Effect of Heat Transfer on Magnetohydrodynamic Axisymmetric Flow Between Two Stretching Sheets

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This investigation describes the effects of heat transfer on magnetohydrodynamic (MHD) axisymmetric flow of a viscous fluid between two radially stretching sheets. Navier-Stokes equations are transformed into the ordinary differential equations by utilizing similarity variables. Solution computations are presented by using the homotopy analysis method. The convergence of obtained solutions is checked. Skin friction coefficient and Nusselt number are given in tabular form. The dimensionless velocities and temperature are also analyzed for the pertinent parameters entering into the problem.

Key words: Heat Transfer; Porous Medium; Skin Friction Coefficient; Nusselt Number.

1. Introduction

It is noted from the existing literature that not much attention has been focused to the flow between radially stretching sheets. To the best of the author’s knowledge even no such attempt is available for a viscous fluid. Therefore, this paper provides the MHD flow analysis of a viscous fluid between two radially stretching sheets. The fluid is electrically conducting and the sheets are not conducting. An incompressible fluid saturates the porous medium. In addition, heat transfer is considered. The nonlinear equations are solved by a homotopy analysis method [20–34]. Salient features of the constructed solutions are discussed.

2. Mathematical Formulation

We consider the MHD steady and axisymmetric flow of an incompressible viscous fluid between two radially stretching sheets at \( z = \pm L \). An incompressible fluid fills the porous medium. Motion in fluid is due to the linear radial stretching of the both upper and lower sheets. We denote the velocity components \((u, v, w)\) in the cylindrical coordinate system \((r, \theta, z)\). The axial symmetry is considered about \( z = 0 \) [35]. A constant magnetic field \( B_0 \) is applied in the \( z \)-direction. The induced magnetic field is neglected under the assumption of small magnetic Reynolds number. There is no external electric field. Both the sheets have constant temperature \( T_w \). The effects of viscous dissipation and Joule heating are taken into account. In view of the axial symmetry, the velocity field is defined as

\[
V = [u(r, z), 0, w(r, z)].
\]

With above form of velocity field, the continuity, Navier-Stokes, and energy equations are

\[
\begin{align*}
\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} &= 0, \\
\frac{u}{r} &+ \frac{\partial u}{\partial r} + \frac{\partial v}{\partial z} = \frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} \right) - \left( \frac{\alpha B_0^2}{\rho} + \frac{v \varphi_m}{k} \right) u, \\
\frac{\partial v}{\partial r} + \frac{w}{r} &+ \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} \right) - \nu \varphi_m \frac{v}{k} w,
\end{align*}
\]

Here \( u \) and \( w \) are the velocity components in radial \((r)\) and axial \((z)\) directions, respectively, \( \rho \) is the density, \( \nu \) the kinematic viscosity, \( \sigma \) the electrical conductivity of the fluid, \( p \) the pressure, \( \mu \) the dynamic viscosity, \( c_p \) the specific heat, \( \varphi_m \) the porosity of the medium, \( T \) the temperature, \( K \) the thermal conductivity, and \( k \) the permeability of the porous medium.

The prescribed boundary conditions are

\[
\begin{align*}
\frac{\partial u}{\partial z} &= 0, \quad w = 0, \quad \frac{\partial T}{\partial z} = 0 \quad \text{at} \quad z = 0, \\
u &= a r, \quad w = 0, \quad T = T_w \quad \text{at} \quad z = L, \quad a > 0.
\end{align*}
\]

From (2)–(6), after employing the transformations

\[
u = raF' (\eta), w = -2aL F (\eta), \theta = \frac{T}{T_w}, \eta = \frac{z}{L},
\]

and eliminating the pressure terms, one obtains

\[
\begin{align*}
F'' (\eta) - Re (M + \phi) F'''' (\eta) + 2 Re F (\eta) F'''' (\eta) &= 0, \\
F (0) &= 0, \quad F' (1) = 1, \quad F'' (0) = 0, \\
\theta'' (\eta) + 2 Re Pr F (\eta) \theta' (\eta) + Re M Pr Ec [F' (\eta)]^2 &= 0,
\end{align*}
\]

in which

\[
Re = \frac{a L^2}{\nu}, \quad M = \frac{\sigma B_0^2}{\rho a}, \quad Pr = \frac{\mu}{\nu}, \quad Ec = \frac{a^2 r^2}{\varphi_m}, \quad \phi = \frac{v \varphi_m}{ka}, \quad \delta = \frac{\mu^2}{\nu L^2}.
\]

denote, respectively, the Reynolds number, Hartman number, Prandtl number, local Eckert number, and the porosity parameter.

