Construction of Quasi-Periodic Wave Solutions for Differential-Difference Equation

Y. C. Hon[a] and Qi Wang[b]

[a] Department of Mathematics, Tat Chee Avenue 80, City University of Hong Kong, Hong Kong, PR China

[b] Department of Applied Mathematics, Shanghai University of Finance and Economics, Shanghai 200433, PR China

Reprint requests to Q. W.; E-mail: wangqee@gmail.com

Z. Naturforsch. **67a,** 21 – 28 (2012) / DOI: 10.5560/ZNA.2011-0063
Received August 25, 2011

Based on the use of the Hirota bilinear method and the Riemann theta function, we develop in this paper a constructive method for obtaining explicit quasi-periodic wave solutions of a new integrable generalized differential-difference equation. Analysis on the asymptotic property of the quasi-periodic wave solutions is given, and it is shown that the quasi-periodic wave solutions converge to the soliton solutions under certain conditions.

Key words: Hirota Bilinear Method; Riemann Theta Function; Quasi-Periodic Wave Solutions.
PACS numbers: 03.65.Ge; 02.30.Ik

1. Introduction

Exact solutions of nonlinear equations have proven to be useful in simulating many real physical phenomena. The bilinear method developed by Hirota provides a direct approach to construct exact solutions of nonlinear equations. In other words, if a nonlinear equation is written in bilinear forms by using an appropriate dependent variable transformation, its multi-soliton solutions can usually be obtained [1 – 9]. Recently, based on the Hirota bilinear method, Nakamura in his two serial papers [10, 11] proposed a convenient way to construct a kind of quasi-periodic solutions of nonlinear equation from which the quasi-periodic wave solutions of the Korteweg–de Vries (KdV) equation and the Boussinesq equation were obtained. Following this work, Dai, Zhang, Fan, and Ma et al. extended the method to other equations such as Kadomtsev–Petviashvili equation, breaking soliton equation, Boussinesq equation, asymmetric Nizhnik–Novikov–Veselov equation, and Bogoyavlenskii equations [12 – 16]. The success of this method depends on circumventing the complicated algebro-geometric theory to directly give explicit quasi-periodic wave solutions. It can, however, be shown that all of the parameters appearing in the periodic wave solutions are conditionally free variables. In the case of quasi-periodic solutions, it involves some Riemann constants which are difficult to be determined explicitly. To the knowledge of the authors, there is very few work available for constructing quasi-periodic solutions of differential-difference equations [16, 17].

Based on the use of the Riemann theta function, we extend in this paper the Hirota bilinear method to construct quasi-periodic solutions of differential-difference equations. For illustration, a new integrable differential-difference equation [18] is chosen to demonstrate the feasibility of the proposed construction method. It will be shown that the quasi-periodic wave solutions converge to the soliton solutions under certain conditions.

This paper is organized as follows. In Section 2, we briefly introduce the bilinear form of differential-difference equation and the Riemann theta function. The Hirota bilinear method and Riemann theta function are then used in Section 3 to construct the quasi-periodic wave solutions for the differential-difference equation. Finally, we give an analysis on the asymptotic behaviour of the quasi-periodic wave solutions in the last Section 4 in which it has rigorously been shown that the periodic solutions tend to the well-known soliton solutions under a ‘small amplitude’ limit.

2. Bilinear Form and Riemann Theta Function

We first consider an integrable differential-difference equation [18] whose bilinear form is

$$\left[D_tD_x + AD_t\sinh(D_n) - 4\sinh^2\left(\frac{1}{2}D_n\right) + c\right] f(n)\cdot f(n) = 0, \tag{1}$$

where A is an arbitrary constant and c is an integration constant. The bilinear differential operator D_x, D_t, and the difference operator e^{D_n} are defined respectively by

$$\begin{aligned} &D_x^m D_t^{m'} f(x,t)\cdot g(x,t) \\ &\quad = (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^{m'} f(x,t)g(x',t')|_{x'=x,t'=t}, \\ &\mathrm{e}^{\delta D_n} f(n)\cdot g(n) = \mathrm{e}^{\delta(\partial_n - \partial_{n'})} f(n)g(n')|_{n'=n} \\ &\quad = f(n+\delta)g(n-\delta), \\ &\sinh(\delta D_n) f(n)\cdot g(n) = \frac{1}{2}(\mathrm{e}^{\delta D_n} - \mathrm{e}^{-\delta D_n}) f(n)\cdot g(n) \\ &\quad = \frac{1}{2}(f(n+\delta)g(n-\delta) - f(n-\delta)g(n+\delta)). \end{aligned} \tag{2}$$

The bilinear form (1) arises from many famous differential-difference equations. In particular, if $f(n;x,t) = f(n;t)$, (1) becomes a special case of an extended Lotka-Volterra equation [19]; when $A = 0$ and taking the transformation of the solution

$$\tau_n = \frac{f(n-1)f(n+1)}{f^2(n)}, \tag{3}$$

(1) becomes the two-dimensional Toda equation [2]

$$\frac{\partial^2 \tau_n}{\partial x \partial t} = \exp(\tau_{n-1} - \tau_n) - \exp(\tau_n - \tau_{n+1}). \tag{4}$$

It had been shown in [18] that (1) is integrable in the sense of Bäcklund transformation.

