A Modified Adomian Decomposition Method for Solving Higher-Order Singular Boundary Value Problems

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In this paper, we present a reliable modification of the Adomian decomposition method for solving higher-order singular boundary value problems. He's polynomials are also used to overcome the complex and difficult calculation of Adomian polynomials occurring in the application of the Adomian decomposition method. Numerical examples are given to illustrate the accuracy and efficiency of the presented method, revealing its reliability and applicability in handling the problems with singular nature.

Key words: Adomian Decomposition Method; Singular Boundary Value Problems; Higher-Order Ordinary Differential Equation; Modified Adomian Decomposition Method; He's polynomials; Homotopy Perturbation Method.

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1. Introduction

The Adomian decomposition method (ADM) is one of the powerful and reliable methods for solving various kinds of problems arising in applied sciences, linear and nonlinear as well. This method, which accurately computes the series solution, is of great interest to applied sciences [1, 2]. The method provides the solution in a rapidly convergent series solution, if the equation has a unique solution, and it has been successfully applied to a wide class of problems arising in applied sciences, see [1 – 6], and the references therein. The convergence of the decomposition series have been investigated by several authors [7 – 9]. In recent years, the Adomian method has been modified so as to solve boundary value problems with singular nature by several authors [10 – 13]. The difficulty of those singular problems is due to the singularity behaviour that occurs at the point $x = 0$.

In this paper we introduce a new reliable modification of the ADM to overcome the singularity difficulty for higher-order boundary value problems. He's polynomials with the homotopy perturbation method (HPM) are also used to overcome the complex and difficult calculation of Adomian polynomials occurring in the application of the Adomian decom-

position method, demonstrating that He's polynomials can completely replace Adomian's polynomials. The HPM, first proposed by He in 1998, was developed and improved by He in [14 – 16]. This method is a novel and effective method [17 – 19], and has been successfully applied to solve many types of problems. For a comprehensive survey on the method, new interpretation, and its applications, the reader is referred to [17, 18]. Several numerical examples will be considered to illustrate that the proposed framework is well suited to attain an accurate solution to the higher-order singular boundary value problems, revealing its reliability and applicability.

2. Modified Adomian Decomposition Method

Consider the singular boundary value problem of $(n + 1)$-order ordinary differential equation in the form

$$y^{(n+1)} + \frac{m}{x}y^{(n)} + Ny = g(x), \qquad (1)$$

$$y(0) = a_0, y'(0) = a_1, \ldots, y^{(r-1)}(0) = a_{r-1}, y(b) = c_0, y'(b) = c_1, \ldots, \qquad (2)$$
$$y^{(n-r)}(b) = c_{n-r},$$

where N is a nonlinear differential operator of order less than n, $g(x)$ is a given function, $a_0, a_1, \ldots, a_{r-1}$, $c_0, c_1, \ldots, c_{n-r}, b$ are given constants, where $m \leq r \leq n, r \geq 1$.

We first rewrite (1) in the form

$$x^{-2} \frac{d^{n-1}}{dx^{n-1}} \left[x^2 y'' + (m - 2n + 2) xy' \right. \\ \left. + (n-m)(n-1)y \right] + Ny = g(x), \tag{3}$$

or equivalently,

$$x^{-2} \frac{d^{n-1}}{dx^{n-1}} \left[x^{2n-m} \frac{d}{dx} \left(x^{m-2n+2} \frac{dy}{dx} \right) \right] \\ + (n-m)(n-1) x^{-2} \frac{d^{n-1}y}{dx^{n-1}} + Ny = g(x). \tag{4}$$

Equation (4) can be written in the operator form

$$L_2 L_1 y = g(x) + (m-n)(n-1) L_2 y - Ny, \tag{5}$$

where the differential operator L employs the first two derivatives

$$L_1 = x^{2n-m} \frac{d}{dx} \left(x^{m-2n+2} \frac{d}{dx} \right), \tag{6}$$

$$L_2 = x^{-2} \frac{d^{n-1}}{dx^{n-1}} \tag{7}$$

in order to overcome the singularity behaviour at $x = 0$.

