

Periodic Structures Based on the Symmetrical Lucas Function of the (2+1)-Dimensional Dispersive Long-Wave System

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By introducing the Lucas Riccati method and a linear variable separation method, new variable separation solutions with arbitrary functions are derived for a (2+1)-dimensional dispersive long-wave system. The main idea of this method is to express the solutions of this system as polynomials in the solution of the Riccati equation that satisfies the symmetrical Lucas functions. From the variable separation solution and by selecting appropriate functions, some novel Jacobian elliptic wave structures and periodic wave evolutional behaviours are investigated.

Key words: Lucas Functions; Variable Separation Excitations; DLW System; Periodic Structure.

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1. Introduction

Over the past 35 years the concept of solitons has entered into various branches of natural science. There has been noticeable progress in the study of the soliton theory. In recent studies on the soliton theory many powerful approaches were presented. An important approach is the multilinear variable separation [1, 2], which shows that, with a Painlevé-Bäcklund transformation, some significant types of localized excitations, such as dromions, rings, compactons, peakons and loop solitons, can simultaneously exist in a (2+1)-dimensional soliton model [1–6]. The main reason is that there is a rather common formula, namely

$$u = \frac{2(a_2a_1 - a_3a_0)q_y p_x}{(a_0 + a_1p + a_2q + a_3pq)^2}, \quad (1)$$

with two arbitrary functions, $p = p(x, t)$ and $q = q(x, t)$, to describe certain physical quantities for some (2+1)-dimensional models. Another important approach is the so-called mapping transformation method. The basic idea of the algorithm is that, for a given nonlinear partial differential equation (NLPDE) with independent variables $x = (x_0 = t, x_1, x_2, x_3, \dots, x_n)$ and the dependent variable u ,

$$P(u, u_t, u_{x_i}, u_{x_i x_j}, \dots) = 0, \quad (2)$$

where p is in general a polynomial function of its argument, and the subscripts denote the partial derivatives.

By using the traveling wave transformation, (2) possesses the ansätze

$$u = u(\xi), \quad \xi = \sum_{i=0}^m k_i x_i, \quad (3)$$

where k_i ($i = 0, 1, 2, \dots, m$) are all arbitrary constants. Substituting (3) into (2) yields an ordinary differential equation (ODE): $O(u(\xi), u(\xi)_\xi, u(\xi)_{\xi\xi}, \dots) = 0$. Then $u(\xi)$ is expanded into a polynomial in $g(\xi)$:

$$u(\xi) = F(g(\xi)) = \sum_{i=0}^n a_i g^i(\xi), \quad (4)$$

where a_i are constants to be determined and n is fixed by balancing the linear term of the highest order with the nonlinear term in (2). If we suppose $g(\xi) = \tanh \xi$, $g(\xi) = \operatorname{sech} \xi$ and $g(\xi) = \operatorname{sn} \xi$ or $g(\xi) = \operatorname{cn} \xi$, respectively, then the corresponding approach is usually called the tanh-function method, the sech-function method and the Jacobian-function method. Although the Jacobian elliptic function method is more improved than the tanh-function method and the sech-function method, the repeated calculations are often tedious since the different function $g(\xi)$ should be treated in a repeated way. The main idea of the mapping approach is that $g(\xi)$ is not assumed to be a specific function, such as tanh, sech, sn and cn, but a solution of a mapping equation, such as the Riccati equation ($g_\xi = g^2 + a_0$), or a solution of the cubic nonlinear

Klein Gordon ($g^2 \xi = a_4 g^4 + a_2 g^2 + a_0$) or a solution of the general elliptic equation ($g_{xi}^2 = \sum_{i=0}^4 a_i g^i$), where a_i ($i = 0, 1, \dots, 4$) are all arbitrary constants. Using the mapping relation (4) and the solutions of these mapping equations, one can obtain many explicit and exact traveling wave solutions of (2). Now an interesting or important question is whether all above localized excitations based on the former multilinear approach can be derived by the latter mapping approach, which is usually used to search for traveling wave solutions. The crucial technique is how to obtain some solutions of (2) also with certain arbitrary functions. Using (1), the quite rich localized excitations, such as lumps, dromions, peakons, compactons, foldons, ring solitons, fractal solitons, chaotic solitons [2, 7–20], are obtained, and the novel interactive behaviour among the same types and various types of soliton excitations is revealed.

