A Numerical Study of the Nonlinear Reaction-Diffusion Equation with Different Type of Absorbent Term by Homotopy Analysis Method

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In this paper, based on the homotopy analysis method (HAM), a new powerful algorithm is used for the solution of the nonlinear reaction-diffusion equation. The algorithm presents the procedure of constructing a set of base functions and gives the high-order deformation equation in a simple form. Different from all other analytic methods, it provides us with a simple way to adjust and control the convergence region of the solution series by introducing an auxiliary parameter h. The solutions of the problem of presence and absence of absorbent term and external force for different particular cases are presented graphically.

Key words: Homotopy Analysis Method; Nonlinear Reaction-Diffusion Equation; Partial Differential Equation; External Force; Reaction Term.
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1. Introduction

The nonlinear reaction diffusion equation as well as a system of nonlinear reaction-diffusion equations are very attractive in recent years, since they have practical applications in many fields of science and engineering. The general form of the governing equations involves temporal and spatial derivatives of the field plus a source/sink or forcing term. In underwater acoustics, for example, the source term explains the effects of wave refraction and spreading due to a discrepancy in the ocean refractive index. In the prediction of shoreline evolution this term stands for the supply or removal of sediment along the shoreline through cross-shore movement.

In the early eighties, Rose [1, 2] discussed a nonlinear advection-diffusion problem for flow in porous media, where he defined a continuous, piecewise linear finite element Galerkin method and derived rates of convergence based on assumed asymptotic rates of degeneracy. Magenes et al. [3] considered a class of problems including the Stephan problem in the porous medium. During the calculation, they rest the strict equality by using the asymptotically correct Chernoff formulation. In 1988, Nochetto and Verdi [4] developed a parallel degenerate parabolic equation and described a continuous, piecewise linear finite element Galerkin method and proved its convergence; in addition, they removed error estimates in measure for the free boundaries that emerge in the solutions. In this continuation, Barrett and Knabner [5] and Arbogastr et al. [6] considered the problem of solute transport. Here, they also defined a similar linear finite element Galerkin method, and the authors used a regularization of the problem to obtain their results. Pao [7] has solved a reaction-diffusion equation with nonlinear boundary conditions.

In 1996, Tang et al. [8] developed the model for El Niño events with a nonlinear convection-reaction-diffusion equation. In this model, the effects of convection, turbulent diffusion, linear feedback, and nonlinear radiation on the anomaly of sea surface temperature are considered. After some time, Wang [9] defined a new technique (monotone technique) for the diffusion equation with nonlinear diffusion coefficients and find the solution.

In 2010, Yıldırım and Pınar [10] discussed the application of the exp-function method for solving the nonlinear reaction diffusion equation which is arising in the mathematical biology. Mishra and Kumar [11] introduced the exact solutions for a variable coefficient nonlinear diffusion-reaction equation with a nonlinear convective term. In this model, the authors found the solitary wave equation and this type of equation is arisen in a variety of contexts in physical and biological problems. At the same time, Rida [12] found the approximate analytical solutions of generalized linear and nonlinear reaction-diffusion equations in an infinite domain. The solutions are obtained in compact and elegant forms in terms of generalized Mittag–Leffler functions, which are suitable for numerical computation.

In this paper, the authors present an application of a new iterative scheme based on the homotopy analysis method (HAM) for the reaction diffusion-equation

$$\frac{\partial u(x,t)}{\partial t}+\left(u(x,t)\frac{\partial u(x,t)}{\partial x}\right)=v\frac{\partial^2 u(x,t)}{\partial x^2} -\frac{\partial}{\partial x}(F(x)\cdot u(x,t))-\int_0^t\left(a(t-\xi)\frac{\partial u(x,\xi)}{\partial x}\right)\mathrm{d}\xi \tag{1}$$

with initial condition

$$u(x,0)=g(x), \tag{2}$$

where $u(x,t)$ is the displacement, v the constant that defines the kinematic viscosity, $F(x)$ the external force, and $a(t)$ the absorbent term.

The homotopy analysis method which provides an analytical approximation solution is applied to various nonlinear problems [13–15]. The motivation of this paper is to extend the application of HAM proposed by Liao [16] and applied by several authors [17–21] in different mathematical/physical/biological problems to solve nonlinear differential equations. In the next section, we discuss a new reliable approach of HAM. This new modification can be implemented for integer/fractional-order nonlinear differential equations [22].

