Dynamic Analysis of Nonlinear Oscillator Equation Arising in Double-Sided Driven Clamped Microbeam-Based Electromechanical Resonator

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In this paper, three different analytical methods have been successfully used to study a nonlinear oscillator equation arising in the microbeam-based electromechanical resonator. These methods are: variational approach, Hamiltonian approach, and amplitude-frequency formulation. The governing equation is based on the Euler–Bernoulli hypothesis and the partial differential equation (PDE) is simplified into an ordinary differential equartion (ODE) by using the Galerkin method. A frequency analysis is carried out, and the relationship between the angular frequency and the initial amplitude is obtained in closed analytical form. A comparison of the present solutions is made with the existing solutions and excellent agreement is noted.

Key words: Microelectromechanical System; Nonlinear Oscillation; Analytical Methods; Periodic Solution.

1. Introduction

Microelectromechanical systems (MEMS) are a technology that combines computers with tiny mechanical devices such as sensors, valves, gears, mirrors, and actuators embedded in semiconductor chips. These systems can sense, control, and activate mechanical processes on the micro scale, and function individually or in arrays to generate effects on the macro scale. Microelectromechanical systems are small integrated devices or systems that combine electrical and mechanical components. They range in size from the sub-micrometer (or sub-micron) level to the millimeter level and there can be any number, from a few to millions, in a particular system. MEMS extend the fabrication techniques developed for the integrated circuit industry to add mechanical elements such as beams, gears, diaphragms, and springs to devices [1 – 3]. Examples of MEMS device applications include inkjet-printer cartridges, accelerometers, miniature robots, microengines, locks, inertial sensors, microtransmissions, micromirrors, micro actuators, optical scanners, fluid pumps, transducers, and chemical, pressure and flow sensors. New applications are emerging as the existing technology is applied to the miniaturization and integration of conventional devices [4]. However, electrostatic actuation, large deflections and damping caused by different sources give rise to nonlinear behaviour. Nonlinearity in MEMS may cause some difficulties in computations. Although it is difficult to get analytic approximations for different phenomena in MEMS, there are some analytic techniques for nonlinear problems of MEMS [5]. The perturbation methods [6 – 8] are in common use. Perturbation methods are based on the existence of small parameters, the so-called perturbation quantity. Many nonlinear problems do not contain such perturbation quantity, so to overcome the shortcomings, many new techniques have appeared in open literature such as: variational iteration method [9 – 13], energy balance method [14 – 19], Hamiltonian approach [20 – 23], coupled homotopy-variational formulation [24, 25], variational approach [26 – 28], amplitude-frequency formulation [29, 30], and other classical methods [31 – 45].

In this paper, the basic idea of variational approach, Hamiltonian approach and amplitude-frequency formulation are introduced and then their applications are studied for the model of nonlinear oscillations in the Micro-electromechanical systems [5].

2. Formulation of the Problem

The equation of motion whose sketch is shown in Figure 1 that governs the transverse deflection $w(x,t)$ with driving force per unit length, resulting from electrostatic excitation is written as [5]

$$\bar{E}I\frac{\partial^4 w}{\partial x^4}+\rho S\frac{\partial^2 w}{\partial t^2}=\left[\bar{N}+\frac{ES}{2l}\int_0^l\left(\frac{\partial w}{\partial x}\right)^2\right]\frac{\partial^2 w}{\partial x^2}+\frac{\varepsilon_v b v^2}{2}\left[\frac{1}{(g_0-w)^2}-\frac{1}{(g_0+w)^2}\right], \tag{1}$$

and the relevant boundary conditions [5] are

$$x=0,l:\ \ w=\frac{\partial w}{\partial x}=0, \tag{2}$$

where $\bar{E}$ is the effective modulus, l the length, h the width band thickness, and $ES=bh$ the area of the cross-section with $I=bh^3/12$ the moment of inertia of the cross-section. Further, $\bar{N}$ is the tensile or compressive axial load, v the Poisson ratio, g_0 the initial gap, and ε_v the dielectric constant of the gap medium.

Upon making use of the following substitutions [5]

$$\xi=\frac{x}{l},\ W=\frac{w}{g_0},\ \tau=\sqrt{\frac{\bar{E}I}{\rho bhl^4}}t,\ \alpha=6\left(\frac{g_0}{h}\right)^2,\quad N=\frac{\bar{N}l^2}{\bar{E}I},\ V^2=\frac{24\varepsilon_v l^4 v^2}{\bar{E}h^3 g_0^3}, \tag{3}$$

