Percolation in a Hierarchical Lattice

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We study the percolation in the hierarchical lattice of order N where the probability of connection between two nodes separated by a distance k is of the form $\min\{\alpha\beta^{-k},1\}$, $\alpha \geq 0$ and $\beta > 0$. We focus on the vertex degrees of the resulting percolation graph and on whether there exists an infinite component. For fixed β, we show that the critical percolation value $\alpha_c(\beta)$ is non-trivial, i.e., $\alpha_c(\beta) \in (0,\infty)$, if and only if $\beta \in (N,N^2)$.

Key words: Percolation; Random Graph; Degree; Hierarchical Lattice; Phase Transition.
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1. Introduction and the Model

Percolation issues in the Euclidean lattice $\mathbb{Z}^d$ were promoted in the mathematics literature by Broadbent and Hammersley a half century ago [1]. The infinity of the space of sites (or nodes) and its symmetric geometry are two principal features of this model (see e.g. [2, 3] for background). Some questions of percolation in other non-Euclidean infinite systems are formulated in [4]. The study of long-range percolation on $\mathbb{Z}^d$ started with the work [5] and leads to a range of interesting results in mathematical physics [6 – 11]. On the other hand, various hierarchical structures take an essential role in many applications in the physical, biological, and social sciences due to the multi-scale organization of many natural objects [12 – 15].

Recently, long-range percolation is studied on the hierarchical lattice Ω_N of order N (to be defined below), where classical methods for the usual lattice break down. In fact, the analysis of percolation on $\mathbb{Z}^d$ and other homogeneous graphs heavily relies on the symmetry assumptions see e.g. [2, 3, 16, 17], while on the hierarchical lattice these assumptions no longer exist and some techniques pertaining to ultra metric are needed. Besides, the percolation on Ω_N is possible only in the form of long range percolation. The asymptotic long-range percolation on Ω_N is addressed in [18] for $N \to \infty$. The work [19] and [20] analyze the long-range percolation on Ω_N for finite N using different connection probabilities and methodologies. The contact process on Ω_N for fixed N has been investigated in [21]. Following the above series of work, in this paper, we focus on the vertex degree of long-range percolation graph on Ω_N for fixed N as well as the phase transition of emerging an infinite component. To our knowledge, the probabilities of connection used here have not been considered in the study of phase transition on hierarchical lattices before.

For an integer $N \geq 2$, we define the set

$$\Omega_N := \Big\{\mathbf{x} = (x_1, x_2, \ldots) : x_i \in \{0,1,\ldots,N-1\},\ i = 1,2,\ldots,\ x_i \neq 0 \text{ only for finitely many } i\Big\}, \tag{1}$$

and define a metric d on it:

$$d(\mathbf{x},\mathbf{y}) = \begin{cases} 0, & \mathbf{x} = \mathbf{y}, \\ \max\{i : x_i \neq y_i\}, & \mathbf{x} \neq \mathbf{y}. \end{cases} \tag{2}$$

We remark here that $\Omega_N \subseteq \mathbb{R}^\infty$ for every integer N, where $\mathbb{R}^\infty$ is thought of as a space with all the sequences eventually end with 0. The pair (Ω_N, d) is called the hierarchical lattice of order N, which may be thought of as the set of leaves at the bottom of an infinite regular tree without a root, where the distance between two nodes is the number of levels (generations) from the bottom to their most recent common ancestor, see Figure 1.

Such a distance d satisfies the strong triangle inequality

$$d(\mathbf{x},\mathbf{y}) \leq \max\{d(\mathbf{x},\mathbf{z}), d(\mathbf{z},\mathbf{y})\} \tag{3}$$