2. Solution of the Problem

Consider the reaction-diffusion equation with external force and absorbent term $a(t)=\alpha\frac{t^{\beta-1}}{\Gamma(\beta)}$, where α is the absorbent coefficient:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=v\frac{\partial^2 u}{\partial x^2}-\frac{\partial}{\partial x}[F(x)\cdot u] -\frac{\alpha}{\Gamma(\beta)}\int_0^t(t-\xi)^{\beta-1}\frac{\partial u}{\partial x}\mathrm{d}\xi,\ \ t>0, \tag{3}$$

subject to the initial condition

$$u(x,0)=g(x). \tag{4}$$

In view of the algorithm presented in the previous section, if we select the auxiliary linear operator as $L=\frac{\partial}{\partial t}$, we can construct the homotopy

$$\begin{aligned}(1-q)&L[u(x,t;q)-u_0(x,t)]\\ &=q\hbar H(t)\left[\frac{\partial u(x,t;q)}{\partial t}\right.\\ &+u(x,t;q)\frac{\partial u(x,t;q)}{\partial x}-v\frac{\partial^2 u(x,t;q)}{\partial x^2}\\ &+\frac{\partial}{\partial x}[F(x)\cdot u(x,t;q)]\\ &\left.+\frac{\alpha}{\Gamma(\beta)}\int_0^t(t-\xi)^{\beta-1}\frac{\partial u(x,\xi;q)}{\partial x}\mathrm{d}\xi\right]\end{aligned} \tag{5}$$

According to HAM, we have the initial guess

$$u_0(x,t)=g(x). \tag{6}$$

Taking $H(t)=1$ with the initial guess (6) into the homotopy equation (5), then equating the terms with identical powers of q, we obtain the following set of linear differential equation:

$$\begin{aligned}q^1: D_t u_1&=\hbar\left(D_t u_0+u_0\frac{\partial u_0}{\partial x}-v\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial}{\partial x}[F(x)\cdot u_0]\right.\\ &\quad\left.+\frac{\alpha}{\Gamma(\beta)}\int_0^t(t-\xi)^{\beta-1}\frac{\partial u_0}{\partial x}\mathrm{d}\xi\right),\\ q^2: D_t u_2&=D_t u_1+\hbar\left(D_t u_1+u_0\frac{\partial u_1}{\partial x}+u_1\frac{\partial u_0}{\partial x}\right.\\ &\quad-v\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial}{\partial x}[F(x)\cdot u_1]\\ &\quad\left.+\frac{\alpha}{\Gamma(\beta)}\int_0^t(t-\xi)^{\beta-1}\frac{\partial u_1}{\partial x}\mathrm{d}\xi\right),\\ q^3: D_t u_3&=D_t u_2+\hbar\left(D_t u_2+u_0\frac{\partial u_2}{\partial x}+u_1\frac{\partial u_1}{\partial x}\right.\\ &\quad+u_2\frac{\partial u_0}{\partial x}-v\frac{\partial^2 u_2}{\partial x^2}+\frac{\partial}{\partial x}[F(x)\cdot u_2]\\ &\quad\left.+\frac{\alpha}{\Gamma(\beta)}\int_0^t(t-\xi)^{\beta-1}\frac{\partial u_2}{\partial x}\mathrm{d}\xi\right),\\ &\ \ \vdots\end{aligned}$$

$$q^n : D_t u_n = D_t u_{n-1} + \hbar \left(D_t u_{n-1} + \sum_{i=0}^{n-1} u_i \frac{\partial u_{n-1-i}}{\partial x} - v \frac{\partial^2 u_{n-1}}{\partial x^2} + \frac{\partial}{\partial x}[F(x) \cdot u_{n-1}] + \frac{\alpha}{\Gamma(\beta)} \int_0^t (t-\xi)^{\beta-1} \frac{\partial u_{n-1}}{\partial x} \mathrm{d}\xi \right). \quad (7)$$

Now, we use its initial solution $u(x,0) = g(x)$ as an initial approximation by the iteration formula (7), and we get the values of $u_1, u_2, u_3, \ldots$ and so on.

