Thermosolutal Convection in a Couple-Stress Fluid in Uniform Vertical Magnetic Field

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The effect of a uniform vertical magnetic field on thermosolutal convection in a layer of an electrically conducting couple-stress fluid heated and soluted from below is considered. For the case of stationary convection, the stable solute gradient, magnetic field, and couple-stress parameter have stabilizing effect on the system. It is also observed that a stable solute gradient and a magnetic field introduce oscillatory modes in the system, but in the absence of a stable solute gradient and a magnetic field, oscillatory modes are not allowed and the principle of exchange of stabilities is valid.

Key words: Thermosolutal Convection; Couple-Stress Fluid; Uniform Vertical Magnetic Field.

1. Introduction

A detailed account of the theoretical and experimental results of the onset of thermal instability (Bénard convection) in a fluid layer under varying assumptions of hydrodynamics and hydromagnetics has been given in the celebrated monograph by Chandrasekhar [1]. Veronis [2] has investigated the problem of thermohaline convection in a layer of a fluid heated from below and subjected to a stable salinity gradient. The buoyancy forces can arise not only from density differences due to variations in temperature but also from those due to variations in solute concentration. Thermosolutal convection problems arise in oceanography, limnology, and engineering. The investigation of thermosolutal convection is motivated by its interesting complexities as a double diffusion phenomena as well as its direct relevance to geophysics and astrophysics. Stomell et al. [3] did the pioneering work regarding the investigation of thermosolutal convection.

This work was elaborated in different physical situations by Stern [4] and Nield [5].

Examples of particular interest are provided by ponds built to trap solar heat [6] and some Antarctic lakes [7]. The physics is quite similar in the stellar case in that Helium acts like salt in raising the density and in diffusing more slowly than heat. The conditions under which convective motion in double-diffusive convection are important (e.g. in lower parts of the Earth's atmosphere, astrophysics, and several geophysical situation) are usually far removed from the consideration of a single component fluid and rigid boundaries, and therefore it is desirable to consider a fluid acted on by a solute gradient and free boundaries. A double-diffusive instability that occurs when a solution of a slowly diffusing protein is layered over a denser solution of more rapidly diffusing sucrose has been explained by Brakke [8]. Nason et al. [9] found that this instability, which is deleterious to certain biochemical separations, can be suppressed by rotation in the ultra centrifuge.

The problem of thermosolutal convection in a couple-stress fluid is of importance in geophysics, soil sciences, ground water hydrology, and astrophysics. The theory of couple-stress fluid has been formulated by Stokes [10]. One of the applications of couple-stress fluids is its use to the study of the mechanisms of lubrications of synovial joints, which has become the object of scientific research. A human joint is a dynamically loaded bearing which has articular cartilage as the bearing and synovial fluid as the lubricant. When a fluid is generated, squeeze-film action is capable of providing considerable protection to the cartilage surface. The shoulder, ankle, knee, and hip joints

are the loaded-bearing synovial joints of the human body and these joints have a low friction coefficient and negligible wear.

The normal synovial fluid is a viscous, non-Newtonian fluid, and is generally clear or yellowish. According to the theory of Stokes [10], couple-stresses appear in noticeable magnitudes in fluids with very large molecules. Since the long chain hyaluronic acid molecules are found as additives in synovial fluids, Walicki and Walicka [11] modelled the synovial fluid as a couple-stress fluid. The synovial fluid is the natural lubricant of joints of the vertebrates. The detailed description of the joint lubrication has very important practical implications. Practically all diseases of joints are caused by or connected with a malfunction of the lubrication. Goel et al. [12] have studied the hydromagnetic stability of an unbounded couple-stress binary fluid mixture under rotation with vertical temperature and concentration gradients. Sharma et al. [13] have considered a couple-stress fluid with suspended particles heated from below. They have found that for stationary convection, couple-stress has a stabilizing effect whereas suspended particles have a destabilizing effect. In another study, Sunil et al. [14, 15] have considered a couple stress fluid heated from below in a porous medium in the presence of a magnetic field and rotation and also studied on superposed couple-stress fluids in porous medium in hydromagnetics. Kumar et al. [16–18] have considered the thermal instability of a layer of a couple-stress fluid acted on by a uniform rotation; they have also studied on the stability of superposed viscous-viscoelastic (couple-stress) fluids through porous media and double-diffusive magneto-rotatory convection in couple-stress fluids through porous media. In another study, Kumar and Singh [19] have considered the rotatory thermosolutal convection in a couple-stress fluid and Singh and Kumar [20, 21] have studied magneto thermal convection in a compressible couple-stress fluid and magneto and rotatory thermosolutal convection in couple-stress fluids in porous media.