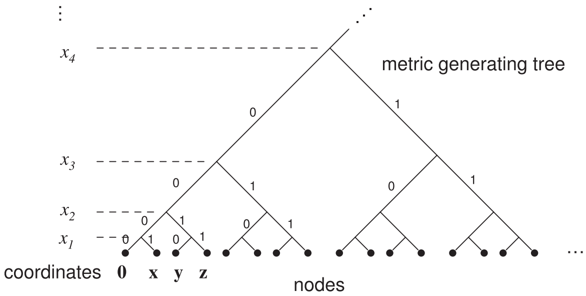

Fig. 1. Illustration of hierarchical lattice Ω_2 of order 2. The distances between three nodes $\mathbf{0} = (0,0,0,\dots)$ (the origin), $\mathbf{x} = (1,0,0,\dots)$ and $\mathbf{y} = (0,1,0,\dots)$ are $d(\mathbf{0},\mathbf{x}) = 1$ and $d(\mathbf{0},\mathbf{y}) = d(\mathbf{x},\mathbf{y}) = 2$. The numbers on the metric generating tree indicate the coordinates of nodes.

for any triple $\mathbf{x},\mathbf{y},\mathbf{z} \in \Omega_N$. Hence, (Ω_N,d) is an ultrametric (or non-Archimedean) space [22]. There are some interesting (but often counterintuitive) characteristic properties of an ultrametric space such as every triangle in it is isosceles and every point inside a ball is its center. From the ultrametricity of Ω_N, it is clear that for every $\mathbf{x} \in \Omega_N$ there are $(N-1)N^{k-1}$ nodes at distance k from it. The random walks in Ω_N can go far only by means of long-range jumps, which is clearly not the case in $\mathbb{Z}^d$; c.f. [19, 23].

Now consider a long-range percolation on Ω_N. For each $k \geq 1$, the probability of connection between $\mathbf{x}$ and $\mathbf{y}$ such that $d(\mathbf{x},\mathbf{y}) = k$ is given by

$$p_k = \min\left\{\frac{\alpha}{\beta^k}, 1\right\}, \tag{4}$$

where $0 \leq \alpha < \infty$ and $0 < \beta < \infty$, all connections being independent. Two vertices $\mathbf{x},\mathbf{y} \in \Omega_N$ are in the same component if there exists a finite sequence $\mathbf{x} = \mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n = \mathbf{y}$ of vertices such that each pair $(\mathbf{x}_{i-1}, \mathbf{x}_i)$, $i = 1,\dots,n$, of vertices forms an edge.

The rest of the letter is organized as follows. In Section 2, we present the vertex degree and phase transition results. Section 3 is devoted to the proofs. Concluding remarks are given in Section 4.

2. Main Results

For $\mathbf{x} \in \Omega_N$, denote by $D_\mathbf{x}$ the degree of node $\mathbf{x}$ in the resulting percolation graph. Let $\mathbf{0}$ be the origin in the space Ω_N with all the components being zero. Since $D_\mathbf{x}$ has the same distribution for every $\mathbf{x} \in \Omega_N$, we may study $D_\mathbf{0}$ instead of $D_\mathbf{x}$. Let $\mathbb{N}$ be the non-negative integers including 0, and denote by $\ell := \min\{k \in \mathbb{N} : \alpha \leq \beta^{k+1}\}$.

Theorem 1. *(Vertex degree)*

(i) If $\beta \leq N$ and $\alpha > 0$, then $P(D_\mathbf{0} = \infty) = 1$;
(ii) If $\beta > N$, then

$$ED_\mathbf{0} = N^\ell - 1 + \frac{\alpha(N-1)N^\ell}{(\beta - N)\beta^\ell}\,. \tag{5}$$

Let $|S|$ be the size of a set S. The connected component containing the node $\mathbf{x}$ is denoted by $C(\mathbf{x})$. Since, for every $\mathbf{x} \in \Omega_N$, $|C(\mathbf{x})|$ has the same distribution, it suffices to consider only $|C(\mathbf{0})|$. The percolation probability is defined as

$$\theta(\alpha,\beta) := P(|C(\mathbf{0})| = \infty)\,, \tag{6}$$

and the critical percolation value is defined as

$$\alpha_c(\beta) := \inf\{\alpha \geq 0 : \theta(\alpha,\beta) > 0\}\,. \tag{7}$$

The phase transition is established in the following result.