In view of (6) and (7), the inverse operators L_1^{-1} and L_2^{-1} are the integral operators defined by

$$L_1^{-1}(.) = \int_b^x x^{2n-m-2} \int_0^x x^{m-2n}(.) dx dx, \tag{8}$$

$$L_2^{-1}(.) = \underbrace{\int_0^x \cdots \int_0^x}_{n-1 \text{ times}} x^2 (.) dx \cdots dx. \tag{9}$$

By applying L_2^{-1} on (5), we have

$$L_1 y = \Psi_1(x) + L_2^{-1} g(x) - L_2^{-1} Ny, \tag{10}$$

such that

$$L_2 \Psi_1(x) = 0. \tag{11}$$

By applying L_1^{-1} on (10), we have

$$y(x) = \Psi_2(x) + L_1^{-1} \Psi_1(x) + L_1^{-1} L_2^{-1} g(x) \\ - L_1^{-1} L_2^{-1} Ny, \tag{12}$$

such that

$$L_1 \Psi_2(x) = 0. \tag{13}$$

The Adomian decomposition method introduces the solution $y(x)$ and the nonlinear function Ny by infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \tag{14}$$

and

$$Ny = \sum_{n=0}^{\infty} A_n(y_0, y_1, \ldots, y_n), \tag{15}$$

where the Adomian polynomials A_n are defined as

$$A_n = \left[\frac{1}{n!} \frac{d^n}{d\lambda^n} N \left(\sum_{i=0}^{n} y_i(\lambda)^i \right) \right]_{\lambda=0}. \tag{16}$$

The components $y_n(x)$ of the solution $y(x)$ will be determined recurrently, and Adomian polynomials can be constructed for various classes of nonlinearity according to specific algorithms presented in [6] and [20].

Substituting (14) and (15) into (12) gives

$$\sum_{n=0}^{\infty} y_n = \Psi_2(x) + L_1^{-1} \Psi_1(x) + L_1^{-1} L_2^{-1} g(x) \\ - L_1^{-1} L_2^{-1} \sum_{n=0}^{\infty} A_n. \tag{17}$$

Identifying $y_0 = \Psi_2(x) + L_1^{-1} \Psi_1(x) + L_1^{-1} L_2^{-1} g(x)$, the Adomian method admits the use of the recursive relation

$$y_0 = \Psi_2(x) + L_1^{-1} \Psi_1(x) + L_1^{-1} L_2^{-1} g(x), \\ y_{n+1} = -L^{-1} A_n, \quad n \geq 0, \tag{18}$$

which gives

$$y_0 = \Psi(x) + L^{-1} g(x), \\ y_1 = -L^{-1} A_0, \\ y_2 = -L^{-1} A_1, \\ y_3 = -L^{-1} A_2, \\ \vdots \tag{19}$$

This will lead to the complete determination of the components $y_n(x)$ of $y(x)$. The series solution of $y(x)$

defined by (14) follows immediately. For numerical purposes, the n-term approximant

$$\Psi_n = \sum_{k=0}^{n-1} y_k \qquad (20)$$

can be used to approximate the exact solution.

3. Numerical Examples

Example 1. Consider the nonlinear boundary value problem

$$y'' - \frac{1}{x}y' = \frac{x^2}{3}y^5, \qquad (21)$$

$$y(0) = 1, \quad y'(1) = -\frac{\sqrt{3}}{8}. \qquad (22)$$

We define

$$L_1 = x^3 \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-1}\frac{\mathrm{d}}{\mathrm{d}x}\right), \qquad (23)$$

so that

$$L_1^{-1}(.) = \int_0^x x \int_1^x x^{-3}(.)\mathrm{d}x\mathrm{d}x. \qquad (24)$$

After multiplying (21) by x^2, in an operator form, it becomes

$$L_1 y = \frac{x^4}{3}y^5. \qquad (25)$$

Applying L_1^{-1} on both sides of (25) we find

$$y(x) = y(0) + \frac{1}{2}y'(1)x^2 + L_1^{-1}\frac{x^4}{3}y^5. \qquad (26)$$

Using the decomposition series for the linear function $y(x)$ and the polynomial series for the nonlinear term y^5, we obtain

$$\sum_{n=0}^{\infty} y_n = y(0) + \frac{1}{2}y'(1)x^2 + L_1^{-1}\left(\frac{x^4}{3}\sum_{n=0}^{\infty} A_n\right). \qquad (27)$$