The skin friction coefficient \( C_F \) and Nusselt number \( Nu \) are defined as follows:

\[
C_F = \frac{\tau_{rz} \bigg|_{z=0}}{\rho (ar)^2} = \left. \mu \left( \frac{\partial u}{\partial z} + \frac{\partial u}{\partial r} \right) \right|_{z=L} = \frac{1}{Re} F'' (1),
\]

\[
Nu = \frac{L q_w}{K_T w} = - \left. \frac{L k F'}{K_T} \right|_{z=L} = - \theta' (1),
\]

in which

\[
\begin{align*}
\tau_{rz} &= \frac{\partial u}{\partial z} + \frac{\partial u}{\partial r}, \\
q_w &= \frac{\partial T}{\partial z}, \quad F' &= \frac{\partial F}{\partial \eta}, \quad \theta' &= \frac{\partial \theta}{\partial \eta}.
\end{align*}
\]
where $Re = (ar^2/v)$ indicates the local Reynolds number.

3. Solutions of the Problems

In order to obtain the homotopy solutions, $F(\eta)$ and $\theta(\eta)$ in the form of base functions

$$\{\eta^{2n+1}, n \geq 0\}, \{\eta^{2n}, n \geq 0\}$$  \hspace{1cm} (11)

are written as

$$F(\eta) = \sum_{n=0}^{\infty} a_n \eta^{2n+1}, \quad \theta(\eta) = \sum_{n=0}^{\infty} b_n \eta^{2n},$$  \hspace{1cm} (12)

in which $a_n$ and $b_n$ are coefficients to be determined. Further, initial guesses $F_0(\eta)$, $\theta_0(\eta)$, and linear operators $L_i$ $(i = 1, 2)$ are selected from the forms given below

$$F_0(\eta) = \frac{1}{2} \eta(\eta^2 - 1), \quad \theta_0(\eta) = \eta^2,$$  \hspace{1cm} (13)

$$L_1[F(\eta)] = \frac{\partial^2 F}{\partial \eta^2}, \quad L_2[\theta(\eta)] = \frac{\partial^2 \theta}{\partial \eta^2}.$$  \hspace{1cm} (14)

We note that the operators $L_i$ $(i = 1, 2)$ satisfy the following properties:

$$L_1[C_1 + C_2 \eta + C_3 \eta^2 + C_4 \eta^3] = 0,$$

$$L_2[C_5 + C_6 \eta] = 0,$$  \hspace{1cm} (15)

where $C_i$ $(i = 1 - 6)$ are the constants.

3.1. Zeroth-Order Deformation Problems

The deformation problems at the zeroth order are

$$(1 - q) L_1[\Phi(\eta; q) - F_0(\eta)] = q \vartheta_1 N_1[\Phi(\eta; q)],$$

$$(1 - q) L_2[\Psi(\eta; q) - \theta_0(\eta)] = q \vartheta_2 N_2[\Psi(\eta; q)],$$  \hspace{1cm} (16)

$$\Phi(0; q) = 0, \quad \Phi(1; q) = 0, \quad \frac{\partial \Phi(\eta; q)}{\partial \eta} \bigg|_{\eta=1} = 1,$$

$$\frac{\partial^2 \Phi(\eta; q)}{\partial \eta^2} \bigg|_{\eta=0} = 0,$$  \hspace{1cm} (17)

$$\Psi(0; q) = 1, \quad \frac{\partial \Psi(\eta; q)}{\partial \eta} \bigg|_{\eta=0} = 0,$$

$$N_1[\Phi(\eta; q)] = \frac{\partial^4 \Phi(\eta; q)}{\partial \eta^4} - Re(M + \phi) \frac{\partial^2 \Phi(\eta; q)}{\partial \eta^2} + 2Re \Phi(\eta; q) \frac{\partial^3 \Phi(\eta; q)}{\partial \eta^3}$$  \hspace{1cm} (18)