It is a well-known fact that two mathematical constants of Nature, the π - and e-numbers, play a great role in mathematics and physics. Their importance consists in the fact that they “generate” the main classes of so-called “elementary functions”: sin, cosine (the π -number), exponential, logarithmic and hyperbolic functions (the e-number). It is impossible to imagine mathematics and physics without these functions. For example, there is the well-known greatest role of the classical hyperbolic functions in geometry and in cosmological researches. However, there is the one more mathematical constant playing a great role in modeling processes in living nature termed the Golden Section, Golden Proportion, Golden Ratio, Golden Mean [21–25]. However, we should certify that the role of this mathematical constant is sometimes undeservedly humiliated in modern mathematics and mathematical education. There is the well-known fact that the basic symbols of esoteric (pentagram, pentagonal star, platonic solids etc.) are closely connected to the Golden Section. Moreover, the “materialistic” science together with its “materialistic” education had decided to “throw out” the Golden Section. However, in modern science, an attitude towards the Golden Section connected to its Fibonacci and Lucas numbers is changing very quickly. The outstanding discoveries of modern science based on the Golden Section have a revolutionary importance for the development of modern science. These are enough convincing confirmations of the fact that human science approaches to uncovering one of the most complicated scientific notions, namely the concept of Har-

mony, which is based on the Golden Section. Harmony was opposed to Chaos and means the organization of the Universe. In Euclid’s “The Elements” we find a geometric problem called “the problem of division of a line segment in the extreme and middle ratio”. Often this problem is called the golden section problem [22–25]. The solution of the golden section problem reduces to the algebraic equation $x^2 = x + 1$, which has two roots. We call the positive root, $\alpha = \frac{1+\sqrt{5}}{2}$, the golden proportion, golden mean, or golden ratio.

In the present paper, we construct symmetrical Lucas functions and we find new solutions of the Riccati equation by using these functions. Also, we devise an algorithm called Lucas Riccati method to obtain new exact solutions of NLPDEs. Along with the above line, i. e., in order to derive some new solutions with certain arbitrary functions, we assume that these solutions in the form

$$u(x) = \sum_{i=0}^n \alpha_i(x) F^i(x), \quad (5)$$

with

$$F' = A + BF^2, \quad (6)$$

where $x = (x_0 = t, x_1, x_2, x_3, \dots, x_n)$, A, B are constants, and the prime denotes differentiation with respect to ξ . To determine u explicitly, one may take the following steps: First, similar to the usual mapping approach, determine n by balancing the highest nonlinear terms and the highest-order partial terms in the given NLPDE. Second, substitute (5) and (6) into the given NLPDE and collect coefficients of polynomials of F , then eliminate each coefficient to derive a set of partial differential equations (PDEs) of a_i ($i = 0, 1, 2, \dots, n$) and ξ . Third, solve the system of partial differential equations to obtain a_i and ξ . Substitute these results into (4), then a general formula of solutions of (2) can be obtained. Choose properly A and B in ODE (6) such that the corresponding solution $F(\xi)$ is one of the symmetrical Lucas functions given below. Some definitions and properties of the symmetrical Lucas functions are given in the appendix.

Case 1. If $A = \ln \alpha$ and $B = -\ln \alpha$, then (6) possesses the solutions

$$\text{tLs}(\xi), \quad \text{cotLs}(\xi).$$

Case 2. If $A = \frac{\ln \alpha}{2}$ and $B = -\frac{\ln \alpha}{2}$, then (6) possesses the solution

$$\frac{t \text{Ls}(\xi)}{1 \pm \sec \text{Ls}(\xi)}.$$

Case 3. If $a = \ln \alpha$ and $B = -4 \ln \alpha$, then (6) possesses the solution

$$\frac{t \text{Ls}(\xi)}{1 \pm t \text{Ls}^2(\xi)}.$$

In the following section we apply the Lucas Riccati method to obtain new localized exaltations. Also, we pay our attention to some novel Jacobian elliptic wave structures and periodic wave evolutional behaviours.