This gives the recursive relationship

$$y_0(x) = y(0) + \frac{1}{2}y'(1)x^2, \qquad (28)$$

$$y_{k+1}(x) = L_1^{-1}\left(\frac{x^4}{3}A_k\right), \quad k \geq 0. \qquad (29)$$

The Adomian polynomials for the nonlinear term y^5 are computed as follows:

$$\begin{aligned}
A_0 &= y_0^5, \\
A_1 &= 5y_0^4 y_1, \\
A_2 &= 5y_2 y_0^4 + 10y_0^3 y_1^2, \\
A_3 &= 5y_3 y_0^4 + 20y_2 y_1 y_0^3 + 10y_1^3 y_0^2, \\
&\vdots
\end{aligned} \qquad (30)$$

Substituting (30) into (29) gives the components

$$\begin{aligned}
y_0 &= 1 - 0.108253x^2, \\
y_1 &= -0.0637827x^2 + 0.0416666x^4 \\
&\quad - 0.0075176x^6 + 0.00081380x^8 \\
&\quad - 0.00005286x^{10} + 1.9073486\cdot10^{-6}x^{12} \\
&\quad - 2.9496649\cdot10^{-8}x^{14} + \cdots, \\
y_2 &= 0.0062577x^2 - 0.0044294x^6 \\
&\quad + 0.0024057x^8 - 0.0006259x^{10} \\
&\quad + 0.0001017x^{12} - 0.0000114x^{14} + \cdots, \\
y_3 &= -0.0010668x^2 + 0.0004346x^6 \\
&\quad + 0.0001884x^8 - 0.0003596x^{10} \\
&\quad + 0.0001864x^{12} - 0.0000553x^{14} + \cdots, \\
&\vdots
\end{aligned} \qquad (31)$$

The solution in a series form is thus approximately given by

$$\begin{aligned}
y &= y_0 + y_1 + y_2 + y_3 \\
&= 1 - 0.1668450x^2 + 0.0416667x^4 \\
&\quad - 0.0115124x^6 + 0.0034079x^8 \\
&\quad - 0.0010384x^{10} + 0.0002900x^{12} \\
&\quad - 0.0000668x^{14} + \cdots,
\end{aligned} \qquad (32)$$

which is in good agreement with the Taylor series of the exact solution $y(x) = \frac{1}{\sqrt{1+\frac{x^2}{3}}}$ given by

$$\begin{aligned}
y(x) &= 1 - 0.166667x^2 + 0.0416667x^4 \\
&\quad - 0.0115741x^6 + 0.00337577x^8 \\
&\quad - 0.0010123x^{10} + 0.0003094x^{12} \\
&\quad - 0.0000958x^{14} + \cdots.
\end{aligned} \qquad (33)$$

For the above and later computations, the Maple package was used and the digits environment variable was

set to 20 as the number of digits that Maple uses when calculating with software floating-point numbers. The results rounded off to seven decimal places are presented for the sake of simplicity.

Example 2. Consider the nonlinear boundary value problem

$$y'' + \frac{\rho}{x}y' = g(x), \tag{34}$$

$$y(0) = 0, \; y'(b) = b^{-\rho}(1-\rho)\cos b - b^{1-\rho}\sin b, \tag{35}$$

where

$$g(x) = -x^{1-\rho}\cos x - (2-\rho)x^{-\rho}\sin x. \tag{36}$$

We define

$$L_1 = x^{2-\rho}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^\rho \frac{\mathrm{d}}{\mathrm{d}x}\right), \tag{37}$$

$$L_2 = x^{-2}, \tag{38}$$

so that

$$L_1^{-1}(.) = \int_0^x x^{-\rho}\int_b^x x^{\rho-2}(.)\mathrm{d}x\mathrm{d}x, \tag{39}$$

$$L_2^{-1}(.) = x^2(.). \tag{40}$$

In an operator form, (34) becomes

$$L_2 L_1 y = g(x). \tag{41}$$

We find that

$$\begin{aligned}L_1^{-1}L_1 y &= \int_0^x x^{-\rho}\int_b^x x^{\rho-2}\left[x^{2-\rho}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^\rho \frac{\mathrm{d}y}{\mathrm{d}x}\right)\right]\mathrm{d}x\mathrm{d}x \\ &= \int_0^x x^{-\rho}\left[x^\rho y' - b^\rho y'(b)\right]\mathrm{d}x \\ &= y(x) - y(0) - \frac{b^\rho}{1-\rho}y'(b)x^{1-\rho}.\end{aligned} \tag{42}$$