In the above expressions $q \in [0, 1]$ and $\vartheta_i \neq 0$ $(i = 1, 2)$ are, respectively, the embedding and auxiliary parameters and $\Phi(\eta; 0) = F_0(\eta)$, $\Psi(\eta; 0) = \theta_0(\eta)$ and $\Phi(\eta; 1) = F(\eta)$, $\Psi(\eta; 1) = \theta(\eta)$. When $q$ varies from 0 to 1, then $\Phi(\eta; q)$ varies from the initial guess $F_0(\eta)$ to $F(\eta)$, $\Psi(\eta; q)$ varies from the initial guess $\theta_0(\eta)$ to $\theta(\eta)$ and $N_i$ $(i = 1, 2)$ are nonlinear operators.

By Taylors’ series, one obtains

$$\Phi(\eta; q) = F_0(\eta) + \sum_{m=1}^{\infty} F_m(\eta) q^m,$$  \hspace{1cm} (20)

$$\Psi(\eta; q) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta) q^m,$$  \hspace{1cm} (21)

$$F_m(\eta) = \frac{1}{m!} \frac{\partial^m \Phi(\eta; q)}{\partial q^m} \bigg|_{q=0},$$

$$\theta_m(\eta) = \frac{1}{m!} \frac{\partial^m \Psi(\eta; q)}{\partial q^m} \bigg|_{q=0}.$$  \hspace{1cm} (22)

3.2. Higher-Order Deformation Problems

Writing

$$F_m(\eta) = \{F_0(\eta), F_1(\eta), F_2(\eta), F_3(\eta), \ldots, F_m(\eta)\},$$

$$\theta_m(\eta) = \{\theta_0(\eta), \theta_1(\eta), \theta_2(\eta), \theta_3(\eta), \ldots, \theta_m(\eta)\},$$  \hspace{1cm} (23)

the deformation problems at the $m$th order are

$$L_1[F_m(\eta) - \chi_m F_{m-1}(\eta)] = h_1 R_1(F_{m-1}(\eta)),$$

$$F_m(0) = 0, \quad F_m(1) = 0, \quad F'_m(0) = 0,$$  \hspace{1cm} (24)

$$L_2[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = h_2 R_2(\theta_{m-1}(\eta)),$$

$$\theta'_m(0) = 0, \quad \theta_m(1) = 0, \quad \chi_m = \\begin{cases} 0, m \leq 1, \\ 1, m > 1, \end{cases}$$

$$R_1(F_{m-1}(\eta)) = F'_{m-1}(\eta) - Re(M + \phi) F''(\eta) + 2 Re \sum_{n=0}^{m-1} F_n(\eta) F'_{m-1-n}(\eta).$$  \hspace{1cm} (25)
The general solutions of the problems (23) and (24) are

\[ F(\eta) = F^*(\eta) + C_1^m \eta + C_2^m \eta^2 + C_3^m \eta^3, \]
\[ \theta(\eta) = \theta^*(\eta) + C_1^\theta \eta + C_2^\theta \eta^2, \]

(27)

where \( F^*(\eta) \) and \( \theta^*(\eta) \) are particular solutions of problems (23) and (24).

4. Results and Discussion

Note that the analytical solutions (27) strongly depend upon the non-zero auxiliary parameters \( \tilde{h}_i \) (i = 1, 2). These auxiliary parameters play an important role in the convergence of the derived series solutions. Therefore, for this objective \( \tilde{h}_1 \)-curves are plotted (Fig. 1 and 2). The admissible ranges of \( \tilde{h}_1 \) and \( \tilde{h}_2 \) are \(-1.2 \leq \tilde{h}_1 \leq -0.8 \) and \(-1.6 \leq \tilde{h}_2 \leq -0.4 \). The whole analysis is carried out when \( \tilde{h}_1 = \tilde{h}_2 = -0.9 \).

