# Triple Spin Interaction and Entanglement

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Z. Naturforsch. **68a,** 172 – 177 (2013) / DOI: 10.5560/ZNA.2012-0087
Received September 20, 2012 / published online February 15, 2013

We study a Hamilton operator $\hat{H}$ for spin-1/2 with triple spin interactions. The eigenvalues and eigenvectors are determined and the unitary matrices $\exp(-\mathrm{i}\hat{H}t/\hbar)$ are found. Entanglement of the eigenvectors is investigated. A Hamilton operator $\hat{K}$ for spin-1 and triple spin interaction is also discussed.

*Key words:* Spin Interactions; Hamilton Operators; Entanglement.

## 1. Introduction

In quantum theory Hamilton operators with spin-interactions have a long history [1 – 5]. Triple spin interaction have been studied by several authors [6 – 19]. Iglói [6] investigated an Ising model with three-spin interaction by finite-size scaling and applying free boundary conditions. Vanderzande and Iglói [7] studied the critical behaviour and logarithmic corrections of a quantum model with three-spin interaction. Alcaraz and Barber [8] and Wittlich [9] studied a one-dimensional Ising quantum chain with $3N$ sites with staggered three-spin coupling and periodic boundary conditions. Somma et al. [10] studied the unitary operator $U(t) = \exp(\mathrm{i}\omega t \sigma_1 \otimes \sigma_3 \otimes \sigma_2)$. Here $\sigma_1$, $\sigma_2$, $\sigma_3$ are the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Pachos and Plenio [11] studied three-spin interactions in optical lattices. Three qubit Hamilton operators and Riemannian geometry has been discussed by Brandt [12]. Using the mean field method Jiang and Kong [13] studied a spin-1 quantum Ising model with three-spin interaction. Wang et al. [14] investigated the bifurcation in ground-state fidelity for a one-dimensional spin model with competing two-spin and three-spin interactions. Lanyon et al. [15] considered among others the triple-spin operator $\sigma_3 \otimes \sigma_1 \otimes \sigma_1$ for universal digital quantum simulation with trapped ions. Topilko et al. [16] considered magnetocaloric effects in spin-1/2 XX chains with three-spin interactions. Zhang et al. [17] investigated the geometric phase of a qubit symmetrically coupled to a XY-spin chain with three spin interaction in a transverse magnetic field. The Greenberger–Horne–Zeilinger (GHZ) state and triple-spin operators $\sigma_1 \otimes \sigma_2 \otimes \sigma_2$, $\sigma_2 \otimes \sigma_1 \otimes \sigma_2$, $\sigma_2 \otimes \sigma_2 \otimes \sigma_1$, $\sigma_1 \otimes \sigma_1 \otimes \sigma_1$ have been discussed by Aravind [18] in connection with Bell's theorem without inequalities. A nonlinear eigenvalue problem with triple-spin interaction has been solved by Steeb and Hardy [19]. Here we consider triple-spin interaction and entanglement for spin-1/2 systems. Spin-1 systems will also be discussed.

We consider the Hamilton operator with triple-spin interaction of spin-1/2,

$$\begin{aligned}\hat{H} = {} & \hbar\omega_1(\sigma_1 \otimes I_2 \otimes I_2 + I_2 \otimes \sigma_2 \otimes I_2 + I_2 \otimes I_2 \otimes \sigma_3) \\ & + \hbar\omega_2(\sigma_1 \otimes \sigma_2 \otimes I_2 + \sigma_1 \otimes I_2 \otimes \sigma_3 + I_2 \otimes \sigma_2 \otimes \sigma_3) \\ & + \hbar\omega_3(\sigma_1 \otimes \sigma_2 \otimes \sigma_3),\end{aligned}$$

where $\sigma_1$, $\sigma_2$, $\sigma_3$ are the Pauli spin matrices, and $I_2$ is the $2 \times 2$ unit matrix. Thus the Hamilton operator $\hat{H}$ acts in the Hilbert space $\mathbb{C}^8$. The Pauli matrices are hermitian and unitary with $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I_2$. The eigenvalues are $+1$ and $-1$ with the corresponding normalized eigenvectors