Theorem 2. *(Critical value)*

(i) If $\beta \leq N$, then $\alpha_c(\beta) = 0$;
(ii) If $N < \beta < N^2$, then $0 < \alpha_c(\beta) < \infty$;
(iii) If $\beta \geq N^2$, then $\alpha_c(\beta) = \infty$.

We should mention that a similar result has been established in [20, Theorem 1], where a different connection probability formation is used.

3. Proofs

In this section, we prove Theorems 1 and 2, respectively. Before proceeding, we introduce some notations.

For $\mathbf{x} \in \Omega_N$, define $B_r(\mathbf{x})$ the ball of radius r around $\mathbf{x}$, that is, $B_r(\mathbf{x}) = \{\mathbf{y} : d(\mathbf{x},\mathbf{y}) \leq r\}$. From this we make the following observations. Firstly, for any $\mathbf{x} \in \Omega_N$, $B_r(\mathbf{x})$ contains N^r vertices. Secondly, $B_r(\mathbf{x}) = B_r(\mathbf{y})$ if $d(\mathbf{x},\mathbf{y}) \leq r$. Finally, for any $\mathbf{x}$, $\mathbf{y}$ and r, we either have $B_r(\mathbf{x}) = B_r(\mathbf{y})$ or $B_r(\mathbf{x}) \cap B_r(\mathbf{y}) = \emptyset$.

For a set S of vertices, denote by $\bar{S} = \Omega_N \backslash S$ its complement. Let $C_n(\mathbf{x})$ be the component of vertices that are connected to $\mathbf{x}$ by a path using only vertices within $B_n(\mathbf{x})$. For disjoint sets $S_1, S_2 \subseteq \Omega_N$, we denote by

$S_1 \leftrightarrow S_2$ the event that at least one edge joins a vertex in S_1 to a vertex in S_2. $S_1 \not\leftrightarrow S_2$ means the event that such an edge does not exist, that is, S_1 and S_2 are not directly connected. Let $C_n^m(\mathbf{x})$ be the largest components in $B_n(\mathbf{x})$. If there are more than one such components, just take any one of them as $C_n^m(\mathbf{x})$. It is clear that $|C_n^m(\mathbf{x})| = \max_{\mathbf{y}\in B_n(\mathbf{x})} |C_n(\mathbf{y})|$.

Proof of Theorem 1. We begin with (i). Let E_k be the event that the origin $\mathbf{0}$ connects by an edge to at least one node at distance k. Therefore, by (4) and the fact that there are $(N-1)N^{k-1}$ nodes at distance k from $\mathbf{0}$, we have

$$P(E_k) = 1-(1-p_k)^{(N-1)N^{k-1}} = 1-\left(1-\min\left\{\frac{\alpha}{\beta^k},1\right\}\right)^{(N-1)N^{k-1}} . \tag{8}$$

Exploiting the inequality $1-\frac{1}{x} < \exp\left(-\frac{1}{x}\right)$ for $x > 0$, we obtain for $k \geq \ell+1$, $P(E_k) > 1 - \exp\left(-\frac{\alpha}{\beta^k}(N-1)N^{k-1}\right)$, while for $k \leq \ell$, $P(E_k) = 1$. If $\beta \leq N$, we can see that the sum $\sum_{k=1}^{\infty} P(E_k)$ diverges for any $\alpha > 0$. Since the events $\{E_k\}_{k\geq 1}$ are independent, it then follows from the Borel–Cantelli lemma that infinitely many of the event E_k occur with probability 1. Consequently, $P(D_{\mathbf{0}} = \infty) = 1$.