3. Particular Cases

In this section, we are considering some cases of the equations (1) and (2) to demonstrate the reliability of the method and its wider applicability for solving nonlinear diffusion equations with external force and absorbent term.

Case 1. Here the expressions of the displacement $u(x,t)$ are deduced for different particular cases for initial condition $g(x) = x$.

A. For $k = 1$, $\alpha = 1$, i.e. the one-dimensional nonlinear diffusion equation in the presence of an external force and a reaction term:

$$u(x,t) = \left[x - \hbar(3+3\hbar+\hbar^2)tx + \hbar^2 x \frac{t^3}{3} + \hbar(3+3\hbar+\hbar^2)\frac{t^{1+\beta}}{\Gamma(2+\beta)} - \hbar^2(3+2\hbar)\frac{t^{2+\beta}}{\Gamma(3+\beta)} - \hbar^3 \frac{t^{3+\beta}}{(3+\beta)\Gamma(2+\beta)} + \cdots \right]. \quad (8)$$

B. For $k = 0$, $\alpha = 1$, i.e. the one-dimensional nonlinear diffusion equation in the presence of an absorbent term and the absence of an external force:

$$u(x,t) = \left[x + \hbar(3+3\hbar+\hbar^2)xt + \hbar^2(3+2\hbar)xt^2 + \hbar^2 xt^3 + \hbar(3+3\hbar+\hbar^2)\frac{t^{1+\beta}}{\Gamma(2+\beta)} + 2\hbar^2(3+2\hbar)\frac{t^{2+\beta}}{\Gamma(3+\beta)} + 4\hbar^3 \frac{t^{3+\beta}}{\Gamma(4+\beta)} + \hbar^3 \frac{t^{3+\beta}}{(3+\beta)\Gamma(2+\beta)} + \cdots \right]. \quad (9)$$

C. For $k = 1$, $\alpha = 0$, i.e. the one-dimensional nonlinear diffusion equation in the absence of an absorbent term and the presence of an external force:

$$u(x,t) = \left[x - \hbar(3+3\hbar+\hbar^2)xt + \hbar^2 x \frac{t^3}{3} + \cdots \right]. \quad (10)$$

Case 2. Here the expressions of the displacement $u(x,t)$ are deduced for different particular cases for initial condition $g(x) = x^2$.

A. For $k = 1$, $\alpha = 1$, i.e. the one-dimensional nonlinear diffusion equation in the presence of an external force and a reaction term:

$$u(x,t) = \left[x^2 + \hbar(3+3\hbar+\hbar^2)(-2-3x^2+2x^3)t + \hbar^2(3(4-8x+9x^2-20x^3+10x^4) + 2\hbar(2-4x+9x^2-20x^3+10x^4))\frac{t^2}{2!} + \hbar^2(-26+192x-219x^2+152x^3-210x^4 + 84x^5)\frac{t^3}{3!} + 2\hbar(3+3\hbar+\hbar^2)x\frac{t^{1+\beta}}{\Gamma(2+\beta)} + 2\hbar^2(3+2\hbar)(-5+6x)x\frac{t^{2+\beta}}{\Gamma(3+\beta)} + 2\hbar^3(-20+19x-63x^2+44x^3)\frac{t^{3+\beta}}{\Gamma(4+\beta)} + 2\hbar^3(-2-9x^2+8x^3)\frac{t^{3+\beta}}{(3+\beta)\Gamma(2+\beta)} + 2\hbar^2(3+2\hbar)\frac{t^{2+2\beta}}{\Gamma(3+2\beta)} + 4\hbar^3(-3+7x)\frac{t^{3+2\beta}}{\Gamma(4+2\beta)} + \hbar^3 x \frac{t^{3+2\beta}}{(3+2\beta)\Gamma(2+\beta)^2} + \cdots \right]. \quad (11)$$