Applying L_2^{-1}, L_1^{-1} on both sides of (41), we find after imposing the given boundary conditions

$$\begin{aligned}y(x) &= y(0) + \frac{b^\rho}{1-\rho}y'(b)x^{1-\rho} \\ &\quad + \int_0^x x^{-\rho}\int_b^x [-x\cos x - (2-\rho)\sin x]\,\mathrm{d}x\mathrm{d}x = \\ &\quad \frac{1}{1-\rho}[(1-\rho)\cos b - b\sin b]x^{1-\rho} \\ &\quad + [\cos x - \cos b]x^{1-\rho} + \frac{b\sin b}{1-\rho}x^{1-\rho} = x^{1-\rho}\cos x,\end{aligned} \tag{43}$$

which is the exact solution of the problem (34) and (35).

Example 3. Consider the nonlinear boundary value problem

$$y''' + \frac{3}{x}y'' - y^3 = g(x), \tag{44}$$

$$y(0) = 0, \quad y'(0) = 0, \quad y(1) = \mathrm{e}, \tag{45}$$

where $g(x) = 24\mathrm{e}^x + 36x\mathrm{e}^x + 12x^2\mathrm{e}^x + x^3\mathrm{e}^x - x^9\mathrm{e}^{3x}$. We use the Taylor series of $g(x)$ with order 10,

$$\begin{aligned}g(x) &\approx g_T = 24 + 60x + 60x^2 + 35x^3 + 14x^4 + \frac{21}{5}x^5 \\ &\quad + x^6 + \frac{11}{56}x^7 + \frac{11}{336}x^8 - \frac{30097}{30240}x^9 - \frac{64787}{21600}x^{10}.\end{aligned} \tag{46}$$

We define

$$L_1 = x\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right), \tag{47}$$

$$L_2 = x^{-2}\frac{\mathrm{d}}{\mathrm{d}x}, \tag{48}$$

so that

$$L_1^{-1}(.) = \int_0^x x^{-1}\int_0^x x^{-1}(.)\mathrm{d}x\mathrm{d}x. \tag{49}$$

$$L_2^{-1}(.) = \int_0^x x^2(.)\mathrm{d}x. \tag{50}$$

In an operator form, (44) becomes

$$L_2 L_1 y = g_T(x) + L_2 y + y^3. \tag{51}$$

Applying L_2^{-1}, L_1^{-1} on both sides of (51) and then incorporating the given boundary conditions, we find

$$y(x) = L_1^{-1}L_2^{-1}g_T(x) + L_1^{-1}y + L_1^{-1}L_2^{-1}y^3. \tag{52}$$

Using the decomposition series for the linear function $y(x)$ and the polynomial series for the nonlinear term y^3, we obtain

$$\begin{aligned}\sum_{n=0}^\infty y_n &= L_1^{-1}L_2^{-1}g_T(x) + L_1^{-1}\left(\sum_{n=0}^\infty y_n\right) \\ &\quad + L_1^{-1}L_2^{-1}\left(\sum_{n=0}^\infty A_n\right).\end{aligned} \tag{53}$$

This gives the recursive relationship

$$y_0(x) = L_1^{-1}L_2^{-1}g_T(x), \tag{54}$$

$$y_{k+1}(x) = L_1^{-1}y_k + L_1^{-1}L_2^{-1}A_k, \quad k \geq 0. \tag{55}$$

The Adomian polynomials for the nonlinear term y^3 are computed as follows:

$$A_0 = y_0^3,$$
$$A_1 = 3y_0^2 y_1,$$
$$A_2 = 3y_2 y_0^2 + 3y_0 y_1^2, \tag{56}$$
$$A_3 = 3y_3 y_0^2 + 6y_0 y_1 y_2 + y_1^3,$$
$$\vdots$$