2. New Variable Separation Solutions of the (2+1)-Dimensional Dispersive Long-Wave System

We consider the (2+1)-dimensional dispersive long-wave (DLW) system

$$\begin{aligned} u_{yt} + v_{xx} + u_x u_y + uu_{xy} &= 0, \\ v_t + u_x + vu_x + uv_x + u_{xxy} &= 0. \end{aligned} \quad (7)$$

System (7) was first obtained by Boiti et al. [26] as a compatibility condition for a “weak” Lax Pair. Paquin and Winternitz [27] showed that the symmetry algebra of (7) is infinite-dimensional and has a Kac-Moody-Virasoro structure. Some special similarity solutions were also given in [27] by using symmetry algebra and classical theoretical analysis. Abundant propagating localized excitations were also derived by Tang et al. [28] and Bai and Zhao [29] with the help of the Painlevé-Bäcklund transformation and a multilinear variable separation approach. Searching for more types of solutions to the system (7) is of fundamental interest in fluid dynamics.

Now we apply the Lucas Riccati method to (7). By the balancing procedure, we have

$$\begin{aligned} u(x, y, t) &= a_0(x, y, t) + a_1(x, y, t)F(\varphi(x, y, t)), \\ v(x, y, t) &= b_0(x, y, t) + b_1(x, y, t)F(\varphi(x, y, t)) \\ &\quad + b_2(x, y, t)F^2(\varphi(x, y, t)), \end{aligned} \quad (8)$$

where $a_0(x, y, t) \equiv a_0$, $a_1(x, y, t) \equiv a_1$, $b_0(x, y, t) \equiv b_0$, $b_1(x, y, t) \equiv b_1$, $b_2(x, y, t) \equiv b_2$, and $\varphi(x, y, t)\varphi$ are arbitrary functions of x, y, t to be determined. Substituting (8) with (6) into (7), and equating each of the coefficients of $F(\varphi)$ to zero, we obtain a system of PDEs.

Solving this system of PDEs, with the help of Maple, we obtain the following solution:

$$a_0(x, y, t) = -\frac{\varphi_{xx}(x, y, t) + \varphi_t(x, y, t)}{\varphi_x(x, y, t)}, \quad (9)$$

$$a_1(x, y, t) = -2B\varphi_x(x, y, t), \quad (10)$$

$$\begin{aligned} b_0(x, y, t) &= \frac{-1}{\varphi_x^2(x, y, t)} (\varphi_{ty}(x, y, t)\varphi_x(x, y, t) \\ &\quad + \varphi_x(x, y, t)\varphi_{xy}(x, y, t) - \varphi_{xy}(x, y, t)\varphi_{xx}(x, y, t) \\ &\quad - \varphi_{xy}(x, y, t)\varphi_t(x, y, t) + 2AB\varphi_x^3(x, y, t)\varphi_y(x, y, t) \\ &\quad + \varphi_x^2(x, y, t)), \end{aligned} \quad (11)$$

$$b_1(x, y, t) = -2B\varphi_{xy}(x, y, t), \quad (12)$$

$$b_2(x, y, t) = -2B^2\varphi_y(x, y, t)\varphi_x(x, y, t), \quad (13)$$

$$\varphi(x, y, t) = f(x, t) + g(y), \quad (14)$$

where $f(x, t) \equiv f$ and $g(y) = g$ are two arbitrary functions of x, t and y , respectively.

Now, based on the solutions of (6), one can obtain new types of localized excitations of the (2+1)-dimensional DLW system. We obtain the general formulae of the solutions of the (2+1)-dimensional DLW system

$$u = -\frac{f_{xx} + f_t}{f_x} - 2Bf_x F(f + g), \quad (15)$$

$$v = -1 - 2ABf_x g_y - 2B^2 f_x g_y F^2(f + g). \quad (16)$$

By selecting the special values of A, B and the corresponding function F , we have the following solutions of the (2+1)-dimensional DLW system:

$$u_1 = -\frac{f_{xx} + f_t}{f_x} + 2f_x t \text{Ls}(f + g) \ln \alpha, \quad (17)$$

$$v_1 = -1 + 2f_x g_y \ln \alpha^2 - 2f_x g_y t \text{Ls}^2(f + g) \ln \alpha^2, \quad (18)$$