Keeping in mind the importance in geophysics, soil sciences, ground water hydrology, astrophysics, and various applications mentioned above, the thermosolutal convection in a couple-stress fluid in the presence of a uniform magnetic field has been considered in the present paper.

2. Formulation of the Problem and Perturbation Equations

Consider an infinite horizontal layer of an electrically conducting couple-stress fluid of depth d, which is acted on by a uniform vertical magnetic field intensity $\vec{H} = (0,0,H)$ and gravity force $\vec{g} = (0,0,-g)$. This layer is heated and soluted from below such that a steady adverse temperature gradient $\beta(= |\mathrm{d}T/\mathrm{d}z|)$ and solute concentration gradient $\beta'(= |\mathrm{d}C/\mathrm{d}z|)$ are maintained (see Fig. 1).

The hydromagnetic equations [1, 2, 10]), relevant to the problem and following the Boussinesq approximation, are

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q} = -\frac{1}{\rho_0}\nabla p + \vec{g}\left(1+\frac{\delta\rho}{\rho_0}\right) + \frac{\mu_e}{4\pi\rho_0}\left(\nabla\times\vec{H}\right)\times\vec{H} + \left(\nu - \frac{\mu'}{\rho_0}\nabla^2\right)\nabla^2\vec{q}, \tag{1}$$

$$\nabla\cdot\vec{q} = 0, \tag{2}$$

$$\nabla\cdot\vec{H} = 0, \tag{3}$$

$$\frac{\partial \vec{H}}{\partial t} = (\vec{H}\cdot\nabla)\vec{q} + \eta\nabla^2\vec{H}, \tag{4}$$

$$\frac{\partial T}{\partial t} + (\vec{q}\cdot\nabla)T = \chi\nabla^2 T, \tag{5}$$

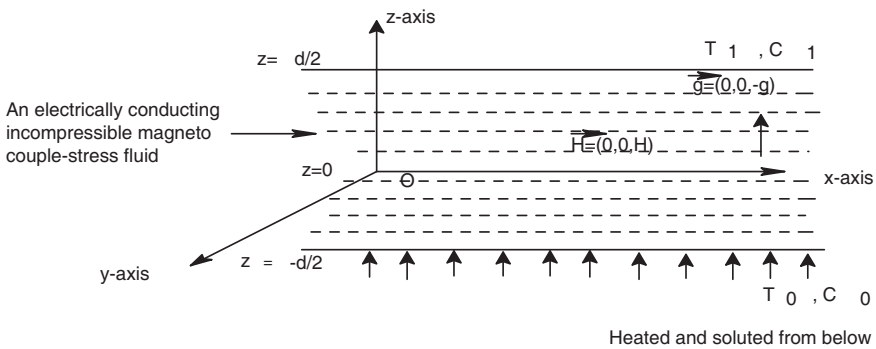

Fig. 1. Geometrical configuration.

$$\frac{\partial C}{\partial t} + (\vec{q}\cdot\nabla)C = \chi'\nabla^2 C. \tag{6}$$

Here ρ, p, T, C, μ_e, $\vec{q} = (u,v,w)$, $\vec{H} = (0,0,H)$, and $\vec{g} = (0,0,-g)$ stand for density, pressure, temperature, solute mass concentration, magnetic permeability, velocity, magnetic field intensity, and gravitational acceleration, respectively. The viscosity μ, couple-stress viscosity μ', kinematic viscosity ν, thermal diffusivity χ, analogous solute diffusivity χ', and electrical resistivity η are each assumed to be constant.

