Dynamic Properties of the Predator–Prey Discontinuous Dynamical System

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In this work, we study the dynamic properties (equilibrium points, local and global stability, chaos and bifurcation) of the predator–prey discontinuous dynamical system. The existence and uniqueness of uniformly Lyapunov stable solution will be proved.

Key words: Discontinuous Dynamical Systems; Predator–Prey Discontinuous Dynamical System; Existence and Uniqueness; Uniform and Local Stability; Equilibrium Points; Chaos and Bifurcations.

1. Introduction

The dynamical properties of the predator–prey discrete dynamical system have been intensively studied by some authors, see for example [1 – 9] and the references therein.

Here we are concerned with the predator–prey discontinuous dynamical system

$$\begin{aligned} x(t) &= ax(t-r_1)(1-x(t-r_1)) - bx(t-r_1)y(t-r_2) \\ &= f(x(t-r_1), y(t-r_2)), \quad t \in (0,T], \end{aligned} \tag{1}$$

$$\begin{aligned} y(t) &= -cy(t-r_2) + dx(t-r_1)y(t-r_2) \\ &= g(x(t-r_1), y(t-r_2)), \quad t \in (0,T], \end{aligned} \tag{2}$$

with the initial values

$$x(t) = x_0, \;\; y(t) = y_0, \;\; t \le 0, \tag{3}$$

where $x, y \ge 0$ and a, b, c, d, r_1, and r_2 are positive constants and $T < \infty$.

We study the dynamic properties (equilibrium points, local and global stability, chaos and bifurcation) of the discontinuous dynamical system (1) – (3). The existence of a unique uniformly stable solution is also proved.

2. Discontinuous Dynamical Systems

The discrete dynamical system

$$x_n = ax_{n-1}, \;\; n = 1, 2, \ldots, \tag{4}$$

$$x_0 = c, \tag{5}$$

has the discrete solution

$$x_n = a^n x_0, \;\; n = 1, 2, \ldots. \tag{6}$$

The more general dynamical system

$$x(t) = ax(t-r), \;\; t \in (0,T] \text{ and } r > 0, \tag{7}$$

$$x(t) = x_0, \;\; t \le 0, \tag{8}$$

has the discontinuous (integrable) solution

$$x(t) = a^{1+[\frac{t}{r}]} x_0 \in L_1(0,T], \tag{9}$$

where $[.]$ is the bract function.

The nonlinear discrete dynamical system

$$x_n = f(x_{n-1}), \;\; n = 1, 2, \ldots, \tag{10}$$

with the initial data (5) has the discrete solution

$$x_n = f^n(x_0), \;\; n = 1, 2, \ldots, \tag{11}$$

but the nonlinear problem

$$x(t) = f(x(t-r)), \;\; r > 0, \tag{12}$$

with the initial data (8) is more general than the problem (10) – (5) and has the discontinuous (integrable) solution

$$x_n = f^{1+[\frac{t}{r}]}(x_0) \in L_1(0,T]. \tag{13}$$

So, we can call the systems (7)–(8) and (12)–(8) discontinuous dynamical systems (see [4]).

Definition. The discontinuous dynamical system is the problem of the retarded functional equation

$$x(t) = f(t, x(t-r)), \ \ r,t > 0, \tag{14}$$
$$x(t) = g(t), \ \ t \in (-\infty, 0]. \tag{15}$$

3. Existence and Uniqueness

The problem (1)–(3) can be written in the matrix form

$$\begin{aligned}&(x(t), y(t))^{\mathrm{T}} = \\ &(ax(t-r_1)(1-x(t-r_1)) - bx(t-r_1)y(t-r_2), \\ &-cy(t-r_2) + dx(t-r_1)y(t-r_2))^{\mathrm{T}}\end{aligned} \tag{16}$$

and

$$(x(t), y(t))^{\mathrm{T}} = (x_0, y_0)^{\mathrm{T}}, \ \ t \le 0, \tag{17}$$

where T is the transpose of the matrix.

Let $L^1[0,T]$ be the class of Lebesgue integrable functions defined on $[0,T]$.

Let X be the class of columns vectors $(x(t), y(t))^{\mathrm{T}}$, $x, y \in L^1[0,T]$ with the equivalent norm

$$\begin{aligned}||(x,y)^{\mathrm{T}}||_X &= ||x|| + ||y|| \\ &= \int_0^T \mathrm{e}^{-Nt}|x(t)|\,\mathrm{d}t + \int_0^T \mathrm{e}^{-Nt}|y(t)|\,\mathrm{d}t, \\ &N > 0.\end{aligned}$$

Let $D \subset R^+$, $D = \{x, y \ge 0, \max\{x,y\} \le A\}$, and $a_1 = \max\{a,b,c,d\}$.