The operators D_x, D_t, e^{D_n}, and $\sinh(\delta D_n)$ have the following nice properties when acting on exponential functions:

$$\begin{aligned} &D_x^m D_t^{m'} \mathrm{e}^{\xi_1}\cdot \mathrm{e}^{\xi_2} = (\alpha_1 - \alpha_2)^m (\omega_1 - \omega_2)^{m'} \mathrm{e}^{\xi_1+\xi_2}, \\ &\mathrm{e}^{\delta D_n}\mathrm{e}^{\xi_1}\cdot \mathrm{e}^{\xi_2} = \mathrm{e}^{\delta(\nu_1-\nu_2)}\mathrm{e}^{\xi_1+\xi_2}, \\ &\sinh(\delta D_n)\mathrm{e}^{\xi_1}\cdot \mathrm{e}^{\xi_2} = \sinh[\delta(\nu_1-\nu_2)]\mathrm{e}^{\xi_1+\xi_2}, \end{aligned} \tag{5}$$

where $\xi_j = \alpha_j x + \omega_j t + \nu_j n + \sigma_j$, $j = 1,2$. More generally, we have

$$\begin{aligned} &G(D_x, D_t, \sinh(\delta D_n))\mathrm{e}^{\xi_1}\cdot \mathrm{e}^{\xi_2} \\ &\quad = G(\alpha_1-\alpha_2, \omega_1-\omega_2, \sinh(\delta(\nu_1-\nu_2)))\mathrm{e}^{\xi_1+\xi_2}, \end{aligned} \tag{6}$$

where $G(D_x, D_t, \sinh(\delta D_n))$ is a polynomial function with respect to the operators D_x, D_t, and $\sinh(\delta D_n)$.

In the special case of $c = 0$, (1) admits the following one-soliton solution:

$$f(n) = 1 + \exp\left(\alpha x + 4\frac{\sinh^2\left(\frac{1}{2}\nu\right)}{\alpha + A\sinh(\nu)}t + \nu n + \sigma\right). \tag{7}$$

The following one-dimensional Riemann theta function plays a central role in the quasi-periodicity of the solutions:

$$\begin{aligned} \vartheta(\xi,\varepsilon,s|\tau) = \sum_{m\in\mathbb{Z}} &\exp[2\pi\mathrm{i}(\xi+\varepsilon)(m+s) \\ &- \pi\tau(m+s)^2], \end{aligned} \tag{8}$$

where $m \in \mathbb{Z}$, $s, \varepsilon \in \mathbb{C}$, and $\xi = \alpha x + \omega t + \nu n + \sigma$ is a complex phase variable depending on the continuous variables x, t, and discretized variable n. Here, $\tau > 0$ is called the period matrix of the Riemann theta function. For simplicity, in the case when $s = \varepsilon = 0$, we denote

$$\vartheta(\xi,\tau) = \vartheta(\xi,0,0|\tau). \tag{9}$$

Definition 1. A function $g(t)$ on $\mathbb{C}$ is said to be quasi-periodic in t with fundamental periods $T_1,\ldots,T_k \in \mathbb{C}$ if $T_1,\ldots,T_k$ are linearly dependent over $\mathbb{Z}$ and there exists a function $G(y_1,\ldots,y_k) \in \mathbb{C}^k$, such that

$$\begin{aligned} &G(y_1,\ldots,y_j+T_j,\ldots,y_k) \\ &\quad = G(y_1,\ldots,y_j,\ldots,y_k), \end{aligned} \tag{10}$$

for all $(y_1,\ldots,y_k) \in \mathbb{C}^k$. If we denote

$$G(t,\ldots,t,\ldots,t) = g(t), \tag{11}$$

then $g(t)$ becomes periodic with the period T if and only if $T_j = m_j T$ for some $m_j \in \mathbb{Z}$.