The equation of state is

$$\rho = \rho_0[1 - \alpha(T - T_0) + \alpha'(C - C_0)], \tag{7}$$

where the suffix zero refers to the values at the reference level $z = 0$, and so the change in density $\delta\rho$ caused by the perturbation θ and γ in temperature and concentration is given by

$$\delta\rho = -\rho_0(\alpha\theta - \alpha'\gamma). \tag{8}$$

The equation of state (7) contains a thermal coefficient of expansion α and an analogous solvent coefficient α'.

The steady state solution is

$$\begin{aligned} &\vec{q} = (0,0,0),\ T = T_0 - \beta z,\ C = C_0 - \beta' z, \\ &\rho = \rho_0(1 + \alpha\beta z - \alpha'\beta' z'), \end{aligned} \tag{9}$$

where $\beta = \frac{T_0 - T_1}{d}$ and $\beta' = \frac{C_0 - C_1}{d}$ are the magnitudes of uniform temperature and concentration gradients and are both positive as temperature and concentration decrease upwards. The temperatures and the solute concentrations at the bottom surface $z = 0$ are T_0 and C_0 and at the upper surface $z = d$ are T_1 and C_1, respectively.

Let $\delta\rho$, δp, θ, γ, $\vec{q} = (u,v,w)$, and $\vec{h} = (h_x, h_y, h_z)$ denote the perturbations in density ρ, pressure p, temperature T, solute concentration C, velocity $(0,0,0)$, and magnetic field intensity $\vec{H}$, respectively. Then the linearized hydromagnetic perturbation equations are

$$\begin{aligned} \frac{\partial \vec{q}}{\partial t} &= -\frac{1}{\rho_0}\nabla\delta p - \vec{g}(\alpha\theta - \alpha'\gamma) \\ &+ \frac{\mu_e}{4\pi\rho_0}\left(\nabla\times\vec{h}\right)\times\vec{H} + \left(\nu - \frac{\mu'}{\rho_0}\nabla^2\right)\nabla^2\vec{q}, \end{aligned} \tag{10}$$

$$\nabla\cdot\vec{q} = 0, \tag{11}$$

$$\nabla\cdot\vec{h} = 0, \tag{12}$$

$$\frac{\partial \vec{h}}{\partial t} = (\vec{H}\cdot\nabla)\vec{q} + \eta\nabla^2\vec{h}, \tag{13}$$

$$\frac{\partial \theta}{\partial t} = \beta w + \chi\nabla^2\theta, \tag{14}$$

$$\frac{\partial \gamma}{\partial t} = \beta' w + \chi'\nabla^2\gamma. \tag{15}$$

Here we consider the case in which both the boundaries are free as well as perfect conductors of both heat and solute concentration, and the adjoining medium is electrically nonconducting. The case of two free surfaces is a little artificial except in the case of stellar atmospheres. However, this assumption allows us to obtain the analytical solution without affecting the essential features of the problem. The boundary conditions appropriate for the problem are

$$w = \frac{\partial^2 w}{\partial z^2} = \frac{\partial^4 w}{\partial z^4} = 0,\ \theta = 0,\ \gamma = 0 \tag{16}$$

at $z = 0$ and $z = d$, and $\vec{h}$ is continuous with an external field.

Within the framework of the Boussinesq approximation, (10)–(15) give

$$\begin{aligned} &\frac{\partial}{\partial t}\nabla^2 w - g\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\alpha\theta - \alpha'\gamma) \\ &- \frac{\mu_e H}{4\pi\rho_0}\frac{\partial}{\partial z}\nabla^2 h_z = \left(\nu - \frac{\mu'}{\rho_0}\nabla^2\right)\nabla^4 w, \end{aligned} \tag{17}$$

$$\left(\frac{\partial}{\partial t} - \chi\nabla^2\right)\theta = \beta w, \tag{18}$$

$$\left(\frac{\partial}{\partial t} - \chi'\nabla^2\right)\gamma = \beta' w, \tag{19}$$

$$\left(\frac{\partial}{\partial t} - \eta\nabla^2\right)h_z = H\frac{\partial w}{\partial z}, \tag{20}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