Now we have the following existence theorem.

Theorem 2.1. The problem (16)–(17) has a unique solution $(x,y)^{\mathrm{T}} \in X$.

Proof. Define the operator $F : X \to X$ by

$$\begin{aligned}&F(x(t), y(t))^{\mathrm{T}} = \\ &(ax(t-r_1)(1-x(t-r_1)) - bx(t-r_1)y(t-r_2), \\ &-cy(t-r_2) + dx(t-r_1)y(t-r_2))^{\mathrm{T}}\end{aligned}$$

then by direct calculations, we can get

$$||F(x,y)^{\mathrm{T}} - F(u,v)^{\mathrm{T}}||_X \le K||(x,y)^{\mathrm{T}} - (u,v)^{\mathrm{T}}||_X$$

where $K = a_1(1+5A)\mathrm{e}^{-Nr}$ and $r = \max\{r_1, r_2\}$.

Choose N large enough such that $K < 1$, then by the contraction fixed point theorem ([10]) the problem (16)–(17) has a unique solution $(x,y)^{\mathrm{T}} \in X$.

3.1. Uniform Stability

Here we prove the uniform Lyapunov stability of the solution of the problem (16)–(17).

Theorem 2.2. The solution of the problem (16)–(17) is uniformly Lyapunov stable in the sense that

$$|x_0 - x_0^*| + |y_0 - y_0^*| \le \delta \Rightarrow ||(x,y) - (x^*,y^*)||_X \le \varepsilon,$$

where $(x^*(t), y^*(t))^{\mathrm{T}}$ is the solution of the problem

$$\begin{aligned}&(x(t), y(t))^{\mathrm{T}} = \\ &(ax(t-r_1)(1-x(t-r_1)) - bx(t-r_1)y(t-r_2), \\ &-cy(t-r_2) + dx(t-r_1)y(t-r_2))^{\mathrm{T}}\end{aligned}$$

and

$$(x(t), y(t))^{\mathrm{T}} = (x_o^*, y_o^*)^{\mathrm{T}}, \ \ t \le 0.$$

Proof. Direct calculations give

$$\begin{aligned}||(x,y)^{\mathrm{T}} - (x^*,y^*)^{\mathrm{T}}||_X \le &\frac{1}{N}(|x_0 - x_0^*| + |y_0 - y_0^*|) \\ &+ K||(x,y)^{\mathrm{T}} - (u,v)^{\mathrm{T}}||_X\end{aligned}$$

which implies that

$$\begin{aligned}||F(x,y)^{\mathrm{T}} - F(x^*,y^*)^{\mathrm{T}}||_X \le &\frac{1}{N}(1-K)^{-1}(|x_0 - x_0^*| \\ &+ |y_0 - y_0^*|) \le \varepsilon,\end{aligned}$$

$\varepsilon = \frac{1}{N}(1-K)^{-1}\delta$.

4. Equilibrium Points and Local Stability

The equilibrium solution of the discontinuous dynamical system (1)–(3) is given by

$$\begin{aligned}x_{\mathrm{eq}} &= f(x_{\mathrm{eq}}, y_{\mathrm{eq}}), \\ y_{\mathrm{eq}} &= g(x_{\mathrm{eq}}, y_{\mathrm{eq}}),\end{aligned}$$

which are

$$\begin{aligned}&E_0(0,0), \ \ E_1\left(\frac{a-1}{a}, 0\right), \\ &E_2\left(\frac{1+c}{d}, \frac{a}{b}\left(1 - \frac{1+c}{d}\right) - \frac{1}{b}\right).\end{aligned}$$

The equilibrium solution of the discontinuous dynamical system (1)–(3) is locally asymptotically stable if

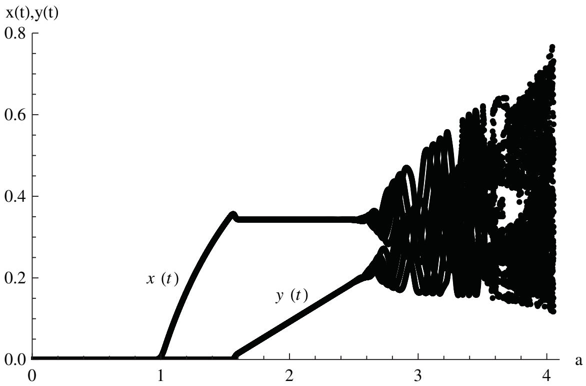

Fig. 1. Bifurcation diagram of (1)–(3) with respect to a, $r_1 = r_2 = 1$, and $t \in [0, 100]$.