As for part (ii), we calculate as follows:

$$\begin{aligned} ED_{\mathbf{0}} &= \sum_{k=1}^{\infty}(N-1)N^{k-1}p_k \\ &= \sum_{k=1}^{\infty}(N-1)N^{k-1}\min\left\{\frac{\alpha}{\beta^k},1\right\} \\ &= \frac{N-1}{N}\sum_{k=1}^{\infty}N^k\min\left\{\frac{\alpha}{\beta^k},1\right\} \\ &= \frac{N-1}{N}\left(\sum_{k=1}^{\ell}N^k+\sum_{k=\ell+1}^{\infty}\frac{\alpha N^k}{\beta^k}\right) \\ &= N^{\ell}-1+\frac{\alpha(N-1)N^{\ell}}{(\beta-N)\beta^{\ell}} , \end{aligned} \tag{9}$$

for $\beta > N$, where the definition of ℓ, i.e., $\ell = \min\{k\in\mathbb{N} : \alpha \leq \beta^{k+1}\}$, is utilized in the last but two equality. $\square$

Proof of Theorem 2. Part (i) is a direct consequence of Theorem 1 (i). In fact, we know that $\theta(\alpha,\beta) = 1$ for any $\alpha > 0$ and $0 < \beta \leq N$. Hence, $\alpha_c(\beta) = 0$ for $0 < \beta \leq N$.

As for part (iii), we only need to show $\alpha_c(N^2) = \infty$ by virtue of the monotonicity. Take $\beta = N^2$, and then for any $\mathbf{x} \in \Omega_N$ and $j \in \mathbb{N}$, we obtain

$$\begin{aligned} P(B_j(\mathbf{x}) \leftrightarrow \overline{B_j(\mathbf{x})}) &= 1-\left(\prod_{k=j+1}^{\infty}(1-p_k)^{(N-1)N^{k-1}}\right)^{N^j} \\ &= 1-\left(\prod_{k=j+1}^{\infty}\left(1-\min\left\{\frac{\alpha}{N^{2k}},1\right\}\right)^{(N-1)N^{k-1}}\right)^{N^j} . \end{aligned} \tag{10}$$

Therefore, if $j+1 \leq \ell$, we have $P(B_j(\mathbf{x}) \leftrightarrow \overline{B_j(\mathbf{x})}) = 1$; if $j+1 > \ell$, we obtain from (10) that

$$\begin{aligned} &P(B_j(\mathbf{x}) \leftrightarrow \overline{B_j(\mathbf{x})}) \\ &= 1-\left(\prod_{k=j+1}^{\infty}\left(1-\frac{\alpha}{N^{2k}}\right)^{(N-1)N^{k-1}}\right)^{N^j} \\ &< 1-\exp\left(-\alpha N^j\frac{(N-1)}{N^2}\sum_{k=1}^{\infty}\frac{N^{j+k-1}}{N^{2(j+k-1)}-\alpha N^{-2}}\right) \end{aligned} \tag{11}$$

involving the inequality $\exp(-\frac{1}{x-1}) < 1-\frac{1}{x}$ for $x > 1$. Note that there exists a constant M (independent of j) sufficiently large such that the following inequality holds for $k = 1$ (and hence for any $k \geq 1$ by monotonicity):

$$N^{2(j+k-1)} - \alpha N^{-2} > \frac{1}{M}N^{2(j+k-1)} . \tag{12}$$

Combining (11) and (12), we have for $j+1 > \ell$,

$$P(B_j(\mathbf{x}) \leftrightarrow \overline{B_j(\mathbf{x})}) < 1-\exp\left(-\frac{\alpha}{N}\right) , \tag{13}$$

which is strictly less than 1 for any finite α.

Let $n_0 = 0$ and $n_{i+1} = \inf\{n \geq n_i : B_{n_i}(\mathbf{0}) \not\leftrightarrow \overline{B_n(\mathbf{0})}\}$. Since

$$\{|C(\mathbf{0})| = \infty\} \subseteq \bigcap_{i=0}^{\infty}\{B_{n_i}(\mathbf{0}) \leftrightarrow \overline{B_{n_i}(\mathbf{0})}\} , \tag{14}$$

it suffices to prove that there a.s. exists an i such that $B_{n_i}(\mathbf{0}) \not\leftrightarrow \overline{B_{n_i}(\mathbf{0})}$. Now that the events $\{B_{n_i}(\mathbf{0}) \leftrightarrow \overline{B_{n_i}(\mathbf{0})}\}$ are independent and all but finitely many of them have the same probability strictly less than 1 as per (13), we obtain

$$P(B_{n_i}(\mathbf{0}) \leftrightarrow \overline{B_{n_i}(\mathbf{0})},\ \text{for any } i \geq 0) = 0 , \tag{15}$$

as desired.