3. Dispersion Relation

We now analyse the disturbances into normal modes, assuming that the perturbation quantities have the space and time dependence of the form

$$\begin{aligned} [w, \theta, h_z, \gamma] = &[W(z), \Theta(z), K(z), \Gamma(z)] \\ &\cdot \exp(\mathrm{i}k_x x + \mathrm{i}k_y y + nt), \end{aligned} \tag{21}$$

where k_x and k_y are the wave numbers along x- and y-directions, respectively, $k = (\sqrt{k_x^2 + k_y^2})$ is the resultant wave number, and n is the growth rate which is, in general, a complex constant.

Using expression (21), (17)–(20) in nondimensional form become

$$\sigma(D^2 - a^2)W - [1 - F(D^2 - a^2)](D^2 - a^2)^2 W - \frac{\mu_e H d}{4\pi\rho_o \nu}(D^2 - a^2)\, DK + \frac{g a^2 d^2}{\nu}(\alpha\Theta - \alpha'\Gamma) = 0\,, \tag{22}$$

$$(D^2 - a^2 - p_1\sigma)\Theta = -\left(\frac{\beta d^2}{\chi}\right)W\,, \tag{23}$$

$$(D^2 - a^2 - q\sigma)\Gamma = -\left(\frac{\beta' d^2}{\chi'}\right)W\,, \tag{24}$$

$$(D^2 - a^2 - p_2\sigma)K = -\left(\frac{H d}{\eta}\right)DW\,, \tag{25}$$

where we have put $a = kd$, $\sigma = \frac{nd^2}{\nu}$, $\frac{x}{d} = x^*$, $\frac{y}{d} = y^*$, $\frac{z}{d} = z^*$, and $D = \frac{\mathrm{d}}{\mathrm{d}z^*}$. Here $p_1 = \frac{\nu}{\chi}$ is the Prandtl number, $p_2 = \frac{\nu}{\eta}$ is the magnetic Prandtl number, $q = \frac{\nu}{\chi'}$ is the Schmidt number, and $F = \frac{\mu'}{\rho_0 d^2 \nu}$ is the dimensionless couple-stress parameter.

Eliminating Θ, Γ, and K between (22)–(25), we obtain

$$\begin{aligned}&(D^2 - a^2)(D^2 - a^2 - p_1\sigma)(D^2 - a^2 - q\sigma)\\ &\cdot\Big[\sigma(D^2 - a^2 - p_2\sigma) + QD^2 - [1 - F(D^2 - a^2)]\\ &\cdot(D^2 - a^2 - p_2\sigma)\Big]^W = (D^2 - a^2 - p_2\sigma)\\ &\cdot[Ra^2(D^2 - a^2 - q\sigma) - Sa^2(D^2 - a^2 - p_1\sigma)]^W\,,\end{aligned} \tag{26}$$

where $R = \frac{g\alpha\beta d^4}{\nu\kappa}$ is the Rayleigh number, $S = \frac{g\alpha'\beta' d^4}{\nu\chi'}$ is the analogous solute Rayleigh number, and $Q = \frac{\mu_e H^2 d^2}{4\pi\rho_0 \nu\eta}$ is the Chandrasekhar number.

The boundary conditions (16) transform to [10]

$$\begin{aligned}&W = D^2W = D^4W = 0\,,\ \ \Theta = 0\,,\ \ \Gamma = 0\,,\\ &\xi = 0 \ \text{ at } \ z^* = 0 \ \text{ and } \ z^* = 1\,,\end{aligned} \tag{27}$$

where $\xi = (\mathrm{Curl}\,\vec{h})_z$ is the z-component of current density.

Dropping the stars for convenience and using the boundary conditions (27), it can be shown that all the even order derivatives of W must vanish on the boundaries and hence the proper solution of (26), characterising the lowest mode, is

$$W = W_0 \sin \pi z\,, \tag{28}$$

where W_0 is a constant.