Substituting (56) into (55) gives the components of the solution

$$\begin{aligned}
y_0 &= 0.8888889x^3 + 0.9375000x^4 + 0.4800000x^5 \\
&\quad + 0.1620370x^6 + 0.0408163x^7 + 0.0082031x^8 \\
&\quad + 0.0013717x^9 + 0.0001964x^{10} + 0.0000246x^{11} \\
&\quad - 0.0005759x^{12} - 0.0013652x^{13},
\end{aligned}$$

$$\begin{aligned}
y_1 &= 0.0987654x^3 + 0.0585938x^4 + 0.0192000x^5 \\
&\quad + 0.0045010x^6 + 0.0008329x^7 + 0.0001282x^8 \\
&\quad + 0.0000169x^9 + 0.0000019x^{10} + 0.0000002x^{11} \\
&\quad + \cdots,
\end{aligned}$$

$$\begin{aligned}
y_2 &= 0.0109739x^3 + 0.0036621x^4 + 0.0007680x^5 \\
&\quad + 0.0001250x^6 + 0.0000169x^7 + 0.0000020x^8 \\
&\quad + 2.0907516 \times 10^{-7} x^9 + 1.9642857 \times 10^{-8} x^{10} \\
&\quad + 1.6799819 \times 10^{-9} x^{11} + \cdots,
\end{aligned}$$

$$\begin{aligned}
y_3 &= 0.0012193x^3 + 0.0002289x^4 + 0.0000307x^5 \\
&\quad + 0.00000347x^6 + 3.4693305 \times 10^{-7} x^7 \\
&\quad + 3.1292439 \times 10^{-8} x^8 + 2.5811748 \times 10^{-9} x^9 \\
&\quad + 1.9642857 \times 10^{-10} x^{10} + 1.3884148 \times 10^{-11} x^{11}, \\
&\quad + \cdots,
\end{aligned}$$

$$\vdots \tag{57}$$

The solution in a series form is thus approximately given by

$$\begin{aligned}
y &= y_0 + y_1 + y_2 + y_3 \\
&= 0.9998476x^3 + 0.9999847x^4 + 0.4999987x^5 \\
&\quad + 0.1666666x^6 + 0.0416667x^7 + 0.0083333x^8 \\
&\quad + 0.0013888x^9 + 0.0001984x^{10} + 0.0000248x^{11} \\
&\quad \cdots, \tag{58}
\end{aligned}$$

which is in good agreement with the Taylor series of the exact solution $y(x) = x^3 e^x$ given by

$$\begin{aligned}
y(x) &= x^3 + x^4 + 0.5x^5 + 0.1666666x^6 \\
&\quad + 0.0416666x^7 + 0.008333x^8 \\
&\quad + 0.0013888x^9 + 0.000198x^{10} \\
&\quad + 0.0000248x^{11} + \cdots. \tag{59}
\end{aligned}$$

Example 4. Consider the nonlinear boundary value problem

$$y^{(4)} + \frac{3}{x} y^{(3)} + y^2 - y^3 = g(x), \tag{60}$$

$$y(0) = 0, \; y'(0) = 0, \; y(1) = e, \; y'(1) = 3e, \tag{61}$$

where $g(x) = 18x^{-1} e^x + 30e^x + 11xe^x + x^2 e^x + x^4 e^{2x} - x^6 e^{3x}$. We use the 11 terms of the Taylor series of $g(x)$,

$$\begin{aligned}
g(x) &\approx \frac{18}{x} + 48 + 50x + 30x^2 + \frac{49}{4} x^3 + \frac{71}{15} x^4 \\
&\quad + \frac{29}{10} x^5 + \frac{33}{28} x^6 - \frac{6599}{4032} x^7 - \frac{9649}{2520} x^8 - \frac{1279991}{302400} x^9. \tag{62}
\end{aligned}$$