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{u}_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$
$$\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}, \quad \mathbf{v}_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix},$$

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{w}_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We calculate the unitary matrix $U(t) = \exp(-\mathrm{i}\hat{H}t/\hbar)$ to solve the Schrödinger and Heisenberg equation of motion. These unitary operators are applied to entangled and unentangled states. Entangled and unentangled states can be found depending on the parameters $\hbar\omega_1$, $\hbar\omega_2$, $\hbar\omega_3$. As entanglement measure for the Hamilton operator $\hat{H}$, we consider the three-tangle.

## 2. A Theorem

For the calculation of the eigenvalues and eigenvectors of the Hamilton operator $\hat{H}$, we utilize the following theorem [20, 21]. Let $A_1, A_2, A_3$ be $n \times n$ matrices over $\mathbb{C}$ and $I_n$ be the $n \times n$ unit matrix. Consider the matrix

$$\begin{aligned} M = {} & c_1(A_1 \otimes I_n \otimes I_n + I_n \otimes A_2 \otimes I_n + I_n \otimes I_n \otimes A_3) \\ & + c_2(A_1 \otimes A_2 \otimes I_n + A_1 \otimes I_n \otimes A_3 + I_n \otimes A_2 \otimes A_3) \\ & + c_3(A_1 \otimes A_2 \otimes A_3), \end{aligned}$$

where $c_1$, $c_2$, $c_3$ are constants. Since the terms in $M$ commutate pairwise, we can write $\exp(M)$ as

$$\begin{aligned} \mathrm{e}^M = {} & \mathrm{e}^{c_1 A_1 \otimes I_n \otimes I_n} \mathrm{e}^{c_1 I_n \otimes A_2 \otimes I_n} \mathrm{e}^{c_1 I_n \otimes I_n \otimes A_3} \mathrm{e}^{c_2 A_1 \otimes A_2 \otimes I_n} \\ & \cdot \mathrm{e}^{c_2 A_1 \otimes I_n \otimes A_3} \mathrm{e}^{c_2 I_n \otimes A_2 \otimes A_3} \mathrm{e}^{c_3 (A_1 \otimes A_2 \otimes A_3)}. \end{aligned}$$

If $|\mathbf{u}\rangle$, $|\mathbf{v}\rangle$, $|\mathbf{w}\rangle$ are eigenvectors of $A_1$, $A_2$, $A_3$, respectively, with eigenvalues $\lambda$, $\mu$, $\nu$, we find the eigenvector $|\mathbf{u}\rangle \otimes |\mathbf{v}\rangle \otimes |\mathbf{w}\rangle$ of $M$ with the eigenvalue

$$c_1(\lambda + \mu + \nu) + c_2(\lambda\mu + \lambda\nu + \mu\nu) + c_3(\lambda\mu\nu).$$

Then $|\mathbf{u}\rangle \otimes |\mathbf{v}\rangle \otimes |\mathbf{w}\rangle$ is also an eigenvector of $\mathrm{e}^M$ with the corresponding eigenvalues

$$c_1(\lambda + \mu + \nu) + c_2(\lambda\mu + \lambda\nu + \mu\nu) + c_3(\lambda\mu\nu).$$

If the matrices $A_1, A_2, A_3$ have the additional properties that $A_1^2 = A_2^2 = A_3^2 = I_n$ (spin-$\frac{1}{2}$ case with $n = 2$), we obtain

$$\begin{aligned} \mathrm{e}^{c_3 A_1 \otimes A_2 \otimes A_3} = {} & (I_n \otimes I_n \otimes I_n) \cosh(c_3) \\ & + (A_1 \otimes A_2 \otimes A_3) \cosh(c_3). \end{aligned}$$