Substituting (28) in (26), we obtain the dispersion relation

$$\begin{aligned}R_1 = {}&\frac{(1+x)(1+x+\mathrm{i}p_1\sigma_1)}{x}[(1+x)\{1+F_1(1+x)\}\\ &+\mathrm{i}\sigma_1] + Q_1\frac{(1+x)}{x}\frac{(1+x+\mathrm{i}p_1\sigma_1)}{(1+x+\mathrm{i}p_2\sigma_1)}\\ &+S_1\frac{(1+x+\mathrm{i}p_1\sigma_1)}{(1+x+\mathrm{i}q\sigma_1)}\,,\end{aligned} \tag{29}$$

where $R_1 = \frac{R}{\pi^4}$, $S_1 = \frac{S}{\pi^4}$, $Q_1 = \frac{Q}{\pi^2}$, $a^2 = \pi^2 x$, $\frac{\sigma}{\pi^2} = \mathrm{i}\sigma_1$, and $F_1 = \pi^2 F$.

4. Results and Discussion

4.1. Stationary Convection

When the instability sets in as stationary convection, marginal state will be characterized by $\sigma = 0$. Putting $\sigma = 0$, the dispersion relation (29) reduces to

$$\begin{aligned}R_1 = {}&\frac{(1+x)}{x}[(1+x)^2\{1+F_1(1+x)\} + Q_1]\\ &+S_1\,.\end{aligned} \tag{30}$$

To study the effect of stable solute gradient, magnetic field, and couple-stress parameter, we examine the nature of $\frac{\mathrm{d}R_1}{\mathrm{d}S_1}$, $\frac{\mathrm{d}R_1}{\mathrm{d}Q_1}$, and $\frac{\mathrm{d}R_1}{\mathrm{d}F_1}$.

Equation (30) yields

$$\frac{\mathrm{d}R_1}{\mathrm{d}S_1} = +1\,, \tag{31}$$

$$\frac{\mathrm{d}R_1}{\mathrm{d}Q_1} = \frac{(1+x)}{x}\,, \tag{32}$$

$$\frac{\mathrm{d}R_1}{\mathrm{d}F_1} = \frac{(1+x)^4}{x}\,, \tag{33}$$

which imply that stable solute gradient, magnetic field, and couple-stress parameter have a stabilising effect on the system. Graphs have been plotted between R_1 and x for various values of Q_1, F_1, and S_1. The stabilising effect is also evident from Figures 2–4.

4.2. Stability of the System and Oscillatory Modes

Here we examine the possibility of oscillatory modes, if any, on the stability problem due to the presence of stable solute gradient and magnetic field. Multiplying (22) by W^*, the complex conjugate of W,

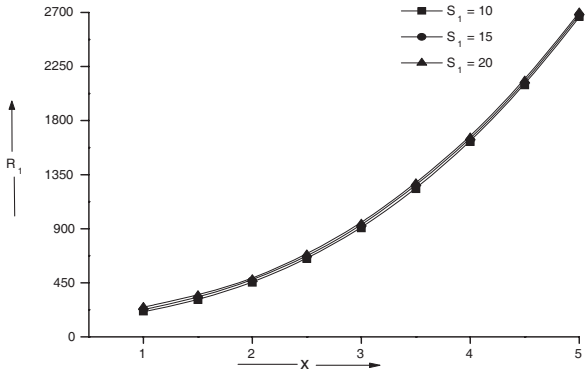

Fig. 2. Variation of R_1 with x for a fixed $F_1 = 10$, $Q_1 = 20$, for different values of S_1 (= 10, 15, 20).