It remains to prove part (ii). The positivity of $\alpha_c(\beta)$ follows from the equality (5). In fact, the expected degree in (5) can be made strictly less than 1 by choosing α small enough (note that $\ell = 0$ when $\alpha \leq \beta$). Hence, by coupling with a subcritical branching process [24], the almost sure finiteness of the percolation cluster follows.

We prove the finiteness of $\alpha_c(\beta)$ in the sequel. The main technique to be used is an iteration involving the tail probability of binomial distributions [20, 25]. Since $\beta < N^2$, we choose $K \in \mathbb{N}$ and $\eta \in \mathbb{R}$ such that

$$\sqrt{\beta} < \eta \leq (N^K - 1)^{1/K}. \tag{16}$$

A ball of radius nK is said to be good if its largest component has size at least η^{nK}. Let s_n represent the probability that a ball of radius nK is good, i.e., $s_n = P(|C^m_{nK}(\mathbf{0})| \geq \eta^{nK})$. We set $s_0 = 1$ by convention. A ball of radius nK is said to be very good if it is good and its largest component connects by an edge to the largest component of the first (from left to right in Figure 1) good sub-ball in the same ball of radius $(n+1)K$. Clearly, the first good sub-ball of radius nK in a ball of radius $(n+1)K$ is very good. From (16), we may conclude that the ball $B_{(n+1)K}(\mathbf{0})$ is good if (a) it contains $N^K - 1$ good sub-balls of radius nK, and (b) all these good sub-balls are very good.

The number of good sub-balls of radius nK in a ball of radius $(n+1)K$ has a binomial distribution $\mathrm{Bin}(N^K, s_n)$ with parameters N^K and s_n. Clearly, given the collection of good sub-balls, the probability that the first such good sub-ball is very good is equal to 1. Fix any of the other good sub-balls B, and we obtain

$$\begin{aligned} P(B \text{ is not very good}) &\leq (1 - p_{(n+1)K})^{\eta^{nK}\eta^{nK}} \\ &= \left(1 - \min\left\{\frac{\alpha}{\beta^{(n+1)K}}, 1\right\}\right)^{\eta^{2nK}} \\ &< \exp\left(-\frac{\alpha}{\beta^K}\left(\frac{\eta^2}{\beta}\right)^{nK}\right) := \varepsilon_n\,, \end{aligned} \tag{17}$$

since the distance between two vertices in a ball of radius $(n+1)K$ is at most $(n+1)K$, and the largest component of a good sub-ball contains at least η^{nK} vertices. Hence, the probability for any of the other good sub-balls B to be very good is at leat $1 - \varepsilon_n$. Consequently, the number of very good sub-balls is stochastically larger than a random variable obeying binomial distribution $\mathrm{Bin}(N^K, s_n(1-\varepsilon_n))$. From the above comments (a) and (b) and the definition of s_n, it follows that

$$s_{n+1} \geq P(\mathrm{Bin}(N^K, s_n(1-\varepsilon_n)) \geq N^K - 1)\,. \tag{18}$$

Generally, we have

$$P(\mathrm{Bin}(n,p) \geq n-1) \geq 1 - \binom{n}{2}(1-p)^2, \tag{19}$$

and then by (18) and writing $\xi_n = 1 - s_n$, we obtain

$$\begin{aligned} \xi_{n+1} &= 1 - s_{n+1} \leq \binom{N^K}{2}(1 - s_n + s_n\varepsilon_n)^2 \\ &\leq \binom{N^K}{2}(1 - s_n + \varepsilon_n)^2 = \binom{N^K}{2}(\xi_n + \varepsilon_n)^2. \end{aligned} \tag{20}$$