We define

$$L_1 = x^3 \frac{d}{dx} \left(x^{-1} \frac{d}{dx} \right), \tag{63}$$

$$L_2 = x^{-2} \frac{d^2}{dx^2}, \tag{64}$$

so that

$$L_1^{-1}(.) = \int_0^x x \int_1^x x^{-3}(.) dx dx, \tag{65}$$

$$L_2^{-1}(.) = \int_0^x \int_0^x x^2 (.) dx dx. \tag{66}$$

In an operator form, (60) then becomes

$$L_2 L_1 y = g(x) + y^3 - y^2. \tag{67}$$

Applying L_2^{-1} on both sides of (67) and then incorporating given boundary conditions, (60) becomes

$$\begin{aligned}
L_1 y &= L_1 y \Big|_{x=0} + \frac{d}{dx} (L_1 y) \Big|_{x=0} \cdot x \\
&\quad + L_2^{-1} g(x) + L_2^{-1} (y^3 - y^2) \\
&= -y'(0)x + L_2^{-1} g(x) + L_2^{-1} (y^3 - y^2) \\
&= L_2^{-1} g(x) + L_2^{-1} (y^3 - y^2). \tag{68}
\end{aligned}$$

Since $L_1^{-1}L_1 y = y(x) - y(0) - \frac{1}{2}y'(1)x^2$, applying L_1^{-1} on both sides of (68) with given boundary conditions yields

$$y(x) = \frac{3e}{2}x^2 + L_1^{-1}L_2^{-1}g(x) \qquad (69)$$
$$+ L_1^{-1}L_2^{-1}(y^3 - y^2).$$

Proceeding as before, we have the recursive relationship

$$y_0(x) = \frac{3e}{2}x^2 + L_1^{-1}L_2^{-1}g(x), \qquad (70)$$

$$y_{k+1}(x) = L_1^{-1}L_2^{-1}A_k - L_1^{-1}L_2^{-1}B_k, \quad k \geq 0. \quad (71)$$

The Adomian polynomials for the nonlinear terms y^3 and y^2 are computed as follows:

$$
\begin{aligned}
A_0 &= y_0^3, \\
A_1 &= 3y_0^2 y_1, \\
A_2 &= 3y_2 y_0^2 + 3y_0 y_1^2, \qquad (72)\\
A_3 &= 3y_3 y_0^2 + 6y_0 y_1 y_2 + y_1^3, \\
&\vdots
\end{aligned}
$$

and

$$
\begin{aligned}
B_0 &= y_0^2, \\
B_1 &= 2y_0 y_1, \\
B_2 &= 2y_2 y_0 + y_1^2, \qquad (73)\\
B_3 &= 2y_3 y_0 + 2y_1 y_2, \\
&\vdots
\end{aligned}
$$

respectively.

Substituting (72) and (73) into (71) gives the components of the solution

$$
\begin{aligned}
y_0 &= 0.9993606x^2 + x^3 + 0.5x^4 + 0.1666667x^5 \\
&\quad + 0.0416667x^6 + 0.0083333x^7 + 0.0017609x^8 \\
&\quad + 0.0006393x^9 + 0.0001637x^{10} - 0.0001503x^{11} \\
&\quad - 0.0002417x^{12} - 0.0001897x^{13}, \\
y_1 &= -0.0007032x^2 - 0.0003715x^8 - 0.0004406x^9 \\
&\quad - 0.0001391x^{10} + 0.0001527x^{11} + \cdots, \\
y_2 &= 2.9469693 \times 10^{-6}x^2 + 5.228892996 \times 10^{-7}x^8 \\
&\quad + 3.1005857 \times 10^{-7}x^9 - 1.9496233 \times 10^{-7}x^{10} \\
&\quad - 3.6567264 \times 10^{-7}x^{11} + \cdots, \\
y_3 &= -1.2055039 \times 10^{-8}x^2 - 2.3752524 \times 10^{-9}x^8 \\
&\quad - 1.2993692 \times 10^{-9}x^9 + 1.0229462 \times 10^{-9}x^{10} \\
&\quad + 1.6686604 \times 10^{-9}x^{11} + \cdots, \\
&\vdots
\end{aligned}
$$
$$(74)$$