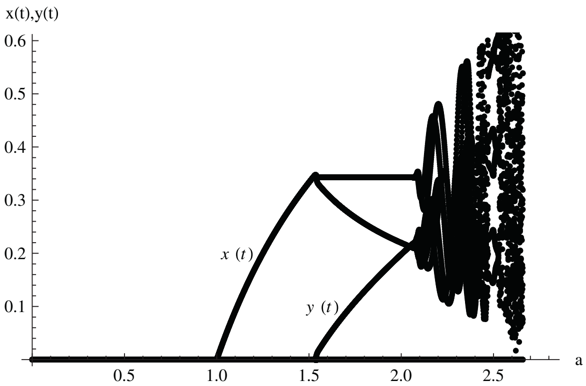

Fig. 2. Bifurcation diagram of (1)–(3) with respect to a, $r_1 = 0.25$, $r_2 = 0.5$, and $t \in [0, 100]$.

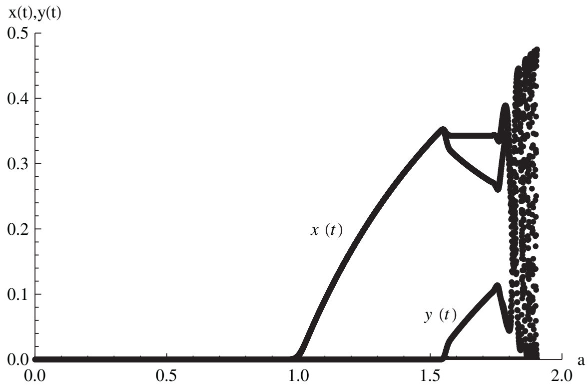

Fig. 3. Bifurcation diagram of (1)–(3) with respect to a, $r_1 = 1$, $r_2 = 0.75$, and $t \in [0, 100]$.

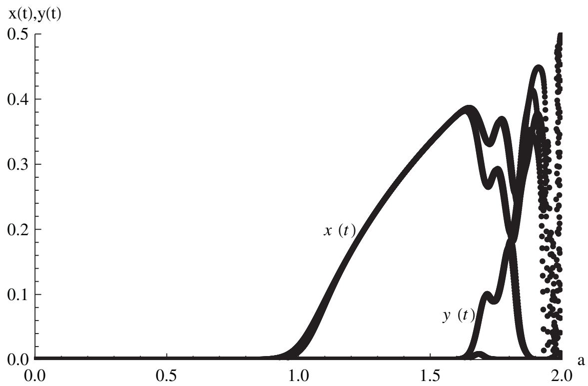

Fig. 4. Bifurcation diagram of (1)–(3) with respect to a, $r_1 = 1$, $r_2 = 0.75$, and $t \in [0, 25]$.

all the roots λ of the following equation satisfy $|\lambda| < 1$, where

$$\left(\lambda^{-r_1}\frac{\partial f(x,y)}{\partial x} - 1\right)\left(\lambda^{-r_2}\frac{\partial g(x,y)}{\partial y} - 1\right) - \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial g}{\partial x}\right)\lambda^{-r_1-r_2} = 0, \tag{18}$$

where all the derivatives in (18) are calculated at the equilibrium values.

5. Bifurcation and Chaos

In this section, some numerical simulation results are presented to show that dynamic behaviours of the discontinuous dynamical system (1)–(3) change for different values of r_1, r_2, and T. To do this, we will use the bifurcation diagrams.

The bifurcation diagrams of (1)–(3) in the $(a - xy)$ plane are showing the dynamical behaviour of the predator–prey systems as a, r_1, r_2 are varying and the fixed parameters are $b = 3.4$, $c = 0.2$, $d = 3.5$, $(x_0, y_0) = (0.1, 0.2)$, see Figures 1–4.

From these figures we deduce that the change of r_1, r_2, and T has an effect on the stability of the system: depending on the parameter set, it occurs a bifurcation point, an aperiodic behaviour or a chaotic behaviour.

6. Conclusions

The discrete dynamical system of the predator–prey model describes the dynamical properties for the case $r_1 = r_2$ and discrete time $t = 1, 2 \ldots$.

On the other hand, the discontinuous dynamical system of the predator–prey model describes the dynamical properties for different values of the delayed parameters r_1 and r_2 and the time continuous.

Figure 1 agrees with standard results. This confirms the correctness of our computation.

The results of the other figures are a new behaviour (there is no analytic explanation for this behaviour). This shows the richness of the models of discontinuous dynamical systems.

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