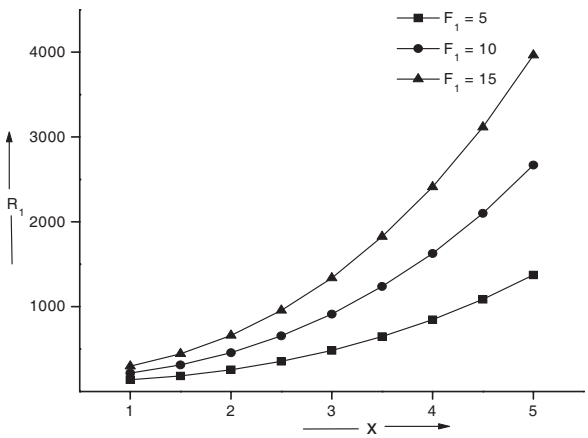

Fig. 3. Variation of R_1 with x for a fixed $S_1 = 10$, $Q_1 = 20$, for different values of F_1 (= 5, 10, 15).

integrating over the range of z, and making use of (23)–(25) together with the boundary conditions (27), we obtain

$$\sigma I_1 + I_2 - \frac{g\alpha\chi a^2}{\nu\beta}(I_3 + p_1\sigma^* I_4) + \frac{g\alpha'\chi' a^2}{\nu\beta'}(I_5 + q\sigma^* I_6) + \frac{\mu_e\eta}{4\pi\rho_0\nu}(I_7 + p_2\sigma^* I_8) + F I_9 = 0\,, \tag{34}$$

where

$$I_1 = \int_0^1 \left(|DW|^2 + a^2|W|^2\right) \mathrm{d}z\,,$$
$$I_2 = \int_0^1 \left(|D^2W|^2 + 2a^2|DW|^2 + a^4|W|^2\right) \mathrm{d}z\,,$$
$$I_3 = \int_0^1 \left(|D\Theta|^2 + a^2|\Theta|^2\right) \mathrm{d}z\,,\ I_4 = \int_0^1 |\Theta|^2\,\mathrm{d}z\,,$$

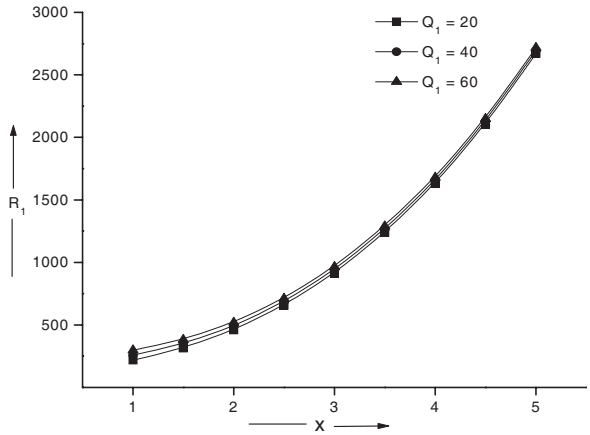

Fig. 4. Variation of R_1 with x for a fixed $S_1 = 10$, $F_1 = 10$, for different values of Q_1 (= 20, 40, 60).

$$I_5 = \int_0^1 \left(|D\Gamma|^2 + a^2|\Gamma|^2\right) \mathrm{d}z\,,\ I_6 = \int_0^1 |\Gamma|^2\,\mathrm{d}z\,,$$
$$I_7 = \int_0^1 \left(|D^2K|^2 + 2a^2|DK|^2 + a^4|K|^2\right) \mathrm{d}z\,,$$
$$I_8 = \int_0^1 \left(|DK|^2 + a^2|K|^2\right) \mathrm{d}z\,,$$
$$I_9 = \int_0^1 \left(|D^3W|^2 + 3a^2|D^2W|^2 + 3a^4|DW|^2 + a^6|W|^2\right) \mathrm{d}z\,,$$

and σ^* is the complex conjugate of σ. The integrals $I_1 - I_9$ are all positive definite.