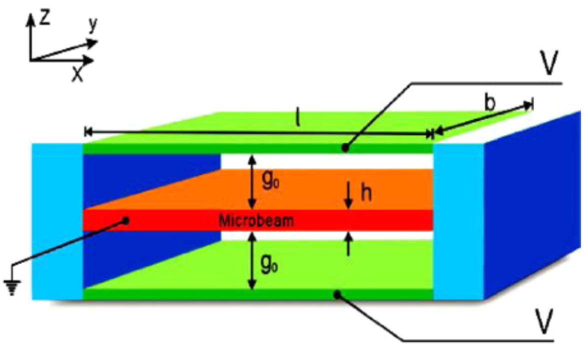

Fig. 1 (colour online). Schematics of a double-sided driven clamped-clamped microbeam-based electromechanical resonator.

the resulting nonlinear partial differential equation is of the form [5]

$$\frac{\partial^4 W}{\partial \xi^4}+\frac{\partial^2 W}{\partial \tau^2}=\left[N+\alpha\int_0^1\left(\frac{\partial W}{\partial \xi}\right)^2\right]\frac{\partial^2 W}{\partial \xi^2}+\frac{V^2}{4}\left[\frac{1}{(1-W^2)}-\frac{1}{(1+W^2)}\right], \tag{4}$$

and the boundary conditions (2) become [5]

$$\xi=0,1:\ \ W=\frac{\partial W}{\partial \xi}=0. \tag{5}$$

The deflection $W(\xi,\tau)$ in (4) is expressed as a sum of spatial shapes that, a priori, satisfy the imposed boundary conditions [5]

$$W(\xi,\tau)=\sum_{i=1}^{n}\phi_i(\xi)u_i(\tau), \tag{6}$$

where n is the number of degrees of freedom and $\phi_i(\xi)$ the ith eigenfunction of the beam. Based on a single degree-offreedom model of the beams $(n=1)$, (4) can be solved with appropriate accuracy [46]. Hence, the solution is constructed by expressing the deflection function $W(\xi,\tau)$ as the product of two separate functions:

$$W(\xi,\tau)=\phi(\xi)u(\tau). \tag{7}$$

As earlier work suggested, here the trial function is $\phi(\xi)=16\xi^2(1-\xi)^2$ [47]. Obviously, (7) satisfies all the boundary conditions listed in (5). In order to avoid division by zero in the electrostatic force term, we multiply (4) by $(1-W^2)^2$.

Substituting (7) into the resulting equation, multiplying by $\phi(\xi)$, and integrating the outcome from 0 to 1, we obtain [5]

$$\ddot{u}\left(a_1u^4+a_2u^2+a_3\right)+a_4u+a_5u^3+a_6u^5+a_7u^7=0, \tag{8}$$

where

$$a_1=\int_0^1\phi^6\,\mathrm{d}\xi,\ a_2=-2\int_0^1\phi^4\,\mathrm{d}\xi,\ a_3=\int_0^1\phi^2\,\mathrm{d}\xi,$$
$$a_4=\int_0^1\left(\phi''''\phi-N\phi''\phi-V^2\phi^2\right)\mathrm{d}\xi,$$
$$a_5=-\int_0^1\Big(2\phi\phi''''\phi^3+2N\phi''\phi^3-\alpha\phi''\phi \tag{9}$$

$$\cdot \int_0^1 (\phi')^2 \, \mathrm{d}\xi \Big) \, \mathrm{d}\xi \,,$$

$$a_6 = \int_0^1 \left(\phi'''' \phi^5 - N\phi'' \phi^5 + 2\alpha \phi'' \phi^3 \int_0^1 (\phi')^2 \, \mathrm{d}\xi \right) \mathrm{d}\xi \,,$$

$$a_7 = -\int_0^1 \left(\alpha \phi'' \phi^5 \int_0^1 (\phi')^2 \, \mathrm{d}\xi \right) \mathrm{d}\xi \,.$$

Here an overdot denotes differentiation with respect to the time variable τ, while a prime indicates the partial differentiation with respect to the coordinate variable ξ.

3. The Application of the Variational Approach (VA)

The variational approach for nonlinear oscillators was proposed in 2007 [26]. Consider the nonlinear oscillator (8). Its variational principle can be obtained by using the semi-inverse method [26]:

$$J(u) = \int_0^{\frac{T}{4}} \left(-\frac{1}{2} \dot{u}^2 (a_1 u^4 + a_2 u^2 + a_3) + \frac{a_4}{2} u^2 + \frac{a_5}{4} u^4 + \frac{a_6}{6} u^6 + \frac{a_7}{8} u^8 \right) \mathrm{d}t \,, \quad T = \frac{2\pi}{\omega} \,, \tag{10}$$

where T is period of the oscillator. Assume that its approximate solution can be expressed as

$$u(t) = A \cos(\omega t) \,. \tag{11}$$

In (11), ω is the frequency to be determined and A is the amplitude of oscillation. Inserting (11) into (10) yields

$$J = \frac{A^2 \pi}{\omega} \left(\frac{5}{192} a_6 A^4 + \frac{35}{2048} a_7 A^6 + \frac{3}{64} a_5 A^2 - \frac{1}{8} a_3 \omega^2 - \frac{1}{64} a_1 \omega^2 A^4 + \frac{1}{8} a_4 - \frac{1}{32} a_2 \omega^2 A^2 \right) . \tag{12}$$