If the matrices $A_1, A_2, A_3$ have the additional properties that $A_j^3 = A_j$ with $j = 1,2,3$ (spin-1 case with $n = 3$), we obtain

$$\begin{aligned} \mathrm{e}^{c_3 A_1 \otimes A_2 \otimes A_3} = {} & I_n \otimes I_n \otimes I_n + (A_1 \otimes A_2 \otimes A_3) \sinh(c_3) \\ & + (A_1^2 \otimes A_2^2 \otimes A_3^2)(\cosh(c_3) - 1). \end{aligned}$$

## 3. Spin-1/2 Case

The eight normalized eigenvectors of $\hat{H}$ can be constructed from the normalized eigenvectors of $\sigma_1$, $\sigma_2$, $\sigma_3$ and the Kronecker products

$$\begin{aligned} \mathbf{e}_{111} &= \mathbf{u}_1 \otimes \mathbf{v}_1 \otimes \mathbf{w}_1, \\ \mathbf{e}_{11-1} &= \mathbf{u}_1 \otimes \mathbf{v}_1 \otimes \mathbf{w}_{-1}, \\ \mathbf{e}_{1-11} &= \mathbf{u}_1 \otimes \mathbf{v}_{-1} \otimes \mathbf{w}_1, \\ \mathbf{e}_{1-1-1} &= \mathbf{u}_1 \otimes \mathbf{v}_2 \otimes \mathbf{w}_2, \\ \mathbf{e}_{-111} &= \mathbf{u}_{-1} \otimes \mathbf{v}_1 \otimes \mathbf{w}_1, \\ \mathbf{e}_{-11-1} &= \mathbf{u}_{-1} \otimes \mathbf{v}_1 \otimes \mathbf{w}_{-1}, \\ \mathbf{e}_{-1-11} &= \mathbf{u}_{-1} \otimes \mathbf{v}_{-1} \otimes \mathbf{w}_1, \\ \mathbf{e}_{-1-1-1} &= \mathbf{u}_{-1} \otimes \mathbf{v}_{-1} \otimes \mathbf{w}_{-1} \end{aligned}$$

with the corresponding eight eigenvalues

$$\begin{aligned} E_{111} &= \hbar(3\omega_1 + 3\omega_2 + \omega_3), \\ E_{11-1} &= \hbar(\omega_1 - \omega_2 - \omega_3), \\ E_{1-11} &= \hbar(\omega_1 - \omega_2 - \omega_3), \\ E_{1-1-1} &= \hbar(-\omega_1 - \omega_2 + \omega_3), \\ E_{-111} &= \hbar(\omega_1 - \omega_2 - \omega_3), \\ E_{-11-1} &= \hbar(-\omega_1 - \omega_2 + \omega_3), \\ E_{-1-11} &= \hbar(-\omega_1 - \omega_2 + \omega_3), \\ E_{-1-1-1} &= \hbar(-3\omega_1 + 3\omega_2 - \omega_3), \end{aligned}$$

where $E_{11-1} = E_{1-11} = E_{-111}$ and $E_{1-1-1} = E_{-11-1} = E_{-1-11}$. The eigenvalues $E_{111}$ and $E_{-1-1-1}$ are not degenerate. Note that the eight normalized eigenvectors are pairwise orthogonal. Thus we have (spectral decomposition)

$$\hat{H} = \sum_{j,k,\ell \in \{1,-1\}} E_{jk\ell} \mathbf{e}_{jk\ell} \mathbf{e}_{jk\ell}^*.$$

Now the unitary matrix $U(t) = \mathrm{e}^{-\mathrm{i}t\hat{H}/\hbar}$ can be constructed from the normalized eigenvectors and eigenvalues of $\hat{H}$ via

$$\mathrm{e}^{-\mathrm{i}t\hat{H}/\hbar} = \sum_{j,k,\ell \in \{1,-1\}} \mathrm{e}^{-\mathrm{i}E_{jk\ell}t/\hbar} \mathbf{e}_{jk\ell} \mathbf{e}_{jk\ell}^*.$$