$$u_2 = -\frac{f_{xx} + f_t}{f_x} + 2f_x \cot \text{Ls}(f + g) \ln \alpha, \quad (19)$$

$$v_2 = -1 + 2f_x g_y \ln \alpha^2 - 2f_x g_y \cot \text{Ls}^2(f + g) \ln \alpha^2, \quad (20)$$

$$u_3 = -\frac{f_{xx} + f_t}{f_x} + \frac{f_x t \text{Ls}(f + g) \ln \alpha}{1 \pm \sec \text{Ls}(f + g)}, \quad (21)$$

$$\begin{aligned} v_3 &= -1 + \frac{f_x g_y \ln \alpha^2}{2} \\ &\quad - \frac{f_x g_y \ln \alpha^2}{2} \left[\frac{t \text{Ls}(f + g)}{1 \pm \sec \text{Ls}(f + g)} \right]^2, \end{aligned} \quad (22)$$

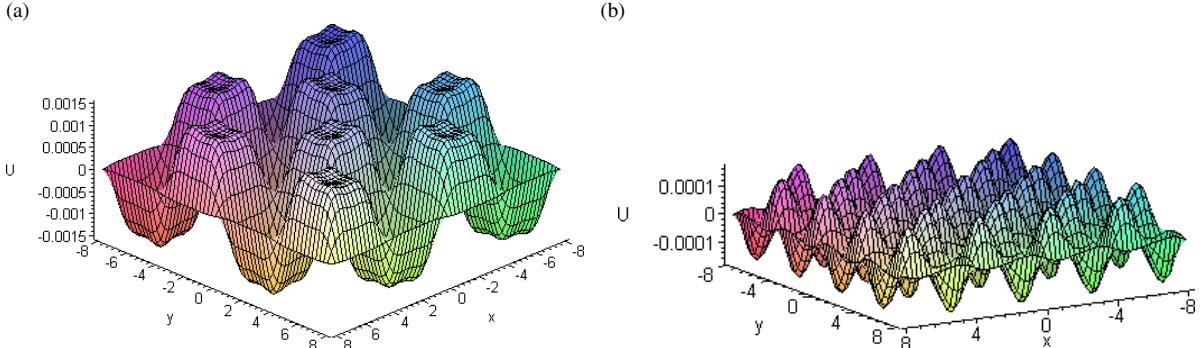


Fig. 1. The structures of the periodic solution (25) with (26) and (27). (a) $m_i = n_i = 0.8$; (b) $m_i = n_i = 1, i = 1, 2, \dots, 5, t = 0$.

$$u_4 = -\frac{f_{xx} + f_t}{f_x} + \frac{8f_x t \text{Ls}(f+g) \ln \alpha}{1 \pm t \text{Ls}^2(f+g)}, \quad (23)$$

$$\begin{aligned} v_4 &= -1 + 8f_x g_y \ln \alpha^2 \\ &\quad + 8f_x g_y \ln \alpha^2 \left[\frac{t \text{Ls}(f+g)}{1 + t \text{Ls}^2(f+g)} \right]^2, \end{aligned} \quad (24)$$

where $f(x, t)$ and $g(y)$ are two arbitrary variable separation functions. Especially, the potential $U = u_{1y}$ has the form

$$U = 8f_x g_y \sec \text{Ls}(f+g) \ln \alpha^2. \quad (25)$$

3. Novel Periodic Structures in the (2+1)-Dimensional DLW System

All rich localized coherent structures, such as non-propagating solitons, dromions, peakons, compactons, foldons, instantons, ghostons, ring solitons, and the interactions between these solitons, can be derived by the quantity U expressed by (25). These abundant localized coherent structures are neglected in the present paper, but we pay our attention to some novel Jacobian elliptic wave structures and periodic wave evolutional behaviours for the field U (25) in (2+1)-dimensions.

