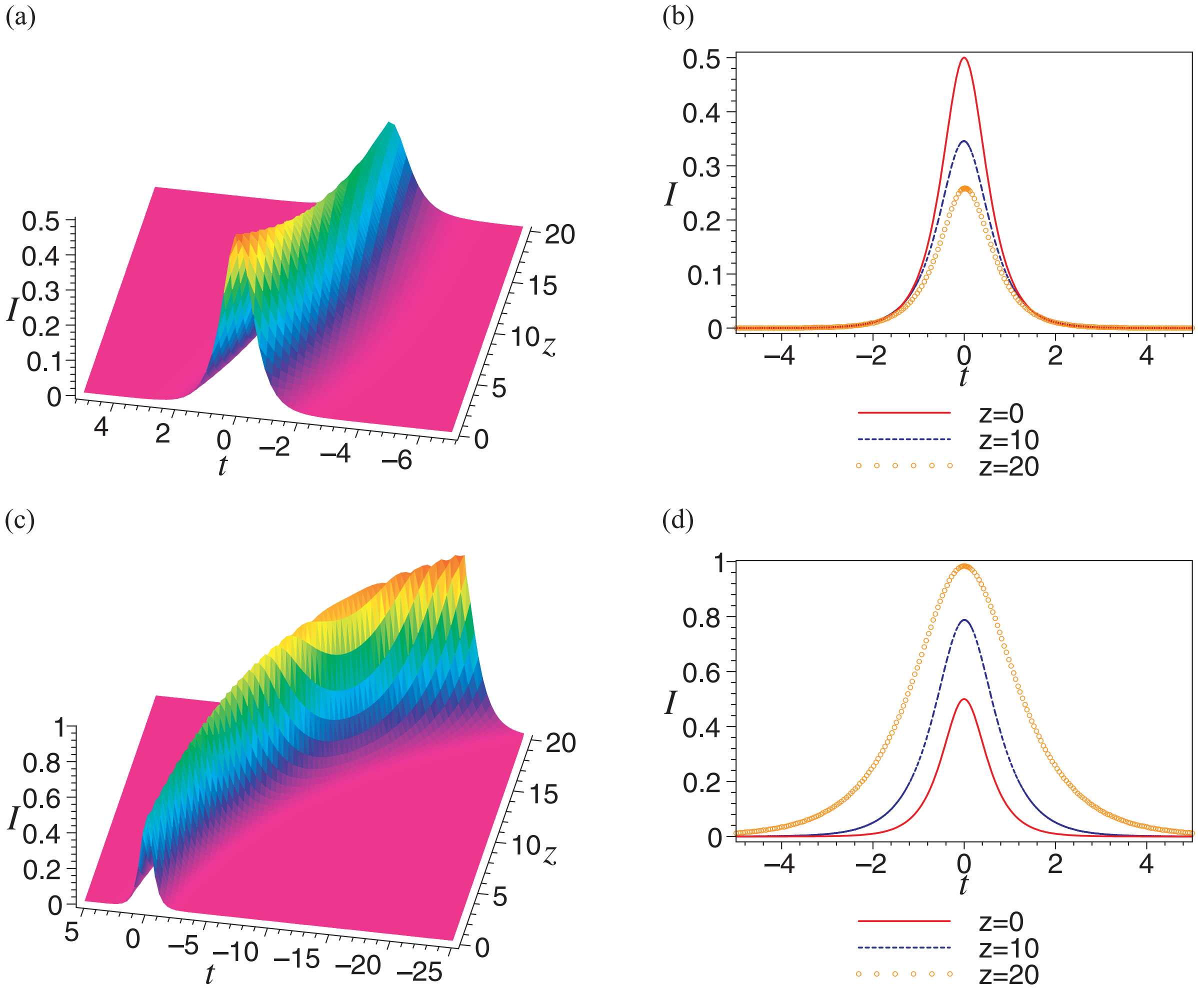


Fig. 1 (colour online). Chirped bright similariton expressed by (7). The parameters are taken as $m = 2$, $B = 2$, $G_2 = g_{20} = 1$, $\beta_0 = W_0 = 0.5, l_0 = 0.1$, $\alpha = -0.05$, $\kappa = -0.3, c_0 = -0.04$ with (a) $\sigma = 0.1$ and (c) $\sigma = -0.1$. (b) and (d): sectional view of (a) and (c) at distances $z = 0, 10, 20$, respectively.