B. For $k = 0$, $\alpha = 1$, i.e. the one-dimensional nonlinear diffusion equation in the presence of an absorbent term and the absence of an external force:

$$u(x,t) = \left[x^2 + 2\hbar(3+3\hbar+\hbar^2)(-1+x^3)t + \hbar^2(10\hbar x^3 + 15x^3 - 4\hbar - 12)xt^2 + 2h^2x^2 \cdot (-16+7x^3)t^3 + 2\hbar(3+3\hbar+\hbar^2)x\frac{t^{1+\beta}}{\Gamma(2+\beta)}\right.$$

$$+12\hbar^2(3+2\hbar)x^2\frac{t^{2+\beta}}{\Gamma(3+\beta)}+\hbar^3(-40+88x^3)\cdot\frac{t^{3+\beta}}{\Gamma(4+\beta)}+\hbar^3(-4+16x^3)\frac{t^{3+\beta}}{(3+\beta)\Gamma(2+\beta)}+2\hbar^2(3+2\hbar)\frac{t^{2+2\beta}}{\Gamma(3+2\beta)}+28\hbar^3x\frac{t^{3+2\beta}}{\Gamma(4+2\beta)}+4\hbar^3x\frac{t^{3+2\beta}}{(3+2\beta)\Gamma(2+\beta)^2}+\cdots\Bigg]. \tag{12}$$

C. For $k=1$, $\alpha=0$, i.e. the one-dimensional nonlinear diffusion equation in the absence of an absorbent term and the presence of an external force:

$$u(x,t)=\Bigg[x^2+h(3+3\hbar+\hbar^2)(-2-3x^2+2x^3)t+\hbar^2(3(4-8x+9x^2-20x^3+10x^4)+2\hbar(2-4x+9x^2+-20x^3+10x^4))\frac{t^2}{2!}+\hbar^2(-26+192x+-219x^2+152x^3-210x^4+84x^5)\frac{t^3}{3!}+\cdots\Bigg]. \tag{13}$$

Case 3. Here the expressions of the displacement $u(x,t)$ are deduced for different particular cases for initial condition $g(x)=\mathrm{e}^x$.

A. For $k=1$, $\alpha=1$, i.e. the one-dimensional nonlinear diffusion equation in the presence of an external force and a reaction term:

$$u(x,t)=\mathrm{e}^x\Bigg[1+\hbar(3+3\hbar+\hbar^3)(-3+\mathrm{e}^x-2x)t+\hbar^2(25+\mathrm{e}^{2x}(9+6h)+44x+12x^2+4\hbar(3+7x+2x^2)-2\mathrm{e}^x(17+12x+2\hbar(5+4x)))\frac{t^2}{2!}+\hbar^2(-79+16\mathrm{e}^{3x}-3\mathrm{e}^{2x}(35+18x)-122x-60x^2-8x^3+4\mathrm{e}^x(47+48x+12x^2))\frac{t^3}{3!}+\hbar(3+3\hbar+\hbar^3)\frac{t^{1+\beta}}{\Gamma(2+\beta)}+4\hbar^2(3+2\hbar)(-2+\mathrm{e}^x-x)\frac{t^{2+\beta}}{\Gamma(3+\beta)}+\hbar^2(3+2\hbar)\frac{t^{2+2\beta}}{\Gamma(3+2\beta)}+\hbar^3(-15+10\mathrm{e}^x-6x)\cdot\frac{t^{3+2\beta}}{\Gamma(4+2\beta)}+\hbar^3\frac{t^{3+3\beta}}{\Gamma(4+3\beta)}+\mathrm{e}^x\hbar^3\cdot\frac{t^{3+2\beta}}{(3+2\beta)\Gamma(2+\beta)^2}+\cdots\Bigg]. \tag{14}$$

B. For $k=0$, $\alpha=1$, i.e. the one-dimensional nonlinear diffusion equation in the presence of an absorbent term and the absence of an external force:

$$u(x,t)=\mathrm{e}^x\Bigg[1+(-1+\mathrm{e}^x)\hbar(3+3\hbar+\hbar^3)t+\hbar^2(1-2\mathrm{e}^x(5+2\hbar)+\mathrm{e}^{2x}(9+6\hbar))\frac{t^2}{2!}+(-1+28\mathrm{e}^x-51\mathrm{e}^{2x}+16\mathrm{e}^{3x})\hbar^2\frac{t^3}{3!}+\hbar(3+3\hbar+\hbar^3)\frac{t^{1+\beta}}{\Gamma(2+\beta)}+2\hbar^2(-1+2\mathrm{e}^x)\hbar^2(3+2\hbar)\cdot\frac{t^{2+\beta}}{\Gamma(3+\beta)}+(3+\mathrm{e}^x(-32+21\mathrm{e}^x))\hbar^3\frac{t^{3+\beta}}{\Gamma(4+\beta)}+\mathrm{e}^x(-2+3\mathrm{e}^x)\hbar^3\frac{t^{3+\beta}}{(3+\beta)\Gamma(2+\beta)}+\hbar^2(3+2\hbar)\cdot\frac{t^{2+2\beta}}{\Gamma(3+2\beta)}+(-3+10\mathrm{e}^x)\hbar^3\frac{t^{3+2\beta}}{\Gamma(4+2\beta)}+\mathrm{e}^x\hbar^3\frac{t^{3+2\beta}}{(3+2\beta)\Gamma(2+\beta)^2}+\hbar^3\frac{t^{3+3\beta}}{\Gamma(4+3\beta)}+\cdots\Bigg]. \tag{15}$$

C. For $k=1$, $\alpha=0$, i.e. the one-dimensional nonlinear diffusion equation in the presence of an absorbent term and the absence of an external force:

$$u(x,t)=\mathrm{e}^x\Bigg[1+\hbar(3+3\hbar+\hbar^2)(-3+\mathrm{e}^x-2x)t+\hbar^2(25+\mathrm{e}^{2x}(9+6\hbar)+44x+12x^2+4\hbar(3+7x+2x^2)-2\mathrm{e}^x(17+12x+2\hbar(5+4x)))\frac{t^2}{2!}+\hbar^2(-79+16\mathrm{e}^{3x}-122x-60x^2-8x^3-3\mathrm{e}^{2x}(35+18x)+4\mathrm{e}^x(47+48x+12x^2))\frac{t^3}{3!}+\cdots\Bigg]. \tag{16}$$

4. Numerical Results and Discussion

In this section, the numerical values of the displacement $u(x,t)$ at the initial condition $g(x)=\mathrm{e}^x$ for various values of t and different ranges of x with the proper choice of $\hbar$ at $v=1$ are obtained, and the results are de-

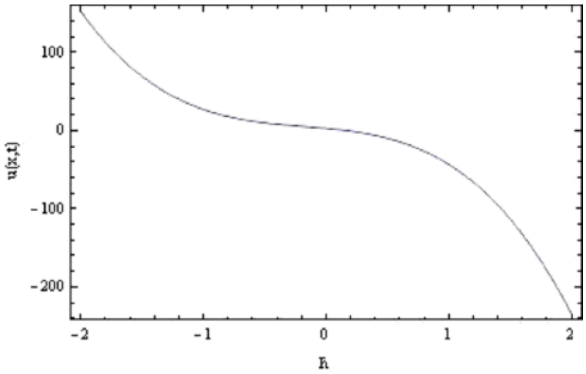

Fig. 1. Plot of $u(x,t)$ vs. $\hbar$ at $\alpha = 1$, $k = 1$, $v = 1$, $\beta = 1$, $x = 1$, and $t = 1$ for $g(x) = e^x$.

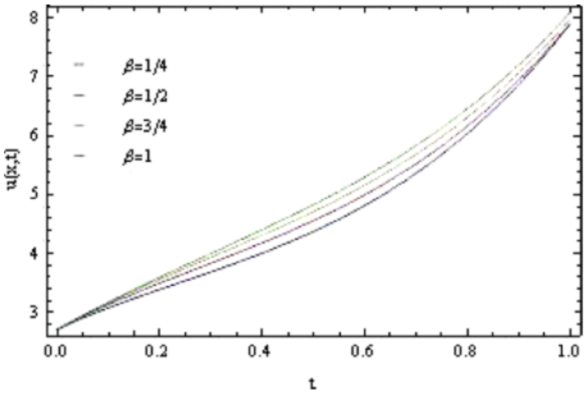

Fig. 2 (colour online). Plot of $u(x,t)$ vs. t at $\alpha = 1$, $k = 1$, $v = 1$, $x = 1$, $\hbar = -0.4$, and $g(x) = e^x$ for different values of β.