We find

$$U(t)=\begin{pmatrix} u_{11}(t) & 0 & u_{13}(t) & 0 & u_{15}(t) & 0 & u_{17}(t) & 0 \\ 0 & u_{22}(t) & 0 & u_{24}(t) & 0 & u_{26}(t) & 0 & u_{28}(t) \\ u_{31}(t) & 0 & u_{33}(t) & 0 & u_{35}(t) & 0 & u_{37}(t) & 0 \\ 0 & u_{42}(t) & 0 & u_{44}(t) & 0 & u_{46}(t) & 0 & u_{48}(t) \\ u_{51}(t) & 0 & u_{53}(t) & 0 & u_{55}(t) & 0 & u_{57}(t) & 0 \\ 0 & u_{62}(t) & 0 & u_{64}(t) & 0 & u_{66}(t) & 0 & u_{66}(t) \\ u_{71}(t) & 0 & u_{73}(t) & 0 & u_{75}(t) & 0 & u_{77}(t) & 0 \\ 0 & u_{82}(t) & 0 & u_{84}(t) & 0 & u_{86}(t) & 0 & u_{88}(t) \end{pmatrix}$$

with

$$\begin{aligned}
u_{11}(t) &= u_{33}(t)=u_{55}(t)=u_{77}(t) \\
&= \mathrm{e}^{-\mathrm{i}E_1t/\hbar}/4+\mathrm{e}^{-\mathrm{i}E_2t/\hbar}/2+\mathrm{e}^{-\mathrm{i}E_4t/\hbar}/4, \\
u_{13}(t) &= -u_{31}(t)=-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}E_1t/\hbar}/4+\mathrm{i}\,\mathrm{e}^{-\mathrm{i}E_4t/\hbar}/4, \\
u_{15}(t) &= u_{51}(t)=\mathrm{e}^{-\mathrm{i}E_1t/\hbar}/4-\mathrm{e}^{-\mathrm{i}E_4t/\hbar}/4, \\
u_{17}(t) &= -u_{71}(t) \\
&= -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}E_1t/\hbar}/4-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}E_4t/\hbar}/4+\mathrm{i}\,\mathrm{e}^{-\mathrm{i}E_2t/\hbar}/2, \\
u_{22}(t) &= u_{44}(t)=u_{66}(t)=u_{88}(t) \\
&= \mathrm{e}^{-\mathrm{i}E_2t/\hbar}/4+\mathrm{e}^{-\mathrm{i}E_4t/\hbar}/2+\mathrm{e}^{-\mathrm{i}E_8t/\hbar}/4, \\
u_{24}(t) &= -u_{42}(t)=-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}E_2t/\hbar}/4+\mathrm{i}\,\mathrm{e}^{-\mathrm{i}E_8t/\hbar}/4, \\
u_{26}(t) &= u_{62}=\mathrm{e}^{-\mathrm{i}E_2t/\hbar}/4-\mathrm{e}^{-\mathrm{i}E_8t/\hbar}/4, \\
u_{28}(t) &= -u_{82} \\
&= -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}E_2t/\hbar}/4+\mathrm{i}\,\mathrm{e}^{-\mathrm{i}E_4t/\hbar}/2-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}E_8t/\hbar}/4, \\
u_{35}(t) &= -u_{53} \\
&= \mathrm{i}\,\mathrm{e}^{-\mathrm{i}E_1t/\hbar}4-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}E_2t/\hbar}/2+\mathrm{i}\,\mathrm{e}^{-\mathrm{i}E_4t/\hbar}/4, \\
u_{37}(t) &= u_{73}=\mathrm{e}^{-\mathrm{i}E_1t/\hbar}/4-\mathrm{e}^{-\mathrm{i}E_4t/\hbar}/4, \\
u_{46}(t) &= -u_{64} \\
&= \mathrm{i}\,\mathrm{e}^{-\mathrm{i}E_2t/\hbar}/4-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}E_4t/\hbar}/2+\mathrm{i}\,\mathrm{e}^{-\mathrm{i}E_8t/\hbar}, \\
u_{48}(t) &= u_{84}=\mathrm{e}^{-\mathrm{i}E_2t/\hbar}/4-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}E_8t/\hbar}/4, \\
u_{57}(t) &= -u_{75}=-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}E_1t/\hbar}/4+\mathrm{i}\,\mathrm{e}^{-\mathrm{i}E_4t/\hbar}/4, \\
u_{68}(t) &= -u_{86}=-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}E_2t/\hbar}/4+\mathrm{i}\,\mathrm{e}^{-\mathrm{i}E_8t/\hbar}/4
\end{aligned}$$