where $\beta_0 > 0$, $g_{m0} > 0$, and $\sigma \neq 0$. In optical system (8), (6) yields the distributed gain

$$\gamma(z) = -\frac{1}{2m}\left\{\sigma + \alpha - \frac{\sigma(m-2)c_0\beta_0 \exp(-\sigma z)}{\sigma + c_0\beta_0[\exp(-\sigma z) - 1]}\right\}. \quad (9)$$

It is clear to see that for the chirped similariton ($c_0 \neq 0$) if the nonlinearity parameter $g_m(z)$ (i. e. $\alpha = 0$) is constant and $m = 3$, $c_0 = \sigma/\beta_0$, or if $m = 2$ and $\sigma = -\alpha$, the gain can be completely balanced out. For chirp-free self-similar waves ($c_0 = 0$), if $\sigma = -\alpha$, then the gain can be also completely balanced out.

The coaction of nonlinearity, dispersion, and gain causes the shape of input pulse to converge asymptotically to a pulse whose shape is self-similar. Figure 1 displays the dynamical behaviours for the intensity $I = |u|^2$ of chirped bright similaritons for quintic media ($m = 2$) in the dispersion decreasing and increasing fibers (8). Note that in the dispersion decreasing fiber, the amplitude of the chirped bright similariton decreases, while it increases in the dispersion increasing fiber. The compression and stretching of similaritons are shown in Figures 1b and d by modulating parameter $\sigma$. Here, for the convenience of comparison, we shifted the center of the similariton to the initial location. Moreover, the amplitude of the bright similariton in the dispersion decreasing fiber decreases

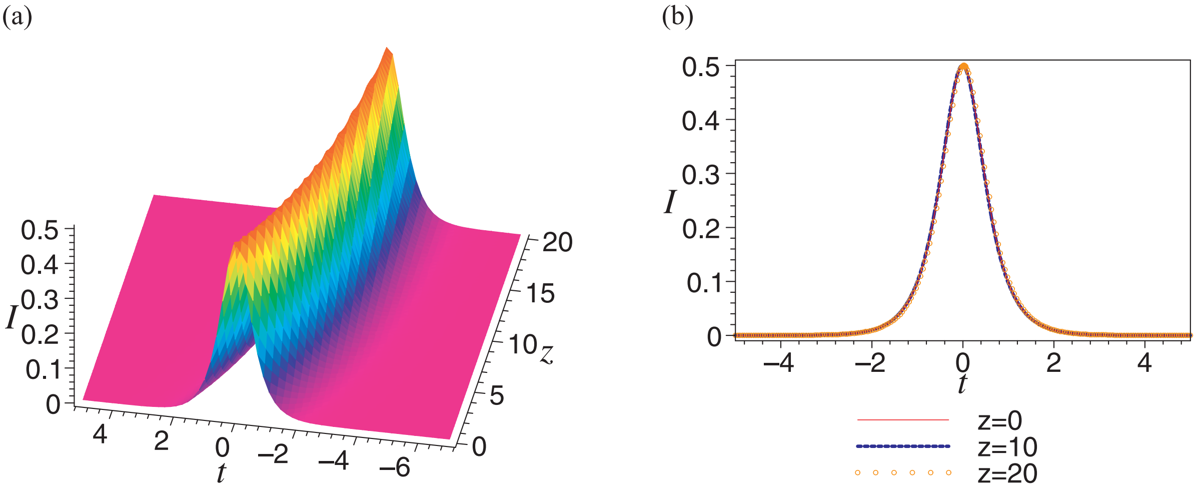


Fig. 2 (colour online). Chirp-free bright soliton with $c_0 = 0$ corresponding to Figure 1a. The parameters are chosen as that in Figure 1a except for $\sigma = -\alpha = 0.1$.

more dramatically than the counterpart in the dispersion increasing fiber. These results obtained in this paper may have potential values for all-optical data-processing schemes and the design of pulse compressors and amplifiers. For the chirp-free bright soliton, due to $c_0 = 0$, the amplitude and width are unchanged from the analytical result (7) and graphically description in Figure 2.