Proposition 1. The Riemann theta function $\vartheta(\xi,\tau)$ defined in (8) has the periodic properties [21, 22]

$$\begin{aligned} &\vartheta(\xi+1+\mathrm{i}\tau,\tau) \\ &\quad = \exp(-2\pi\mathrm{i}\xi + \pi\tau)\vartheta(\xi,\tau). \end{aligned} \tag{12}$$

Proposition 2. The meromorphic functions $F(\xi)$ on $\mathbb{C}$ satisfy:

$$\begin{aligned} &\text{(i)} \quad F(\xi) = \partial_\xi^2 \ln\vartheta(\xi,\tau), \ \ \xi\in\mathbb{C}; \\ &\text{(ii)} \quad F(\xi) = \partial_\xi \ln\frac{\vartheta(\xi+e,\tau)}{\vartheta(\xi+h,\tau)}, \ \ \xi,e,h\in\mathbb{C}; \\ &\text{(iii)} \quad F(\xi) = \frac{\vartheta(\xi+e,\tau)\vartheta(\xi-e,\tau)}{\vartheta(\xi,\tau)^2}, \ \ \xi,e\in\mathbb{C}, \end{aligned} \tag{13}$$

which implies that

$$F(\xi+1+\mathrm{i}\tau)=F(\xi),\ \ \xi\in\mathbb{C}. \tag{14}$$

In other words, the meromorphic functions $F(\xi)$ are quasi-periodic functions with two fundamental periods 1 and $\mathrm{i}\tau$ [17].

3. Quasi-Periodic Solution

Consider the Riemann theta function solution for the differential-difference equation in bilinear form (1),

$$f(n)=\vartheta(\xi,\tau)=\sum_{m=-\infty}^{\infty}\mathrm{e}^{2\pi\mathrm{i}m\xi-\pi m^2\tau},$$

where $m\in\mathbb{Z}$, $\tau>0$, and $\xi=\alpha x+\omega t+\nu n+\sigma$.

Substitute the above $f(n)$ into (1) gives

$$\begin{aligned}
&G(D_x,D_t,\sinh(\delta D_n))f(n)\cdot f(n)=G(D_x,D_t,\sinh(\delta D_n))\sum_{m'=-\infty}^{\infty}\mathrm{e}^{2\pi\mathrm{i}m'\xi-\pi m'^2\tau}\cdot\sum_{m=-\infty}^{\infty}\mathrm{e}^{2\pi\mathrm{i}m\xi-\pi m^2\tau}\\
&=\sum_{m'=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}G(D_x,D_t,\sinh(\delta D_n))\,\mathrm{e}^{2\pi\mathrm{i}m'\xi-\pi m'^2\tau}\cdot\mathrm{e}^{2\pi\mathrm{i}m\xi-\pi m^2\tau}\\
&=\sum_{m'=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}G\left(2\pi\mathrm{i}(m'-m)\alpha,2\pi\mathrm{i}(m'-m)\omega,\sinh[2\pi\mathrm{i}\delta\nu(m'-m)]\right)\mathrm{e}^{2\pi\mathrm{i}(m'+m)\xi-\pi(m'^2+m^2)\tau}\\
&\overset{m'+m=l'}{=}\sum_{l'=-\infty}^{\infty}\sum_{m'=-\infty}^{\infty}G(2\pi\mathrm{i}(2m'-l')\alpha,2\pi\mathrm{i}(2m'-l')\omega,\sinh[2\pi\mathrm{i}\delta\nu(2m'-l')])\,\mathrm{e}^{2\pi\mathrm{i}l'\xi-\pi[m'^2+(l'-m')^2]\tau}\\
&\overset{l'=2l+\mu}{=}\sum_{l=-\infty}^{\infty}\sum_{\mu=0,1}\sum_{m'=-\infty}^{\infty}G\left(4\pi\mathrm{i}\left(m'-l-\frac{\mu}{2}\right)\alpha,4\pi\mathrm{i}\left(m'-l-\frac{\mu}{2}\right)\omega,\sinh\left[4\pi\mathrm{i}\delta\nu\left(m'-l-\frac{\mu}{2}\right)\right]\right)\\
&\qquad\cdot\mathrm{e}^{4\pi\mathrm{i}\left(l+\frac{\mu}{2}\right)\xi-\pi\left[m'^2+(2l+\mu-m')^2\right]\tau}.
\end{aligned} \tag{15}$$