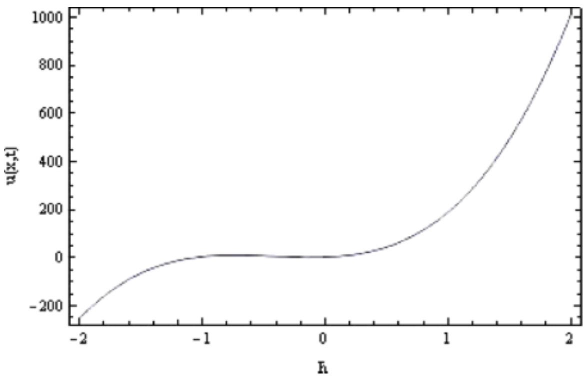

Fig. 3. Plot of $u(x,t)$ vs. $\hbar$ at $\alpha = 1$, $k = 0$, $v = 1$, $\beta = 1$, $x = 1$, and $t = 1$ for $g(x) = e^x$.

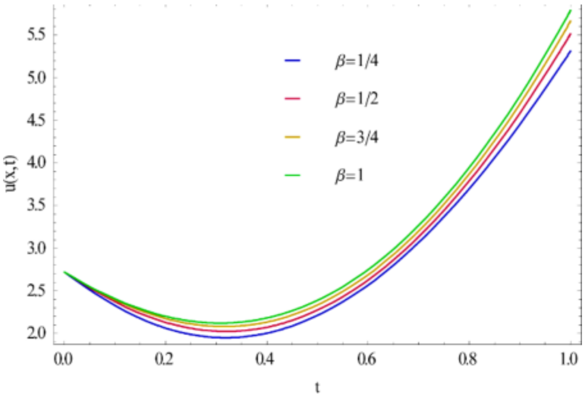

Fig. 4 (colour online). Plot of $u(x,t)$ vs. t at $\alpha = 1$, $k = 0$, $v = 1$, $x = 1$, $\hbar = -0.4$, and $g(x) = e^x$ for different values of β.

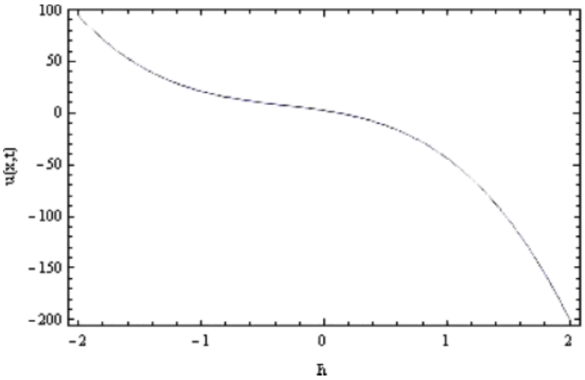

Fig. 5. Plot of $u(x,t)$ vs. $\hbar$ at $\alpha = 0$, $k = 1$, $v = 1$, $\beta = 1$, $x = 1$, and $t = 1$ for $g(x) = e^x$.

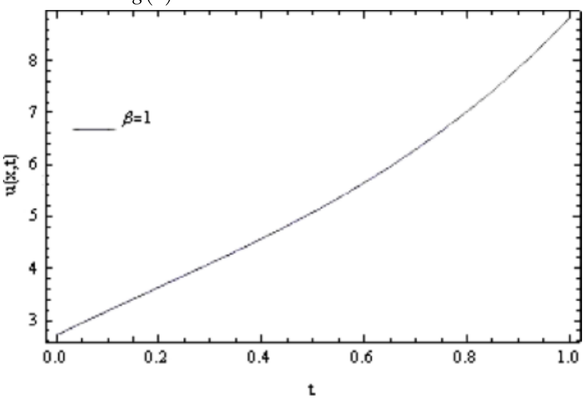

Fig. 6. Plot of $u(x,t)$ vs. t at $\alpha = 0$, $k = 1$, $v = 1$, $x = 1$, $\hbar = -0.4$, and $g(x) = e^x$ for different values of β.

picted through Figures 1 – 7. Here, the authors consider fourth-order term approximation of the series solution throughout the numerical computation. As stated by Liao [16], it is seen that when $\hbar = -1$ the result resounds with the result obtained by the other mathematical tool homotopy perturbation method (HPM) but we cannot say that our solution is always converging at $\hbar = -1$.