Putting $\sigma = \sigma_r + i\sigma_i$ in (34) and equating real and imaginary parts, we have

$$\sigma_r\left(I_1 + \frac{g\alpha'\chi' a^2}{\nu\beta'} q I_6 + \frac{\mu_e\eta}{4\pi\rho_0\nu} p_2 I_8 - \frac{g\alpha\chi a^2}{\nu\beta} p_1 I_4\right) = -\left(I_2 + \frac{g\alpha'\chi' a^2}{\nu\beta'} I_5 + \frac{\mu_e\eta}{4\pi\rho_0\nu} I_7 - \frac{g\alpha\chi a^2}{\nu\beta} I_3 + F I_9\right) \tag{35}$$

and

$$\sigma_i\left(I_1 + \frac{g\alpha\chi a^2}{\nu\beta} p_1 I_4 - \frac{g\alpha'\chi' a^2}{\nu\beta'} q I_6 - \frac{\mu_e\eta}{4\pi\rho_0\nu} p_2 I_8\right) = 0\,. \tag{36}$$

Equation (35) yields that σ_r may be positive or negative, i.e. there may be stability or instability in the pres-

ence of solute gradient and magnetic field in couple-stress fluid. It is clear from (36) that $\sigma_i = 0$ or $\sigma_i \neq 0$, which means that the modes may be nonoscillatory or oscillatory.

From (36) it is clear that σ_i is zero when the quantity multiplying it is not zero and arbitrary when this quantity is zero.

If $\sigma_i \neq 0$, then (36) gives

$$I_1 = \frac{g\alpha'\chi' a^2}{\nu\beta'} q I_6 - \frac{g\alpha\chi a^2}{\nu\beta} p_1 I_4 + \frac{\mu_e \eta}{4\pi\rho_0 \nu} p_2 I_8 . \tag{37}$$

Substituting in (35), we have

$$\begin{aligned} &2\sigma_r I_1 + I_2 + \frac{g\alpha'\chi' a^2}{\nu\beta'} I_5 + \frac{\mu_e \eta}{4\pi\rho_0 \nu} I_7 + F I_9 \\ &= \frac{g\alpha\chi a^2}{\nu\beta} I_3 . \end{aligned} \tag{38}$$

Equation (38) on using Rayleigh–Ritz inequality gives

$$\begin{aligned} &\frac{(\pi^2+a^2)^3}{a^2} \int_0^1 |W| \, dz + \frac{(\pi^2+a^2)}{a^2} \left\{ F I_9 + \frac{\mu_e \eta}{4\pi\rho_0 \nu} I_7 \right. \\ &\left. + \frac{g\alpha'\chi' a^2}{\nu\beta'} I_5 + 2\sigma_r I_1 \right\} \leq \frac{g\alpha\chi}{\nu\beta} \int_0^1 |W| \, dz . \end{aligned} \tag{39}$$

Therefore, it follows from (39) that

$$\begin{aligned} &\left[\frac{27\pi^4}{4} - \frac{g\alpha\chi}{\nu\beta} \right] \int_0^1 |W|^2 \, dz + \frac{(\pi^2+a^2)}{a^2} \\ &\cdot \left\{ F I_9 + \frac{\mu_e \eta}{4\pi\rho_0 \nu} I_7 + \frac{g\alpha'\chi' a^2}{\nu\beta'} I_5 + 2\sigma_r I_1 \right\} \leq 0 , \end{aligned} \tag{40}$$

since the minimum value of $\frac{(\pi^2+a^2)^3}{a^2}$ with respect to a^2 is $\frac{27\pi^4}{4}$.

Now, let $\sigma_r \geq 0$, we necessarily have from (40) that

$$\frac{g\alpha\chi}{\nu\beta} > \frac{27\pi^4}{4} . \tag{41}$$

Hence, if

$$\frac{g\alpha\chi}{\nu\beta} \leq \frac{27\pi^4}{4} , \tag{42}$$

then $\sigma_r < 0$. Therefore, the system is stable.

Therefore, under condition (42), the system is stable and under condition (41) the system becomes unstable.

In the absence of stable solute gradient and magnetic field, equation (36) reduces to

$$\sigma_i \left(I_1 + \frac{g\alpha\chi a^2}{\nu\beta} p_1 I_4 \right) = 0 , \tag{43}$$

and the terms in brackets are positive definite. Thus, $\sigma_i = 0$, which means that oscillatory modes are not allowed and the principle of exchange of stabilities is valid for the couple-stress fluid in the absence of stable solute gradient and magnetic field. The presence of each, the stable solute gradient and the magnetic field brings oscillatory modes (as σ_i may not be zero), which were nonexistent in their absence.