Let $m'=h+l$, and using the relations

$$h+l=\left[h-\frac{\mu}{2}\right]+\left[l+\frac{\mu}{2}\right],\ \ h-l-\mu=\left[h-\frac{\mu}{2}\right]-\left[l+\frac{\mu}{2}\right], \tag{16}$$

we finally obtain that

$$\begin{aligned}
&G(D_x,D_t,\sinh(\delta D_n))f(n)\cdot f(n)\\
&=\sum_{l=-\infty}^{\infty}\left\{\sum_{\mu=0,1}\sum_{h=-\infty}^{\infty}G\left(4\pi\mathrm{i}\left(h-\frac{\mu}{2}\right)\alpha,4\pi\mathrm{i}\left(h-\frac{\mu}{2}\right)\omega,\sinh\left[4\pi\mathrm{i}\delta\nu\left(h-\frac{\mu}{2}\right)\right]\right)\mathrm{e}^{-2\pi\left(h-\frac{\mu}{2}\right)^2\tau}\right\}\\
&\quad\cdot\mathrm{e}^{4\pi\mathrm{i}\left(l+\frac{\mu}{2}\right)\xi-2\pi\left(l+\frac{\mu}{2}\right)^2\tau}\\
&=\sum_{l=-\infty}^{\infty}C(\alpha,\omega,\nu,\mu)\,\mathrm{e}^{4\pi\mathrm{i}\left(l+\frac{\mu}{2}\right)\xi-2\pi\left(l+\frac{\mu}{2}\right)^2\tau},
\end{aligned} \tag{17}$$

where

$$C(\alpha,\omega,\nu,\mu)=\sum_{\mu=0,1}\sum_{h=-\infty}^{\infty}G\left(4\pi\mathrm{i}\left(h-\frac{\mu}{2}\right)\alpha,4\pi\mathrm{i}\left(h-\frac{\mu}{2}\right)\omega,\sinh\left[4\pi\mathrm{i}\delta\nu\left(h-\frac{\mu}{2}\right)\right]\right)\mathrm{e}^{-2\pi\left(h-\frac{\mu}{2}\right)^2\tau}. \tag{18}$$

It can be observed that if the following equations $C(\alpha,\omega,\nu,\mu)=0$ are satisfied, for all possible combinations $\mu=0,1$, then $\vartheta(\xi,\tau)$ is a solution of the bilinear equation (1). On the other hand, the equations $G(\alpha,\omega,\nu,0)=0$ and $G(\alpha,\omega,\nu,1)=0$ can be explicitly written as

$$\sum_{h=-\infty}^{\infty}\{-16\pi^2h^2\omega\alpha+4A\pi\mathrm{i}h\omega\sinh(4\pi\mathrm{i}\nu h)-4\sinh^2(2\pi\mathrm{i}\nu h)+c\}\mathrm{e}^{-2\pi h^2\tau}=0, \tag{19}$$

$$\sum_{h=-\infty}^{\infty}\left\{-16\pi^2\left(h-\frac{1}{2}\right)^2\omega\alpha+4A\pi\mathrm{i}\left(h-\frac{1}{2}\right)\omega\sinh\left[4\pi\mathrm{i}\nu\left(h-\frac{1}{2}\right)\right]-4\sinh^2\left[2\pi\mathrm{i}\nu\left(h-\frac{1}{2}\right)\right]+c\right\}$$
$$\cdot\,\mathrm{e}^{-2\pi\left(h-\frac{1}{2}\right)^2\tau}=0, \tag{20}$$

i.e.

$$\vartheta_1''(0,\lambda)\omega\alpha+A\sinh(D_n)\vartheta_1'(0,\lambda)\omega$$
$$-4\sinh^2\left(\frac{1}{2}D_n\right)\vartheta_1(0,\lambda)+c\vartheta_1(0,\lambda)=0, \tag{21}$$

$$\vartheta_2''(0,\lambda)\omega\alpha+A\sinh(D_n)\vartheta_2'(0,\lambda)\omega$$
$$-4\sinh^2\left(\frac{1}{2}D_n\right)\vartheta_2(0,\lambda)+c\vartheta_2(0,\lambda)=0, \tag{22}$$

where the prime denotes the partial derivate ∂_ξ and

$$\begin{aligned}\lambda&=\mathrm{e}^{-\frac{1}{2}\pi\tau},\\ \vartheta_1(\xi,\lambda)&=\vartheta(2\xi,2\tau)\\ &=\sum_{h=-\infty}^{+\infty}\lambda^{4h^2}\exp(4\pi\mathrm{i}h\xi),\\ \vartheta_2(\xi,\lambda)&=\vartheta\left(2\xi,0,-\frac{1}{2}\middle|2\tau\right)\\ &=\sum_{h=-\infty}^{+\infty}\lambda^{(2h-1)^2}\exp[2\pi\mathrm{i}(2h-1)\xi].\end{aligned} \tag{23}$$