with $1\leftrightarrow 111$, $2\leftrightarrow 11-1$, $3\leftrightarrow 1-11$, $4\leftrightarrow 1-1-1$, $5\leftrightarrow -111$, $6\leftrightarrow -11-1$, $7\leftrightarrow -1-11$, $8\leftrightarrow -1-1-1$.

## 4. Entanglement

Entangled quantum states are an important component of quantum computing techniques such as quantum error-correction, dense coding, and quantum teleportation [21–27]. Entanglement is the characteristic trait of quantum mechanics which enforces its entire departure from classical lines of thought. We consider entanglement of pure states. Entanglement of Hamilton operators with triple-spin interaction has not been considered so far.

The eigenvectors given above are product states and hence not entangled. However, some of the eigenvalues are degenerate ($E_2=E_3=E_5$ and $E_4=E_6=E_7$) and thus we can form linear combinations of these eigenvectors which are again eigenvectors and can be entangled.

An $N$-tangle can be defined for the finite dimensional Hilbert space $\mathcal{H}=\mathbb{C}^{2^N}$, with $N=3$ or $N$ even. Two-level and higher-level quantum systems and their physical realization have been studied by many authors. We consider the finite-dimensional Hilbert space $\mathcal{H}=\mathbb{C}^{2^N}$ and the normalized states

$$|\psi\rangle=\sum_{j_1,j_2,\dots,j_N=0}^{1} c_{j_1,j_2,\dots,j_N}|j_1\rangle\otimes|j_2\rangle\otimes\cdots\otimes|j_N\rangle$$

in this Hilbert space. Here $|0\rangle$, $|1\rangle$ denotes the standard basis. Let $\varepsilon_{jk}$ $(j,k=0,1)$ be defined by $\varepsilon_{00}=\varepsilon_{11}=0$, $\varepsilon_{01}=1$, $\varepsilon_{10}=-1$. Let $N$ be even or $N=3$. Then an $N$-tangle can be introduced by [27]

$$\tau_{1\dots N}=2\left|\sum_{\substack{\alpha_1,\dots,\alpha_N=0\\ \dots\\ \delta_1,\dots,\delta_N=0}}^{1} c_{\alpha_1\dots\alpha_N}c_{\beta_1\dots\beta_N}c_{\gamma_1\dots\gamma_n}c_{\delta_1\dots\delta_N}\cdot\varepsilon_{\alpha_1\beta_1}\varepsilon_{\alpha_2\beta_2}\cdots\varepsilon_{\alpha_{N-1}\beta_{N-1}}\varepsilon_{\gamma_1\delta_1}\varepsilon_{\gamma_2\delta_2}\cdots\varepsilon_{\gamma_{N-1}\delta_{N-1}}\varepsilon_{\alpha_N\gamma_N}\varepsilon_{\beta_N\delta_N}\right|.$$

This includes the definition for the 3-tangle with $N=3$. A computer algebra program to find the $N$-tangle is given by Steeb and Hardy [28].