3.1. Special Jacobian Elliptic Wave Structure

Here we focus on the new periodic structure produced by Jacobian elliptic functions. Actually, it is straightforward to construct the new square lattice by selecting some Jacobian elliptic cosine functions, say, the selection

$$f = e^\theta, \quad \theta = \sum_{i=1}^M a_i \text{cn}(k_i x + \omega_i t + x_0 i; m_i),$$

$$g = e^\eta, \quad \eta = \sum_{i=1}^N b_i \text{cn}(K_i y + y_0 i; n_i) \quad (26)$$

of (25) leads to a periodic solution (square lattice) for the potential U . In (26), $\text{cn}(k_i x + \omega_i t + x_0 i; m_i)$ and $\text{cn}(K_i y + y_0 i; n_i)$ are the Jacobian elliptic cn functions with the modulus m_i , n_i , and a_i , b_i , $x_0 i$, $y_0 i$, k_i , K_i are arbitrary constants. Figure 1 shows the detailed structures of (25) with (26) and

$$\begin{aligned} M &= N = 5, \quad a_i = b_i = 0.1, \\ k_i &= \omega_i = K_i = 1, \quad x_{0i} = y_{0i} = 12 + 4i, \\ i &= 1, 2, \dots, 5, \end{aligned} \quad (27)$$

when $t = 0$. Figure 1a is related to the modulus of the Jacobian cn functions being taken as $m_i = n_i = 0.8, i = 1, 2, \dots, 5$. Figure 1b shows the new structure when we take $m_i, n_i (i = 1, 2, \dots, 5)$ close to 1.

3.2. The Interaction Properties of the Jacobian Elliptic Waves and Dromions

If we take

$$\begin{aligned} f &= \text{sn}(k_1 x + \omega_1 t + x_0 1; m) + \tanh(k_1 x + \omega_1 t + x_0 1), \\ g &= \text{sn}(K_1 y + y_0 1; n) \end{aligned} \quad (28)$$

from (25) we can obtain the interaction between the Jacobian elliptic waves and dromions with different speeds. In order to understand the properties of the interaction between the periodic waves, we illustrate this by several figures. Figure 2 shows a detail of the interaction properties of the periodic wave and dromions with the parameters

$$\begin{aligned} K &= k_1 = 1, \quad k_2 = 2, \quad \omega_1 = -\omega_2 = 1, \\ x_{01} &= x_{02} = y_0, \quad m = 0.7, \quad n = 0.8, \end{aligned} \quad (29)$$

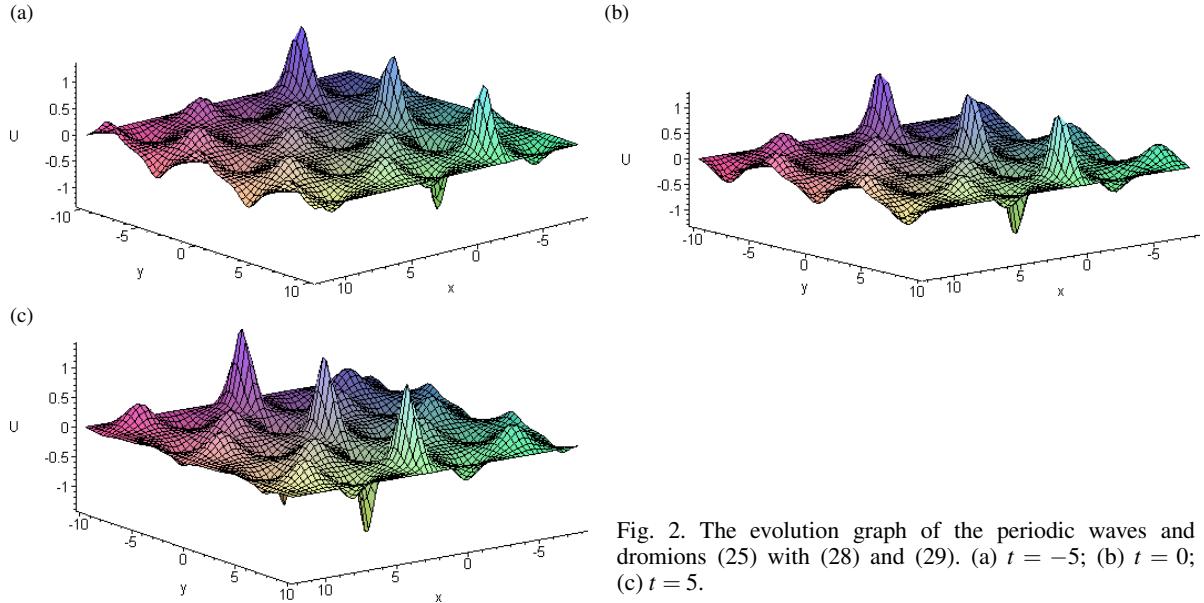


Fig. 2. The evolution graph of the periodic waves and dromions (25) with (28) and (29). (a) $t = -5$; (b) $t = 0$; (c) $t = 5$.