By introducing the notations

$$\begin{aligned}a_{11}&=\vartheta_1''(0,\lambda)\alpha+A\sinh(D_n)\vartheta_1'(0,\lambda),\\ a_{12}&=\vartheta_1(0,\lambda),\\ a_{21}&=\vartheta_2''(0,\lambda)\alpha+A\sinh(D_n)\vartheta_2'(0,\lambda),\\ a_{22}&=\vartheta_2(0,\lambda),\\ b_1&=4\sinh^2\left(\frac{1}{2}D_n\right)\vartheta_1(0,\lambda),\\ b_2&=4\sinh^2\left(\frac{1}{2}D_n\right)\vartheta_2(0,\lambda),\end{aligned} \tag{24}$$

the system (21)–(22) admits an explicit solution

$$\begin{aligned}\omega&=\frac{b_1a_{22}-b_2a_{12}}{a_{11}a_{22}-a_{12}a_{21}},\\ c&=\frac{b_2a_{11}-b_1a_{21}}{a_{11}a_{22}-a_{12}a_{21}}.\end{aligned} \tag{25}$$

Finally, we obtain the quasi-periodic solutions for the integral differential-difference lattice equation (1)

$$f(n)=\sum_{m=-\infty}^{\infty}\mathrm{e}^{2\pi\mathrm{i}m\xi-\pi m^2\tau}, \tag{26}$$

where $\xi=\alpha x+\omega t+\nu n+\sigma$, α, ν, and σ are arbitrary constants, ω and c are given by (25).

4. Asymptotic Properties

In this section, an analysis on the asymptotic properties of the one-periodic wave solution is given. It will be shown that the one-soliton solution can be obtained as a limiting case of the one-periodic wave solution (19). We will directly use the system (25) to analyse the asymptotic properties of the periodic solution, which is easier and more effective than our original method proposed in [14, 15]. The relations between these two solutions are established as follows.

Theorem 1. Suppose that the vector (ω,c) is a solution of the system (25), and for the periodic wave solution (26), we let

$$\alpha'=2\pi\mathrm{i}\alpha,\ \nu'=2\pi\mathrm{i}\nu,\ \sigma'=2\pi\mathrm{i}\sigma-\pi\tau, \tag{27}$$

where α, ν, and δ are given in (26). Then we have the following asymptotic properties:

$$\begin{aligned}&c\to 0,\ 2\pi\mathrm{i}\xi-\pi\tau\to\eta,\\ &\eta=\alpha'x+\frac{4\sinh^2\left(\frac{1}{2}\nu'\right)}{\alpha'-A\sinh(\nu')}t+\nu'n+\sigma',\\ &\theta(\xi,\tau)\to 1+\mathrm{e}^{\eta}\ \text{ as }\ \lambda\to 0.\end{aligned} \tag{28}$$

In other words, the periodic solution (26) tends to the soliton solution (7) under a small amplitude limit.

Proof. Since the coefficients of system (24) are power series about λ, its solution (ω,c) is also a series

about λ. The coefficients of the system (24) can then be explicitly expanded as follows:

$$\begin{aligned} a_{11} &= [-32\pi^2\alpha + 4A\pi \mathrm{i}\sinh(4\pi \mathrm{i} v)]\lambda^4 + \cdots, \\ a_{12} &= 1 + 2\lambda^4 + 2\lambda^{16}\cdots, \\ a_{21} &= [-8\pi^2\alpha + 4A\pi \mathrm{i}\sinh(2\pi \mathrm{i} v)]\lambda + \cdots, \\ a_{22} &= 2\lambda + 2\lambda^3 + 2\lambda^9\cdots, \\ b_1 &= 8\sinh^4(2\pi \mathrm{i} v)\lambda^4 + \cdots, \\ b_2 &= 8\sinh^4(\pi \mathrm{i} v)\lambda + \cdots. \end{aligned} \tag{29}$$

Let the solution of the system (25) be in the form

$$\begin{aligned} \omega &= \omega_0 + \omega_1\lambda + \omega_2\lambda^2 + \cdots = \omega_0 + o(\lambda), \\ c &= c_0 + c_1\lambda + c_2\lambda^2 + \cdots = c_0 + o(\lambda). \end{aligned} \tag{30}$$

Substituting the expansions (29) and (30) into the system (25) and letting $\lambda \to 0$, we immediately obtain the following relations:

$$c_0 = 0, \quad w_0 = \frac{8\sinh^4(\pi \mathrm{i} v)}{-8\pi^2\alpha + 4A\pi \mathrm{i}\sinh(2\pi \mathrm{i} v)}. \tag{31}$$