We thus obtain an approximate solution given by

$$
\begin{aligned}
y &= y_0 + y_1 + y_2 + y_3 \\
&= 0.9986603x^2 + x^3 + 0.5x^4 + 0.1666667x^5 \\
&\quad + 0.0416667x^6 + 0.0083333x^7 \\
&\quad + 0.0013899x^8 + 0.0001990x^9 \\
&\quad + 0.0000244x^{10} + 2.0591302 \times 10^{-6}x^{11} \\
&\quad + \cdots,
\end{aligned}
$$
$$(75)$$

which is in well agreement with Taylor series of the exact solution $y(x) = x^2 e^x$ given by

$$
\begin{aligned}
y(x) &= x^2 + x^3 + 0.5x^4 + 0.1666667x^5 \\
&\quad + 0.0416667x^6 + 0.0083333x^7 \\
&\quad + 0.0013888x^8 + 0.0001984x^9 \qquad (76)\\
&\quad + 0.0000248x^{10} + 2.7557319 \times 10^{-6}x^{11} \\
&\quad + \cdots.
\end{aligned}
$$

4. Homotopy Perturbation Method with He's Polynomials

It should be noted that the calculation of Adomian's polynomials to deal with the nonlinear terms in Example 1, Example 3, and Example 4 is complex and difficult. This aspect is the demerit of the application of Adomian's decomposition method, though this method has been proved to be more powerful and efficient than some existing methods [21]. This difficulty may be easily overcome without using Adomian's polynomials when the so-called He's polynomials are used to solve the problem. We illustrate this in this section by revisiting and solving Example 1 and Example 3 by the homotopy perturbation method (HPM) with He's polynomials [22 – 24].

Revisited Example 1.   Consider the nonlinear boundary value problem

$$y'' - \frac{1}{x}y' = \frac{x^2}{3}y^5, \qquad (77)$$

$$y(0) = 1, \quad y'(1) = -\frac{\sqrt{3}}{8}. \qquad (78)$$

With equation (26), we construct the following homotopy:

$$y(x) - y(0) - \frac{1}{2}y'(1)x^2 - pL_1^{-1}\frac{x^4}{3}N(y) = 0, \quad (79)$$

where $N(y) = y^5$, $p \in [0, 1]$ is the embedding parameter, $L_1^{-1}(\cdot)$ is defined by (24).

According to He's homotopy perturbation method [14, 15], we assume that the solution of (79) is

$$y = y_0 + py_1 + p^2 y_2 + \cdots. \qquad (80)$$

When $p \to 1$, (80) becomes the approximate solution of (77) and (78), i. e.,

$$\lim_{p \to 1} y = y_0 + y_1 + y_2 + \cdots \qquad (81)$$

is the exact solution of (77) and (78). The convergence of the series in (81) to the exact solution is discussed by He in [14, 15].

The nonlinear term $N(y)$ can be expressed in the form

$$N(y) = H(y_0) + pH(y_0, y_1) + p^2 H(y_0, y_1, y_2) + \cdots, \qquad (82)$$

where $H(y_0, \cdots, y_n)$ is called He's polynomial [22] defined by

$$H(y_0, \cdots, y_n) = \frac{1}{n!} \frac{d^n}{dp^n} N\left(\sum_{k=0}^{n} p^k y_k\right)_{p=0}, \qquad (83)$$

$$n = 0, 1, 2, 3 \cdots.$$

Substituting (80) and (82) into (79), and collecting the coefficients of like powers of p, we have

$$\left[y_0 - y(0) - \frac{1}{2} y'(1)x^2\right] p^0 + \left[y_1 - L_1^{-1} \frac{x^4}{3} y_0^5\right] p^1$$
$$+ \left[y_2 - L_1^{-1} \frac{x^4}{3} \left(5y_0^4 y_1\right)\right] p^2 + \left[y_3 - L_1^{-1} \frac{x^4}{3} \left(5y_2 y_0^4\right)\right.$$
$$\left. + 10 y_0^3 y_1^2\right) \right] p^3 + \left[y_4 - L_1^{-1} \frac{x^4}{3} \left(5y_3 y_0^4 + 20 y_2 y_1 y_0^3\right.\right.$$
$$\left.\left. + 10 y_1^3 y_0^2\right) \right] p^4 + \cdots = 0, \qquad (84)$$

from which we have simultaneously the approximate solution to (77) and (78) given in (32) and He's polynomials

$$H(y_0) = y_0^5,$$
$$H(y_0, y_1) = 5y_0^4 y_1,$$
$$H(y_0, y_1, y_2) = 5y_2 y_0^4 + 10 y_0^3 y_1^2, \qquad (85)$$
$$H(y_0, y_1, y_2, y_3) = 5y_3 y_0^4 + 20 y_2 y_1 y_0^3 + 10 y_1^3 y_0^2,$$
$$\vdots$$