Choose $\delta > 0$ small enough so that $4\binom{N^K}{2} \leq \frac{1}{\delta}$, and then choose α large enough so that (c) $\varepsilon_n \leq \delta^{n+1}$ and (d) $\xi_1 \leq \delta^2$ hold. To see (c), note that $\beta < \eta^2$ and

$$\begin{aligned} \varepsilon_n &= \left(\exp\left(-\left(\frac{\eta^2}{\beta}\right)^{nK}\right)\right)^{\alpha\beta^{-K}} \\ &\leq \left(\left(\frac{\beta}{\eta^2}\right)^{\alpha K\beta^{-K}}\right)^n. \end{aligned} \tag{21}$$

To see (d), note that $\lim_{\alpha\to\infty} \varepsilon_0 = 0$, $\xi_0 = 0$ and

$$\xi_1 = 1 - s_1 \leq \binom{N^K}{2}(\xi_0 + \varepsilon_0)^2 \tag{22}$$

using (20). Inductively, if $\xi_n \leq \delta^{n+1}$, then

$$\begin{aligned} \xi_{n+1} &\leq \binom{N^K}{2}(\xi_n + \varepsilon_n)^2 \\ &\leq 4\binom{N^K}{2}(\delta^{n+1})^2 \leq \delta^{2n+1} \leq \delta^{n+2}, \end{aligned} \tag{23}$$

which implies that $\xi_n \leq \delta^{n+1}$ for all $n \in \mathbb{N}$. Therefore, when α is large enough, s_n converges to 1 exponentially fast, and thus, $s_n(1-\varepsilon_n)$ converges to 1 exponentially fast.

Let $t_n := P(|C_{nK}(\mathbf{0})| \geq \eta^{nK})$. We claim that

$$t_{n+1} \geq t_n \cdot P(\mathrm{Bin}(N^K - 1, s_n(1-\varepsilon_n)) \geq N^K - 2)\,. \tag{24}$$

In fact, if $|C_{nK}(\mathbf{0})| \geq \eta^{nK}$, then $B_{nK}(\mathbf{0})$ is the first good sub-ball in the derivation above. If this component is connected to at least $N^K - 2$ other large components in $B_{(n+1)K}(\mathbf{0})$ as above, then the component containing the origin in $B_{(n+1)K}(\mathbf{0})$ is large enough, which has size at least

$$\eta^{nK}(N^K - 1) \geq \eta^{nK}\eta^K = \eta^{(n+1)K}. \tag{25}$$

Thus, the inequality (24) follows.

Recall that a simple coupling gives

$$\begin{aligned} &P(\mathrm{Bin}(N^K - 1, s_n(1-\varepsilon_n)) \geq N^K - 2) \\ &\geq P(\mathrm{Bin}(N^K, s_n(1-\varepsilon_n)) \geq N^K - 1). \end{aligned} \tag{26}$$

Hence, we derive that the right-hand side of (26) converges to 1 exponentially fast by exploiting (19) and the fact that $s_n(1-\varepsilon_n)$ converges to 1 exponentially fast. It then follows from (24) that, for α large enough,

$$\lim_{n\to\infty} t_n > 0, \tag{27}$$

which readily yields $\alpha_c(\beta) < \infty$ as desired. □

4. Concluding Remarks

The use of percolation theory in mathematical physics has long been recognized. In this paper, we characterize the vertex degree of a hierarchical long-range percolation graph as well as the phase transition of this long-rang percolation model. The critical percolation value $\alpha_c(\beta)$ is shown to be non-trivial if and only if $\beta \in (N, N^2)$. One of the important issues left open is the uniqueness of the infinite component. In addition, the graph distance and diameter of the percolation graph are interesting future work. One reviewer suggested a variant model with connection probability of the form

$$p_k = \frac{\alpha\beta^{-k}}{\lambda + \alpha\beta^{-k}} \tag{28}$$

for some $\lambda > 0$. A natural question to ask would be whether this model yields similar phase transition phenomenon in Ω_N? Can we in turn estimate the parameter λ based on some likelihood functions as done in [26]?

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