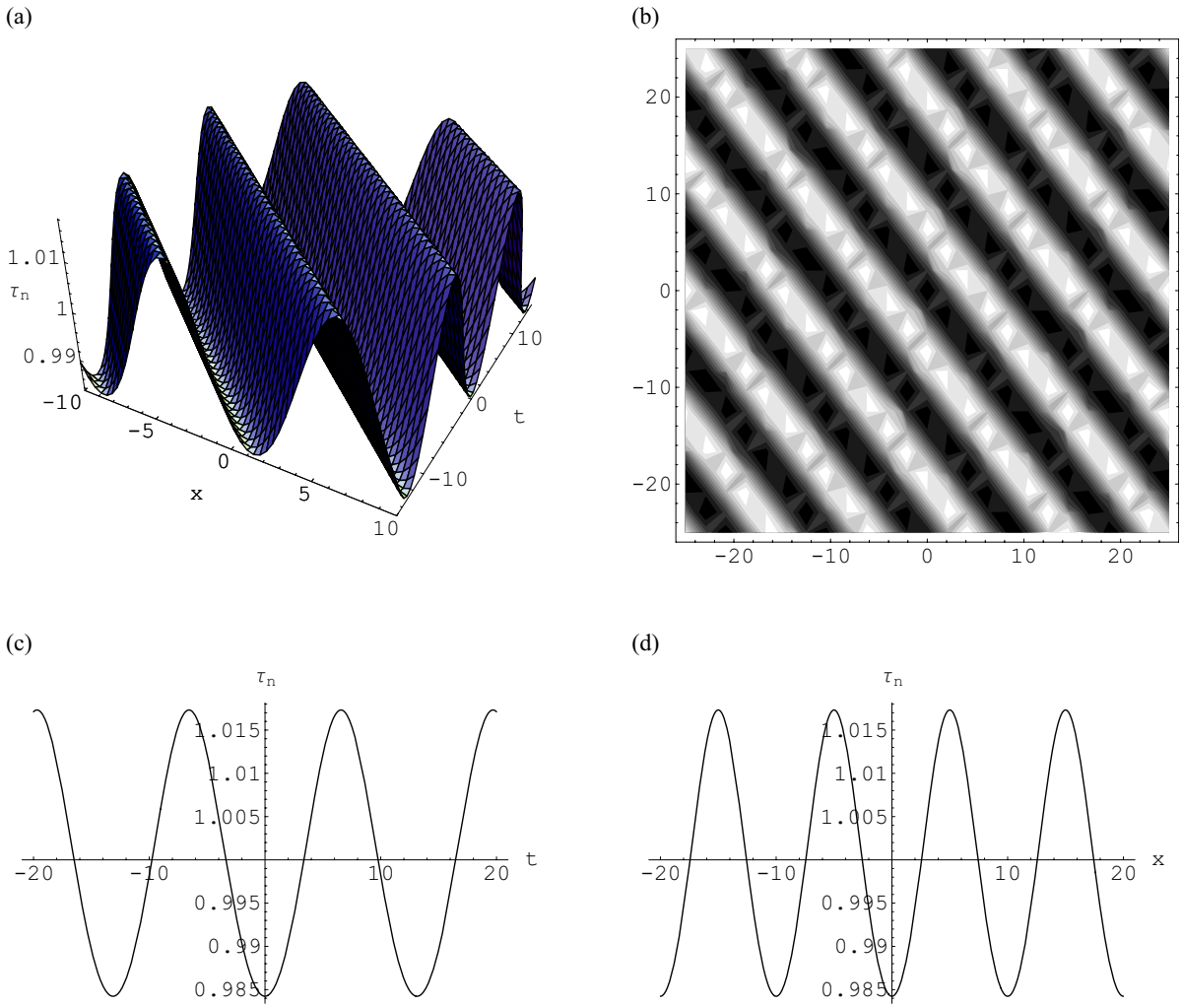

Fig. 1 (colour online). Quasi-periodic wave for the two-dimensional Toda equation (4): (a) perspective view of wave, (b) overhead view of wave, with contour plot shown, (c) along t-axis, (d) along x-axis, where $\alpha = 0.1$, $v = 0.1$, $n = 10$, $\sigma = 0$, and $\tau = 1$.