Consider $N = 3$ and the GHZ-state and the $W$-state

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\1\end{pmatrix}, \qquad |W\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix}0\\1\\1\\0\\1\\0\\0\\0\end{pmatrix}.$$

Then we find for the GHZ-state that $\tau = 1$ and $\tau = 0$ for the $W$-state. Note that the $|W\rangle$ state is not separable.

Consider the triple-spin interaction Hamilton operator

$$\frac{\hat{H}_T}{\hbar\omega} = \sigma_1 \otimes \sigma_2 \otimes \sigma_3 .$$

The eigenvalues are $+1$ (fourfold) and $-1$ (fourfold). A set of normalized eigenvectors are the separable states (and thus not entangled) given above. Owing to the degeneracy of the eigenvalue $+1$ (and analogously for the eigenvalue $-1$), we can form linear combinations of these separable eigenstates that are fully entangled. From the four separable eigenstates with eigenvalue $+1$

$$\begin{pmatrix}1\\1\end{pmatrix} \otimes \begin{pmatrix}1\\1\end{pmatrix} \otimes \begin{pmatrix}1\\1\end{pmatrix} \begin{pmatrix}1\\1\end{pmatrix} \otimes \begin{pmatrix}1\\1\end{pmatrix} \otimes \begin{pmatrix}1\\1\end{pmatrix}$$
$$\begin{pmatrix}1\\1\end{pmatrix} \otimes \begin{pmatrix}1\\1\end{pmatrix} \otimes \begin{pmatrix}1\\1\end{pmatrix} \begin{pmatrix}1\\1\end{pmatrix} \otimes \begin{pmatrix}1\\1\end{pmatrix} \otimes \begin{pmatrix}1\\1\end{pmatrix},$$

we find by linear combinations the fully entangled eigenstates (using the three tangle described above) with eigenvalue $+1$

$$\frac{1}{2}\begin{pmatrix}1\\1\\0\\0\\0\\0\\\mathrm{i}\\-\mathrm{i}\end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix}0\\1\\\mathrm{i}\\0\\1\\0\\0\\-\mathrm{i}\end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix}0\\0\\\mathrm{i}\\-\mathrm{i}\\1\\1\\0\\0\end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix}0\\1\\\mathrm{i}\\0\\1\\0\\0\\\mathrm{i}\end{pmatrix}.$$

The four vectors are also linearly independent. Analogously we can make this construction for the eigenvalue $-1$ to find fully entangled states, i. e. the three tangle is $\tau = +1$.

## 5. Spin-1 Case

Consider the Hamilton operator with triple-spin interaction with spin-1 system with the Hamilton operator

$$\begin{aligned}\hat{K} = \hbar\omega_1 &(s_1 \otimes I_3 \otimes I_3 + I_3 \otimes s_2 \otimes I_3 + I_3 \otimes I_3 \otimes s_3)\\ &+ \hbar\omega_2 (s_1 \otimes s_2 \otimes I_3 + s_1 \otimes I_3 \otimes s_3 + I_3 \otimes s_2 \otimes s_3)\\ &+ \hbar\omega_3 (s_1 \otimes s_2 \otimes s_3),\end{aligned}$$

where $s_1$, $s_2$, $s_3$ are the trace-less and hermitian $3 \times 3$ matrices

$$s_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}, \quad s_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&-\mathrm{i}&0\\\mathrm{i}&0&-\mathrm{i}\\0&\mathrm{i}&0\end{pmatrix},$$
$$s_3 = \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix},$$

and $I_3$ is the $3 \times 3$ identity matrix. The Hamilton operator $\hat{K}$ acts in the Hilbert space $\mathbb{C}^{27}$. Here one can investigate whether the state in $\mathbb{C}^{27}$ can be written as a product state of two vectors in $\mathbb{C}^9$ and $\mathbb{C}^3$, $\mathbb{C}^3$ and $\mathbb{C}^9$, or $\mathbb{C}^3$, $\mathbb{C}^3$ and $\mathbb{C}^3$. Note that $s_j^3 = s_j$ for $j = 1,2,3$ and thus $s_j^4 = s_j^2$ for $j = 1,2,3$. The eigenvalues of the matrices $s_1$, $s_2$, $s_3$ are given by $+1$, $0$, $-1$. The normalized eigenvectors for $s_1$ are

$$\mathbf{u}_1 = \frac{1}{2}\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}, \quad \mathbf{u}_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix},$$
$$\mathbf{u}_{-1} = \frac{1}{2}\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}.$$