Using the Ritz method, it is required

$$\frac{\partial J}{\partial \omega} = 0 \,, \quad \frac{\partial J}{\partial A} = 0 \,. \tag{13}$$

In [26], J. H. He gave a very lucid as well as elementary discussion of the invalidity of the Ritz method. In particular, He used an unheard-of simple procedure to arrive at a surprisingly accurate prediction for the relationship between the frequency and amplitude of a nonlinear oscillator. According to [26], to identify ω one requires

$$\frac{\partial J}{\partial A} = 0 \,, \tag{14}$$

from which the relationship between the amplitude and frequency of the oscillator can be easily obtained:

$$\omega_{VA} = \frac{\sqrt{2}}{4} \sqrt{\frac{(40 a_6 A^4 + 35 a_7 A^6 + 48 a_5 A^2 + 64 a_4)}{3 a_1 A^4 + 4 a_2 A^2 + 8 a_3}} \,. \tag{15}$$

4. The Application of the Hamiltonian Approach (HA)

Previously, He [18] had introduced the energy balance method based on collocation and Hamiltonian. Recently, in 2010 it was developed to the Hamiltonian approach [20]. This approach is a kind of energy method with a vast application in conservative oscillatory systems. In order to clarify this approach, the Hamiltonian of (8) can be written in the form

$$H(u) = \frac{1}{2} \dot{u}^2 (a_1 u^4 + a_2 u^2 + a_3) + \frac{a_4}{2} u^2 + \frac{a_5}{4} u^4 + \frac{a_6}{6} u^6 + \frac{a_7}{8} u^8 \,. \tag{16}$$

Equation (16) implies that the total energy keeps unchanged during the oscillation. According to (16), it is

$$\frac{\partial H}{\partial A} = 0 \,. \tag{17}$$

Introducing a new function, $\overline{H}(u)$, defined as [16]

$$\overline{H}(u) = \int_0^{\frac{T}{4}} \left(\frac{1}{2} \dot{u}^2 (a_1 u^4 + a_2 u^2 + a_3) + \frac{a_4}{2} u^2 + \frac{a_5}{4} u^4 + \frac{a_6}{6} u^6 + \frac{a_7}{8} u^8 \right) \mathrm{d}t = \frac{1}{4} T H \,, \tag{18}$$

it is obvious that

$$\frac{\partial \overline{H}}{\partial T} = \frac{1}{4} H \,. \tag{19}$$

Equation (19) is equivalent to the following one:

$$\frac{\partial}{\partial A} \left(\frac{\partial \overline{H}}{\partial T} \right) = 0 \tag{20}$$

or

$$\frac{\partial}{\partial A}\left(\frac{\partial \overline{H}}{\partial(1/\omega)}\right)=\frac{35}{256}a_7\pi A^7 + \left(\frac{5}{32}a_6-\frac{3}{32}\omega^2 a_1\right)\pi A^5 + \left(\frac{3}{16}a_5-\frac{1}{8}\omega^2 a_2\right)\pi A^3+\left(\frac{1}{4}a_4-\frac{1}{4}\omega^2 a_3\right)\pi A=0\,. \tag{21}$$

Consequently, the approximate frequency can be found from (21):

$$\omega_{\mathrm{HA}}=\frac{\sqrt{2}}{4}\sqrt{\frac{(40a_6A^4+35a_7A^6+48a_5A^2+64a_4)}{3a_1A^4+4a_2A^2+8a_3}}\,. \tag{22}$$

5. The Application of the Amplitude–Frequency Formulation (AFF)

To solve nonlinear problems, an amplitude–frequency formulation for nonlinear oscillators was proposed by He, which was deduced using an ancient Chinese mathematics method [32, 33]. According to He's amplitude-frequency formulation, $u_1 = A\cos t$ and $u_2 = A\cos\omega t$ serve as the trial functions. Substituting u_1 and u_2 into (3) results in the following residuals:

$$R_1=a_7A^7\cos^7 t+\left(-a_1A^5+a_6A^5\right)\cos^5 t + \left(-a_2A^3+a_5A^3\right)\cos^3 t+(-Aa_3+a_4A)\cos t \tag{23}$$

and

$$R_2=a_7A^7\cos^7(\omega t)+(-a_1\omega^2A^5+a_6A^5)\cos^5(\omega t) +(-a_2\omega^2A^3+a_5A^3)\cos^3(\omega t) (-A\omega^2a_3+a_4A)\cos(\omega t)\,. \tag{24}$$