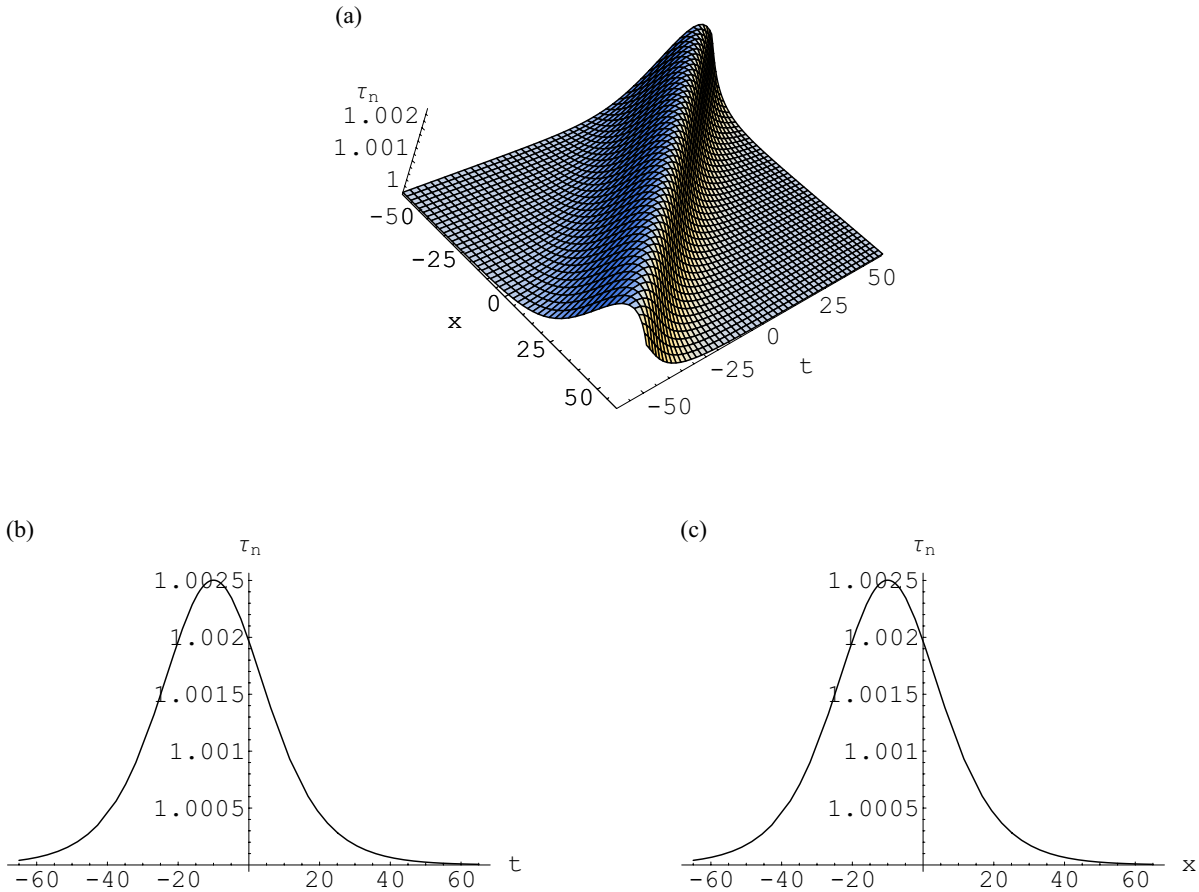

Fig. 2 (colour online). Solitary wave for the two-dimensional Toda equation (4): (a) perspective view of wave, (b) along t-axis, (c) along x-axis, where $\alpha = 0.1$, $\nu = 0.1$, $n = 10$, $\sigma = 0$, and $\tau = 1$.

Combining (30) and (31) leads to

$$\begin{aligned} &2\pi \mathrm{i} w \to 4\frac{\sinh^2\left(\frac{1}{2}\nu'\right)}{\alpha' + A\sinh(\nu')}, \\ &c \to 0 \ \text{ as } \ \lambda \to 0, \end{aligned} \tag{32}$$

or equivalently,

$$\begin{aligned} \hat{\xi} &= 2\pi \mathrm{i}\xi - \pi\tau \\ &= \alpha' x + 4\frac{\sinh^2\left(\frac{1}{2}\nu'\right)}{\alpha' + A\sinh(\nu')}t + \nu' n + \sigma' \to \eta. \end{aligned} \tag{33}$$