We find that different cases appear in higher-order power-law nonlinearity. When $m \le 3$, the amplitude and width of the bright similariton in the dispersion decreasing fiber both decrease along the propagation distance $z$, yet when $m > 3$, the cases are more complicated. From Figure 3, at first, the amplitude and width of the bright similariton in the dispersion decreasing fiber increase, and then the amplitude and width of it gradually decrease along the propagation distance $z$. Moreover, compared with Figure 3a and b, this variation is gradually slow with the increasing of the index of the power-law nonlinearity from $m = 4$ to $m = 5$.

Next, we take the following periodic distributed system with the varying group velocity dispersion and

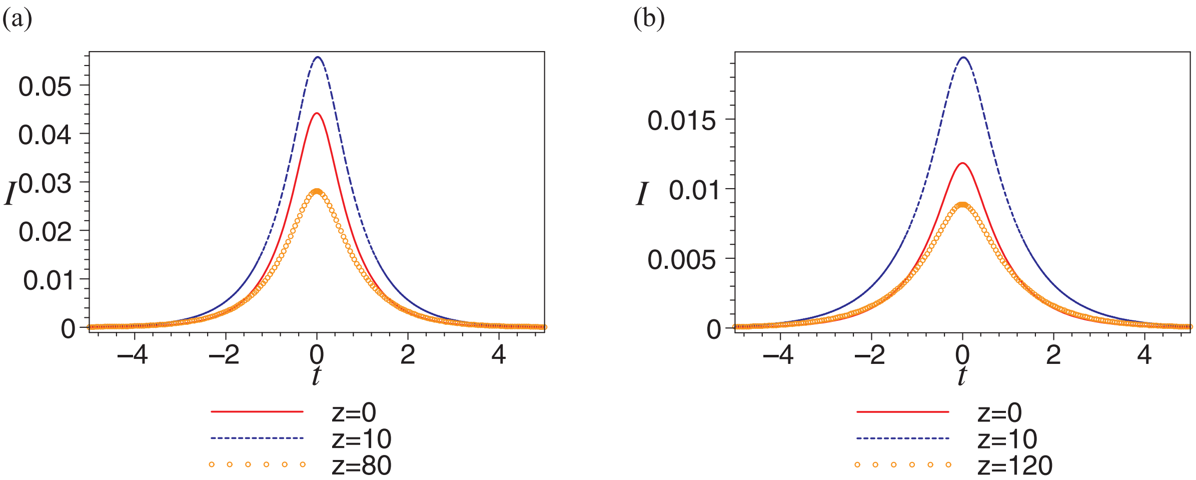


Fig. 3 (colour online). Chirped bright similariton expressed by (7) with (a) $m = 4$ and (b) $m = 5$ in the DDF. The parameters are chosen as that in Figure 1a.

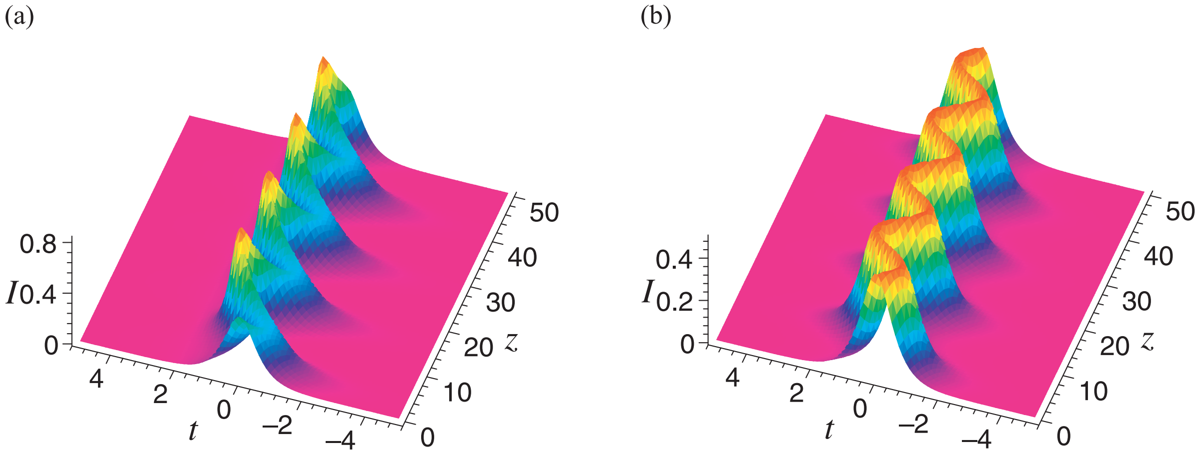


Fig. 4 (colour online). (a) Chirped bright similariton with $c_0 = -0.4$ and (b) chirp-free bright soliton with $c_0 = 0$ expressed by (7) in a periodic distributed system. The parameters are chosen as that in Figure 1a except for $\delta_1 = \delta_2 = 0.5$.