Revisited Example 3. Consider the nonlinear boundary value problem

$$y''' + \frac{3}{x} y'' - y^3 = g(x), \qquad (86)$$

$$y(0) = 0, \quad y'(0) = 0, \quad y(1) = e, \qquad (87)$$

where $g(x) = 24e^x + 36xe^x + 12x^2 e^x + x^3 e^x - x^9 e^{3x}$.

With equation (52), we construct the following homotopy:

$$y(x) - L_1^{-1} L_2^{-1} g_T(x) - L_1^{-1} y - p L_1^{-1} L_2^{-1} N(y) = 0, \qquad (88)$$

where $N(y) = y^3$, $p \in [0, 1]$ is the embedding parameter and, g_T, $L_1^{-1}(\cdot)$ and $L_2^{-1}(\cdot)$ are defined by (46), (49), and (50), respectively.

We assume that the solution of (88) is

$$y = y_0 + py_1 + p^2 y_2 + \cdots. \qquad (89)$$

The nonlinear term $N(y)$ can be expressed in the form

$$N(y) = H(y_0) + pH(y_0, y_1) + p^2 H(y_0, y_1, y_2) + \cdots, \qquad (90)$$

where $H(y_0, \cdots, y_n)$ is the He polynomial defined by (83).

Substituting (89) and (90) into (88), and collecting the coefficients of like powers of p, we have

$$\left[y_0 - L_1^{-1} L_2^{-1} g_T\right] p^0 + \left[y_1 - L_1^{-1} y_0 - L_1^{-1} L_2^{-1} (y_0^3)\right] p^1$$
$$+ \left[y_2 - L_1^{-1} y_1 - L_1^{-1} L_2^{-1} (3y_0^2 y_1)\right] p^2 + \left[y_3 - L_1^{-1} y_2\right.$$
$$\left. - L_1^{-1} L_2^{-1} (3y_0^2 y_2 + 3y_0 y_1^2)\right] p^3 + \left[y_4 - L_1^{-1} y_3\right.$$
$$\left. - L_1^{-1} L_2^{-1} (3y_3 y_0^2 + 6y_0 y_1 y_2 + y_1^3)\right] p^4 + \cdots = 0, \qquad (91)$$

from which we have simultaneously the approximate solution to (86) and (87) given in (58) and He's polynomials

$$H(y_0) = y_0^3,$$
$$H(y_0, y_1) = 3y_0^2 y_1,$$
$$H(y_0, y_1, y_2) = 3y_0^2 y_2 + 3y_0 y_1^2, \qquad (92)$$
$$H(y_0, y_1, y_2, y_3) = 3y_3 y_0^2 + 6y_0 y_1 y_2 + y_1^3,$$
$$\vdots$$

He's polynomials were used in the above examples to overcome the complex and difficult calculation of the Adomian polynomials in the Adomian decomposition method. We calculated He's polynomials and the solutions simultaneously, and observed that He's

polynomials can completely replace Adomian's polynomials. This result reveals the applicability and efficiency of the homotopy perturbation method with higher-order singular boundary value problems.

5. Conclusion

In this paper, we present a reliable modification of the Adomian decomposition method to solve linear and nonlinear problems with singular feature. It is demonstrated that the presented approach can be well suited to attain an accurate solution to the higher-order singular boundary value problems, linear and nonlinear as well. The difficulty of those singular problems, due to the existence of the singular point at $x = 0$, is over-

come in this contribution. He's polynomials are also used to overcome the complex and difficult calculation of Adomian's polynomials occurring in the application of Adomian's decomposition method, illustrating that He's polynomials can completely replace Adomian's polynomials. The other types of differential equations with singular feature can also be similarly handled by the proposed approach.

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