The normalized eigenvectors for $s_2$ are

$$\mathbf{v}_1 = \frac{1}{2}\begin{pmatrix}1\\\sqrt{2}\mathrm{i}\\-1\end{pmatrix}, \quad \mathbf{v}_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix},$$
$$\mathbf{v}_{-1} = \frac{1}{2}\begin{pmatrix}1\\-\sqrt{2}\mathrm{i}\\-1\end{pmatrix}.$$

The normalized eigenvectors for $s_3$ is the standard basis denoted by $\mathbf{w}_1$, $\mathbf{w}_0$, $\mathbf{w}_{-1}$. Thus the 27 normalized eigenvectors (separable states) are given by

$$\mathbf{e}_{jk\ell} = \mathbf{u}_j \otimes \mathbf{v}_k \otimes \mathbf{w}_\ell, \quad j,k,\ell = 1,0,-1$$

with the 27 eigenvalues

$$E_{jk\ell} = \hbar\omega_1(j+k+\ell) + \hbar\omega_2(jk+j\ell+k\ell) + \hbar\omega_3(jk\ell)\,.$$

Now for the unitary matrix $V(t) = \mathrm{e}^{-\mathrm{i}t\hat{K}/\hbar}$, we find

$$V(t) = \sum_{j,k,\ell\in\{1,0,-1\}} \mathrm{e}^{-\mathrm{i}E_{jk\ell}t/\hbar}\mathbf{e}_{jk\ell}\mathbf{e}^*_{jk\ell}\,.$$

Note that for $z \in \mathbb{C}$, we have

$$\mathrm{e}^{zs_1\otimes I_3\otimes I_3} = I_3\otimes I_3\otimes I_3 + (s_1\otimes I_3\otimes I_3)\sinh(z) + (s_1^2\otimes I_3\otimes I_3)(\cosh(z)-1)\,,$$

$$\mathrm{e}^{zs_1\otimes s_2\otimes I_3} = I_3\otimes I_3\otimes I_3 + (s_1\otimes s_2\otimes I_3)\sinh(z) + (s_1^2\otimes s_2^2\otimes I_3)(\cosh(z)-1)\,,$$

$$\mathrm{e}^{zs_1\otimes s_2\otimes s_3} = I_3\otimes I_3\otimes I_3 + (s_1\otimes s_2\otimes s_3)\sinh(z) + (s_1^2\otimes s_2^2\otimes s_3^2)(\cosh(z)-1)\,.$$

With $z = -\mathrm{i}\omega t$, we arrive at

$$\mathrm{e}^{-\mathrm{i}\omega t s_1\otimes s_2\otimes s_3} = I_3\otimes I_3\otimes I_3 - \mathrm{i}(s_1\otimes s_2\otimes s_3)\sin(\omega t) + (s_1^2\otimes s_2^2\otimes s_3^2)(\cos(\omega t)-1)\,.$$

For $\omega_1 = 0$, $\omega_2 = 0$, the eigenvalues are highly degenerate, and we can form eigenvectors which are entangled.

## 6. Conclusion

We have studied spin Hamilton operators with triple-spin interaction. If the eigenvalues are degenerate then by linear combinations, we can construct entangled states from unentangled states.

*Acknowledgement*

The author is supported by the National Research Foundation (NRF), South Africa. This work is based upon research supported by the National Research Foundation. Any opinion, findings and conclusions or recommendations expressed in this material are those of the author(s) and therefore the NRF do not accept any liability in regard thereto.

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