It remains to consider the asymptotic properties of the one-periodic wave solution (26) under the limit $\lambda \to 0$. For this purpose, we expand the Riemann theta function $\vartheta(\xi,\tau)$ and make use of the expression (33) to obtain

$$\begin{aligned} \vartheta(\xi,\tau) &= 1 + \lambda^2\left(\mathrm{e}^{2\pi\mathrm{i}\xi} + \mathrm{e}^{-2\pi\mathrm{i}\xi}\right) \\ &\quad + \lambda^8\left(\mathrm{e}^{4\pi\mathrm{i}\xi} + \mathrm{e}^{-4\pi\mathrm{i}\xi}\right) + \cdots \\ &= 1 + \mathrm{e}^{\hat{\xi}} + \lambda^4\left(\mathrm{e}^{-\hat{\xi}} + \mathrm{e}^{2\hat{\xi}}\right) + \cdots \\ &\to 1 + \mathrm{e}^{\eta} \ \text{ as } \ \lambda \to 0, \end{aligned} \tag{34}$$

which complete the proof for the theorem. We conclude that the periodic solution (26) tends to the soliton solution (7) as the amplitude $\lambda \to 0$. □

The bilinear form (1) in general arises from many famous differential-difference equations. For example, when $A = 0$ and taking the transformation of the solution (3), the bilinear form (1) becomes the two-

dimensional Toda equation (4). From Theorem 1, we can directly obtain the quasi-periodic solution τ_n of the two-dimensional Toda equation (4). The quasi-periodic and the corresponding soliton solutions of the two-dimensional Toda equation have been presented in Figures 1 and 2.

5. Conclusion

In this paper, we give a construction method for obtaining quasi-periodic wave solutions of differential-difference equations. Similarly, multi-periodic wave solutions of differential-difference equations can be constructed by using the following multi-dimensional Riemann theta function:

$$\vartheta(\boldsymbol{\xi},\tau)=\sum_{\boldsymbol{m}\in\mathbb{Z}^N}\exp\{2\pi\mathrm{i}\langle\boldsymbol{\xi},\boldsymbol{m}\rangle-\pi\langle\boldsymbol{\tau}\boldsymbol{m},\boldsymbol{m}\rangle\},\tag{35}$$

where $\boldsymbol{\xi}=(\xi_1,\xi_2,\ldots,\xi_N)^{\mathrm{T}}\in\mathbb{C}^N$, $\boldsymbol{m}=(m_1,m_2,\ldots,m_N)^{\mathrm{T}}\in\mathbb{Z}^N$, $\xi_j=\alpha_j x+\omega_j t+\nu_j n+\sigma_j$, $j=1,\ldots,N$, τ is a $N\times N$ symmetric positive definite matrix. The inner product is defined by

$$\langle\boldsymbol{f},\boldsymbol{g}\rangle=f_1g_1+f_2g_2+\cdots+f_Ng_N,\tag{36}$$

for two vectors $\boldsymbol{f}=(f_1,f_2,\ldots,f_N)^{\mathrm{T}}$ and $\boldsymbol{g}=(g_1,g_2,\ldots,g_N)^{\mathrm{T}}$.

In order that the multi-dimensional Riemann theta function (35) satisfy the bilinear equation (1), from (18) we have

$$\begin{aligned}&\sum_{\mu=0,1}\sum_{h_1,\ldots,h_N=-\infty}^{\infty}G\left(4\pi\mathrm{i}\sum_{j=1}^{N}\left(h_j-\frac{\mu_j}{2}\right)\alpha_j,4\pi\mathrm{i}\sum_{j=1}^{N}\left(h_j-\frac{\mu_j}{2}\right)\omega_j,\sinh\left[4\pi\mathrm{i}\delta\sum_{j=1}^{N}\nu_j\left(h_j-\frac{\mu_j}{2}\right)\right]\right)\\&\cdot\exp\left[-2\pi\sum_{j,l=1}^{N}\left(h_j-\frac{\mu_j}{2}\right)\tau_{jk}\left(h_k-\frac{\mu_k}{2}\right)\right]=0.\end{aligned}\tag{37}$$

Obviously, in the case of differential-difference equations, the number of constraint equations of the type (17) is 2^N. On the other hand, we have parameters $\tau_{jk}=\tau_{kj}$, α_j, ω_j, ν_j, and c whose total number is $\frac{1}{2}N(N+1)+3N+1$. Among which $3N$ parameters τ_{jj}, α_j, and ν_j are taken to be the given parameters related to the amplitudes and wave numbers (or frequencies) of N-periodic waves, and $\frac{1}{2}N(N-1)$ parameters τ_{jk} implicitly appear in series form, which in general cannot be solved explicitly. Hence, the number of the explicit unknown parameters is only $N+1$. The number of equations is larger than the number of unknown parameters in the case when $N\geq 2$. In this paper, we consider the one-periodic wave solution of (1), which belongs to the case when $N=1$. The case when $N\geq 2$ will be considered in our future work.

Acknowledgements

The work described in this paper was partially supported by a grant from CityU (Project No. 7002564). The work was also partially supported by the Leading Academic Discipline Program, 211 Project for Shanghai University of Finance and Economics.

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