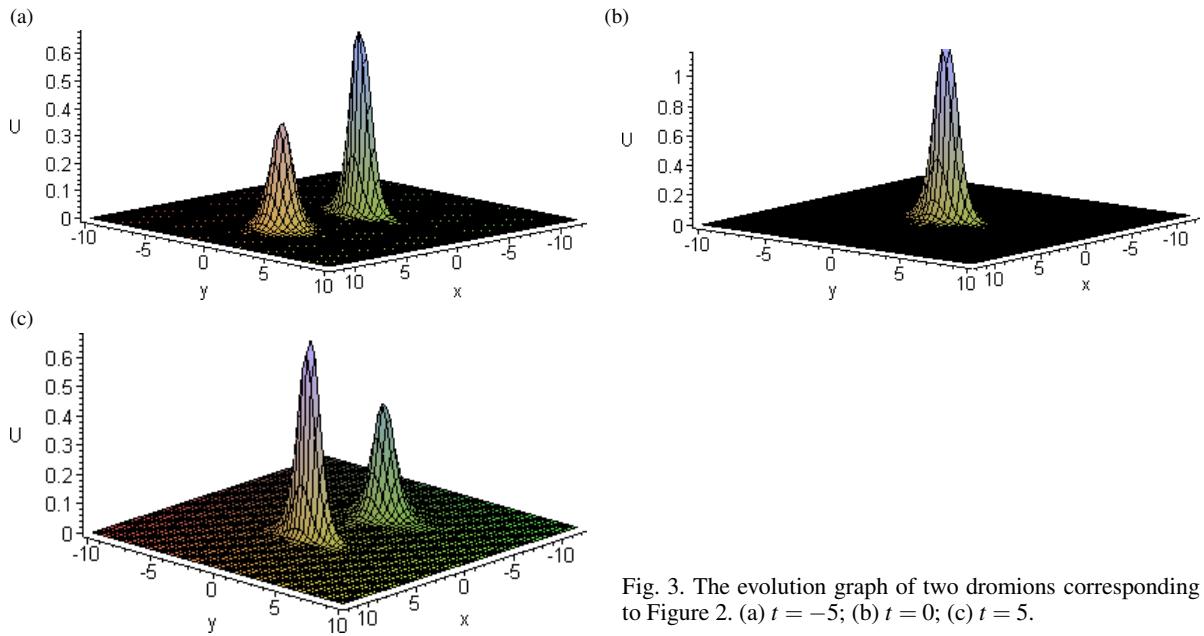


Fig. 3. The evolution graph of two dromions corresponding to Figure 2. (a) $t = -5$; (b) $t = 0$; (c) $t = 5$.

and $t = -5, 0, 5$, respectively. It is very interesting to see the interaction properties of the Jacobian elliptic waves and dromions in various limit cases. As m and n are close to 1, from (25) we can obtain the interaction between two dromions as depicted in Fig. 3 with the same parameters as in Figure 2.

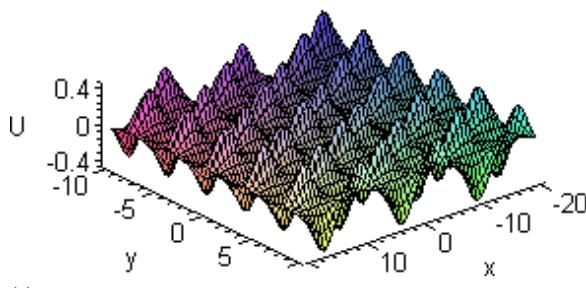
One can see that the interaction of them is completely elastic. That means that there is no exchange

of shapes and velocities, and there are no changes of any physical quantity like the energy and the momentum between the two dromions.