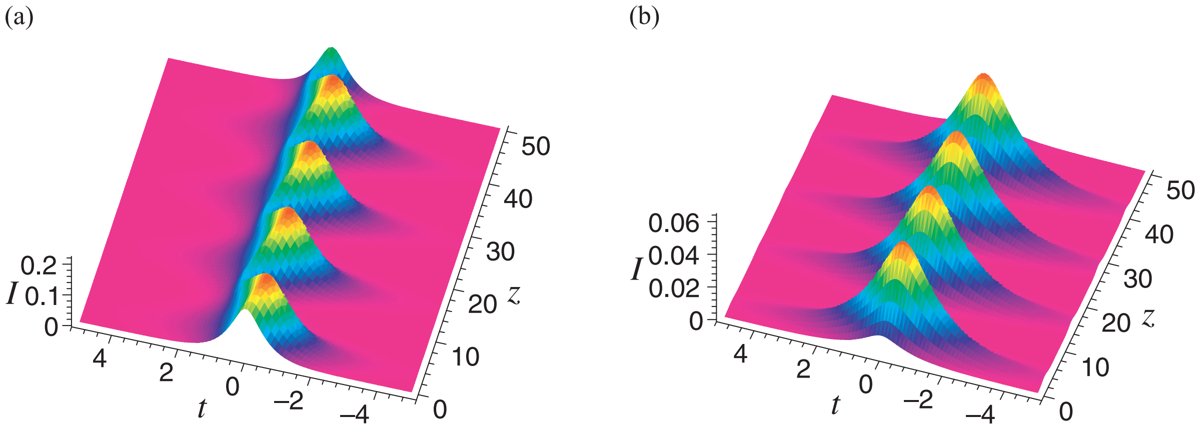


Fig. 5 (colour online). Chirped bright similariton expressed by (6) with (a) $m = 3$ and (b) $m = 5$ in a periodic distributed system. The parameters are chosen as that in Figure 4a.

nonlinear coefficient [24, 34]:

$$\beta(z) = \beta_0 \cos(\delta_1 z)\,, \;\; g_m(z) = g_{m0} \cos(\delta_2 z)\,, \quad (10)$$

where $\beta_0$ and $\delta_1$ describe the dispersion, $g_{m0}$ and $\delta_2$ are the parameters describing the power-law nonlinearity.

In this system, the bright similariton exhibits a snake-like propagation behaviour. From Figure 4a, amplitude and width of the chirped bright similariton vary periodically, while amplitude and width of the chirp-free bright soliton remain with certain values [c. f. Figure 4b]. From Figure 4a with $m = 2$, Figure 5 a with $m = 3$ to Figure 5b with $m = 5$, this snake-like propagation behaviour disappears little by little with the increasing of the index of power-law nonlinearity. When $m = 5$, separated humps appear in Figure 5b. Moreover, the period of waves gradually decreases with the increasing power-law nonlinearity from $m = 2$ to $m = 5$.

## 4. Conclusions

In conclusion, we have derived analytical self-similar solutions for the generalized nonlinear Schrödinger equation with power-law nonlinearity. Based on these solutions, we discussed the dynamical behaviours of the chirped similariton and the chirp-free soliton in the DDF and the periodic distributed system. In the first soliton control system, the pulse is compressed for parameter $\sigma > 0$, while the pulse is stretched for parameter $\sigma < 0$. That is to say, we can control the compression and stretching by modulating the parameter $\sigma$. In the second soliton control system, the snake-like propagation behaviour disappears little by little and the period of waves gradually decreases with the increasing power-law nonlinearity. Compared with the chirped similaritons, the chirp-free solitons remain with the certain amplitude and width in the two systems. These analytical findings here can be expected to assist in areas such as optical fiber compressors, optical fiber amplifiers, nonlinear optical switches, optical communications, and long-haul telecommunication networks for achieving pulse compression. Of course, due to the lack of an experimental and designed basis related to these theoretical results, we could not give further details about the real physical application. More practical implementation of this theoretical method to other important models, such as the Ginzburg–Landau equation arising in Bose–Einstein condensates and some optical materials, might be an interesting task.

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