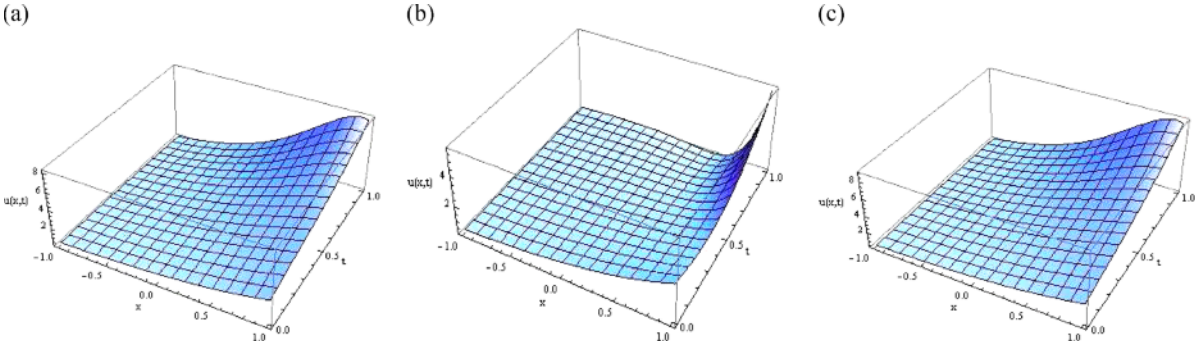

Fig. 7 (colour online). Behaviour of $u(x,t)$ with respect to x and t are obtained when $g(x) = \mathrm{e}^x$, $\hbar = -0.4$; (a) $\alpha = 1$, $k = 1$; (b) $\alpha = 1$, $k = 0$; (c) $\alpha = 0$, $k = 1$.

Figure 1 represents the plot of displacement $u(x,t;\hbar)$ against $\hbar$ for $k = 1$, $\alpha = 1$, $\beta = 1$, $x = 1$, $v = 1$, and $t = 1$. Since $u(x,t;\hbar)$ converges to the exact values for different values of $\hbar$, there exist horizontal line segments shown in the figures, which are usually called $\hbar$-curves and show the validity of the region of convergence of the series solutions of equations (8) – (16). The results justify the statement of Liao [16, 17] that by means of HAM, the convergence region and the rate of series solution can be adjusted and controlled by plotting $\hbar$-curves.

In Section 3, Particular Cases, we have calculated the values of $u(x,t)$ with presence and/or absence of absorbent and external forces and showed that if both terms, i.e. absorbent and external forces ($\alpha = 1$ and $k = 1$), are present the normal solution increases at increase of time but slower than with the presence of only an external force for various values of $\beta = 1/4$, $1/2$, $3/4$, and $\beta = 1$ (Fig. 2). If the absorbent term ($\alpha = 1$ and $k = 0$) is present, the regular solution is rapidly decreasing with the increase of time for various values of $\beta = 1/4$, $1/2$, $3/4$, and $\beta = 1$ (Fig. 4), and if the external force ($\alpha = 0$ and $k = 1$) is present, the regular solution increases at the increase of time (Fig. 6) and it is also physically justified.

It is also observed from Figures 1, 3, and 5, that for smaller time the region of convergence will be better for every value of β. The numerical results $u(x,t)$ for various values of x, t, and $\hbar$ are depicted through Figures 7a, b, and c.

5. Conclusion

The numerical study of this article demonstrates the effect of external force and absorbent term in the nonlinear reaction-diffusion equation. Here the presence of an external force increases the rate of diffusion in the considered nonlinear system. But the rate becomes slower in the presence of an absorbent term. Again in the absence of the external force, the reaction term facilitates the diffusivity of the system much slower which shows that the reaction term controls the system of becoming stable. Therefore, the dynamic response and stability scopes are improved in the presence of an absorbent term which provides a damping force.

Also all the Figures 1 – 7 evidently demonstrate the statement of Liao [16] that the method provides greater freedom to choose an initial approximation, the auxiliary linear operator, and the auxiliary parameter $\hbar$ to ensure the convergence of the series solution. And this article also clearly defined the effect of external force and absorbent term in linear, bilinear, and exponential type of initial conditions. Hence, we have concluded that the diffusion process can be controlled and restricted by the use of the external/absorbent term, respectively.

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