Figures 3–5 show the plots of dimensionless radial velocity \( F'(\eta) \) for the influence of Reynolds number Re, Hartman number M, and porosity parameter \( \phi \).

Figures 6–8 present the variation of the dimensionless axial velocity \( F(\eta) \) for various values of Reynolds number Re, Hartman number M, and porosity parameter \( \phi \). In order to describe the influence of Ec, Pr, M, Re, and \( \phi \) on dimensionless temperature \( \theta(\eta) \), Figures 9–13 have been displayed.

From Figure 3 it is noted that the dimensionless radial velocity \( F'(\eta) \) decreases by increasing Re. Fig-
Fig. 2. $\delta$-curve for 20th-order approximations of $\theta(\eta)$.

Fig. 3. Influence of $\text{Re}$ on $F'(\eta)$.

Fig. 4. Influence of $M$ on $F'(\eta)$.

Fig. 5. Influence of $\phi$ on $F'(\eta)$.

Fig. 6. Influence of $\text{Re}$ on $F(\eta)$.

Fig. 7. Influence of $M$ on $F(\eta)$. 
Fig. 8. Influence of $\phi$ on $F(\eta)$.  

Fig. 9. Influence of $Ec$ on $\theta(\eta)$.  

Fig. 10. Influence of $Pr$ on $\theta(\eta)$.  

Fig. 11. Influence of $M$ on $\theta(\eta)$.  

Fig. 12. Influence of $Re$ on $\theta(\eta)$.  

Fig. 13. Influence of $\phi$ on $\theta(\eta)$.
Figure 4 depicts that \(F' (\mu)\) is a decreasing function of \(M\). This is in view of the fact that magnetic field slows down the motion of the fluid. Figure 5 indicates that the dimensionless radial velocity \(F'(\eta)\) decreases when the porosity parameter \(\phi\) increases. This indicates that the porous medium offers resistance to the flow of the fluid which makes sense. From Figures 6–8 one can conclude that the axial velocity \(F(\eta)\) increases when \(M\), \(Re\), and \(\phi\) are increased. It is obvious from Figures 3–5 that the boundary layer thickness for the radial velocity \(\theta(\eta)\) has increasing trend when \(Ec\), \(Pr\), and \(M\) increase. However, \(\theta(\eta)\) decreases with the increase of Reynolds number \(Re\) and porosity parameter \(\phi\) (Figs. 12 and 13). Table 1 is given to ensure the convergence of homotopy analysis method (HAM) solutions (27). From this table one can see that the convergence is achieved at 35th order of approximation. Table 2 shows the variation of skin friction coefficient \(C_f\) for different values of \(Re\), \(M\), and \(\phi\). This table depicts that the skin friction coefficient \(C_f\) increases when \(Re\), \(M\), and \(\phi\) are increased. Variation of Nusselt number \(Nu\) for different values of physical parameters is shown in Table 3. It indicates that the Nusselt number \(Nu\) is an increasing function of \(Re\), \(M\), \(\phi\), \(Pr\), and \(Ec\).

5. Closing Remarks

This study analyses the flow and heat transfer characteristics on the MHD axisymmetric flow of an electrically conducting viscous fluid in the presence of viscous dissipation and Joule heating. Analytic solutions are developed. The main results can be summarized as follows:

- Effects of \(Re\), \(M\), and \(\phi\) on \(F'(\eta)\) and \(F(\eta)\) are similar in a qualitative sense.
- Influence of \(\phi\) on \(\theta(\eta)\) is similar to that of \(F'(\eta)\) and \(F(\eta)\).
- Qualitatively, the behaviour of \(Ec\) and \(Pr\) on \(\theta(\eta)\) is similar.
- Variations of Mand \(\phi\) on \(\theta(\eta)\), \(F'(\eta)\), and \(F(\eta)\) show opposite behaviour.
- The behaviour of \(Re\), \(M\), \(\phi\), \(Pr\), and \(Ec\) on the skin friction coefficient and the Nusselt number are similar.

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