4.3. Case of Overstability

Here we discuss the possibility of whether instability may occur as overstability. Since we wish to determine the critical Rayleigh number for the onset of instability via a state of pure oscillations, it suffices to find conditions for which (29) will admit a solution with σ_1 real.

Equating the real and imaginary parts of (29), eliminating R_1 between them, and letting $c_1 = \sigma_1^2$, $b = 1 + x$, we obtain

$$A_2 c_1^2 + A_1 c_1 + A_0 = 0 , \tag{44}$$

where

$$\begin{aligned} A_2 &= q^2 p_2^2 b[1 + p_1(1 + F_1 b)] , \\ A_1 &= \left[\left[(p_2^2 + q^2) \left\{ b^3 (1 + \overline{p_1 + F_1 b}) \right\} \right] \right. \\ &\quad \left. + q^2 b Q_1 (p_1 - p_2) + S_1 (b-1) p_2^2 (p_1 - q) \right] , \\ A_0 &= \left[b^5 \left\{ 1 + p_1 (1 + F_1 b) \right\} + S_1 (b-1) b^2 (p_1 - q) \right. \\ &\quad \left. + Q_1 b^3 (p_1 - p_2) \right] . \end{aligned} \tag{45}$$

Since σ_1 is real for overstability, both the values of c_1 $(= \sigma_1^2)$ are positive. Equation (44) is quadratic in c_1 and does not involve any of its roots to be positive if

$$p_1 > p_2 \text{ and } p_1 > q , \tag{46}$$

which imply that

$$\chi < \eta \text{ and } \chi < \chi' . \tag{47}$$

Thus $\chi < \eta$ and $\chi < \chi'$ are the sufficient conditions for the nonexistence of overstability, the violation of which does not necessarily imply the occurrence of overstability.

[1] S. Chandresekhar, Hydrodynamic and Hydromagnetic Stability, Dover Publication, New York 1981.
[2] G. Veronis, J. Marine Res. **23**, 17 (1965).
[3] H. Stomell, A. B. Arons, and D. Balanchord, Deep Sea Res. **3**, 152 (1956).
[4] M. E. Stern, Tellus **12**, 175 (1960).
[5] D. A. Nield, J. Fluid Mech. **29**, 558 (1967).
[6] H. Tabor and R. Matz, Solar Energy **9**, 182 (1965).
[7] T. G. L. Shirtcliffe, J. Geophys. Res. **69**, 5268 (1964).
[8] M. K. Brakke, Arch. Biochem. Biophys. **55**, 190 (1955).
[9] P. Nason, V. Schumaker, B. Halsall, and J. Schwedes, Biopolymers **7**, 249 (1969).
[10] V. K. Stokes, Phys. Fluids **9**, 1715 (1966).
[11] E. Walicki and A. Walicka, Appl. Mech. Eng. **4**, 373 (1999).
[12] A. K. Goel, S. C. Agarwal, and G. S. Agarwal, Indian J. Pure Appl. Math. **30**, 1001 (1999).
[13] R. C. Sharma, Sunil, Y. D. Sharma, and R. S. Chandel, Arch. Mech. **54**, 298 (2002).
[14] Sunil, R. C. Sharma, and M. Pal, J. Porous Media **5**, 158 (2002).
[15] Sunil, R. C. Sharma, and R. S. Chandel, Z. Naturforsch. **57a**, 960 (2002).
[16] P. Kumar, R. Lal, and P. Sharma, Z. Naturforsch. **59**, 411 (2004).
[17] P. Kumar, P. Sharma, and M. Singh, Ganita Sandesh **20**, 178 (2006).
[18] P. Kumar, M. Singh, and G. Singh, Phys. Sep. Sci. Eng. **ID45347**, 7 (2007).
[19] P. Kumar and M. Singh, Z. Naturforsch. **64a**, 7 (2009).
[20] M. Singh and P. Kumar, Z. Naturforsch. **65a**, 220 (2010).
[21] M. Singh and P. Kumar, J. Porous Media **14**, 648 (2011).