According to the amplitude–frequency formulation, the above residuals can be rewritten in the forms of weighted residuals:

$$R_{11}=\frac{4}{T_1}\int_0^{\frac{T_1}{4}} R_1\cos(t)\,\mathrm{d}t,\;\; T_1=2\pi\,, \tag{25}$$

and

$$R_{22}=\frac{4}{T_2}\int_0^{\frac{T_2}{4}} R_2\cos(\omega t)\,\mathrm{d}t,\;\; T_2=\frac{2\pi}{\omega}\,. \tag{26}$$

Applying He's frequency-amplitude formulation

$$\omega^2=\frac{\omega_1^2R_{22}-\omega_2^2R_{11}}{R_{22}-R_{11}}\,, \tag{27}$$

where

$$\omega_1=1\,,\;\; \omega_2=\omega\,. \tag{28}$$

Then the approximate frequency can be obtained:

$$\omega_{\mathrm{AFF}}=\frac{\sqrt{2}}{4}\sqrt{\frac{(40a_6A^4+35a_7A^6+48a_5A^2+64a_4)}{5a_1A^4+6a_2A^2+8a_3}}\,. \tag{29}$$

6. Results and Discussion

In this section, the applicability, accuracy, and effectiveness of the proposed approaches are illustrated by comparing the analytical approximate frequency and

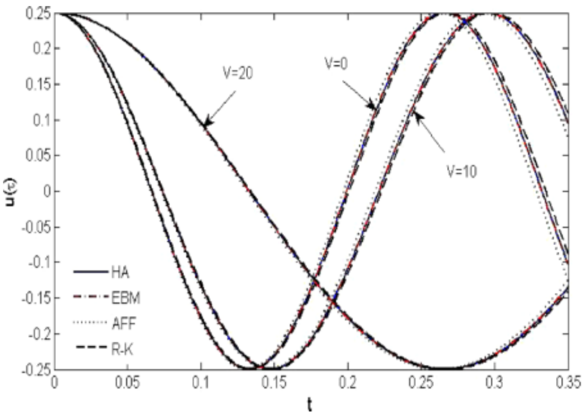

Fig. 2. Comparison between Hamiltonion approach (HA), amplitude–frequency formulation (AFF), energy balance method (EBM), and the fourth-order Runge–Kutta method (R-K); $A = 0.25$.

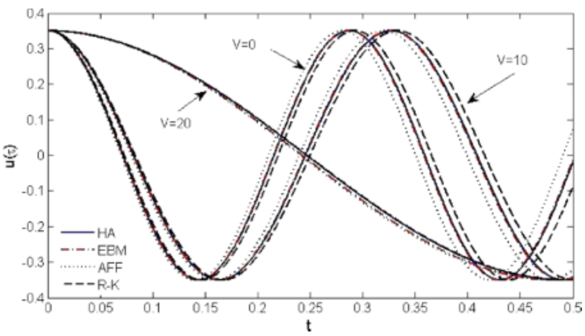

Fig. 3. Comparison between Hamiltonion approach (HA), amplitude–frequency formulation (AFF), energy balance method (EBM), and the fourth-order Runge–Kutta method (R-K); $A = 0.35$.

periodic solution with the energy balance method [5] and fourth-order Runge–Kutta method. Fu et al. obtained a solution for this problem by energy balance method [5]:

$$\omega_{\text{EBM}} = \frac{\sqrt{6}}{12}\sqrt{\frac{\left(56a_6A^4+45a_7A^6+72a_5A^2+96a_4\right)}{a_1A^4+2a_2A^2+4a_3}}. \tag{30}$$

It is found that the obtained results for variational approach and Hamiltonion approach are similar. The comparison between Hamiltonion approach, amplitude–frequency formulation, energy balance method, and fourth-order Runge–Kutta method is plotted in Figures 2 and 3. Herein the values of parameters are taken as $N = 10$ and $\alpha = 25$.

7. Conclusions

In this paper, three powerful and simple methods are applied for solving the equation of motion of a double-sided driven clamped-clamped microbeam-based electromechanical resonator. The proposed techniques are employed without any linearization, discretization or restrictive assumptions. The nonlinear oscillator ODE is applicable in microelectromechanical systems. These new approaches prove to be very rapid, effective, and accurate, and this is proved by comparing the solutions obtained through the proposed methods with the published results in the literature. This paper shows one step in the attempt to develop the nonlinear analytical techniques valid for PDE/ODE problems arising in the microelectromechanical systems, and the proposed procedure can easily be used to find analytical approximate solutions to other strongly nonlinear oscillators.

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