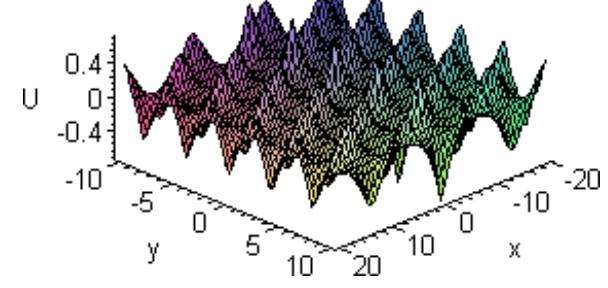
3.3. The Interaction Properties of the Jacobian Elliptic Waves and Peakons

Moreover, we can also discuss the interaction between the Jacobian elliptic wave and peakons. When

(a)



(b)



(c)

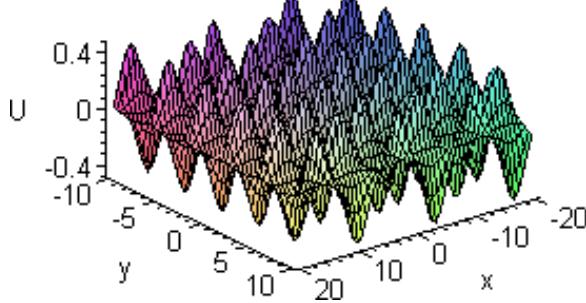


Fig. 4. The evolution graph of the periodic wave and peakons (25) with (30) and (31). (a) $t = -5$; (b) $t = 0$; (c) $t = 5$.

we consider

$$f = \begin{cases} \operatorname{sn}(kx + \omega t + x_0; m) + \sum_{i=1}^M F_i(k_i x + \omega_i t), \\ \quad \text{if } k_i x + \omega_i t \leq 0, \\ \operatorname{sn}(kx + \omega t + x_0; m) - \sum_{i=1}^M F_i(k_i x + \omega_i t) + 2F_i(0), \\ \quad \text{if } k_i x + \omega_i t > 0, \end{cases}$$

$$g = \operatorname{sn}(Ky + y_0; n), \quad (30)$$

the interaction between the Jacobian elliptic sn wave and peakons can be constructed. A simple example of this interaction is depicted in Fig. 4, when choosing

$$\begin{aligned} F_1 &= 0.3e^{x-t}, \quad M = 1, \quad k = 0.5, \quad K = 2, \\ m &= 0.5, \quad n = 0.8, \quad \omega = -1, \quad x_0 = y_0 = 0, \end{aligned} \quad (31)$$

and $t = -5, 0, 5$, respectively. Figure 5 shows the interaction between a dromion and peakon, which is non-completely elastic since their shapes and amplitudes are not completely preserved after interaction. The parameters of Fig. 5 are the same as those in Fig. 4 except that the modulus m and n are taken as the limiting values, $m = n = 1$.

4. Summary and Discussion

The Lucas Riccati method is applied to obtain variable separation solutions of the (2+1)-dimensional

DLW equation. Based on the quantity (25), some novel Jacobian elliptic wave structures and periodic wave evolutional behaviours are found. We hope that in future experimental studies these novel Jacobian elliptic wave structures and periodic wave evolutional behaviours can be realized in some fields. Actually, our present short paper is merely a beginning work, due to a wide variety of potential applications of the soliton theory. Besides the system discussed in this paper, we think that the variable separation solutions of the (2+1)-dimensional Korteweg-de Vries equation, NNV system and BKK system, can also be obtained by the Lucas Riccati method. The solutions obtained here can be useful to some physical problems in fluid dynamics.

Appendix

The definition and properties of the symmetrical Lucas functions, the symmetrical Lucas sine function (sLs), the symmetrical Lucas cosine function (cLs) and the symmetrical Lucas tangent function (tLs), are defined [22] as

$$\begin{aligned} \text{sLs}(x) &= \alpha^x - \alpha^{-x}, \\ \text{cLs}(x) &= \alpha^x + \alpha^{-x}, \\ \text{tLs}(x) &= \frac{\alpha^x - \alpha^{-x}}{\alpha^x + \alpha^{-x}}. \end{aligned} \quad (32)$$

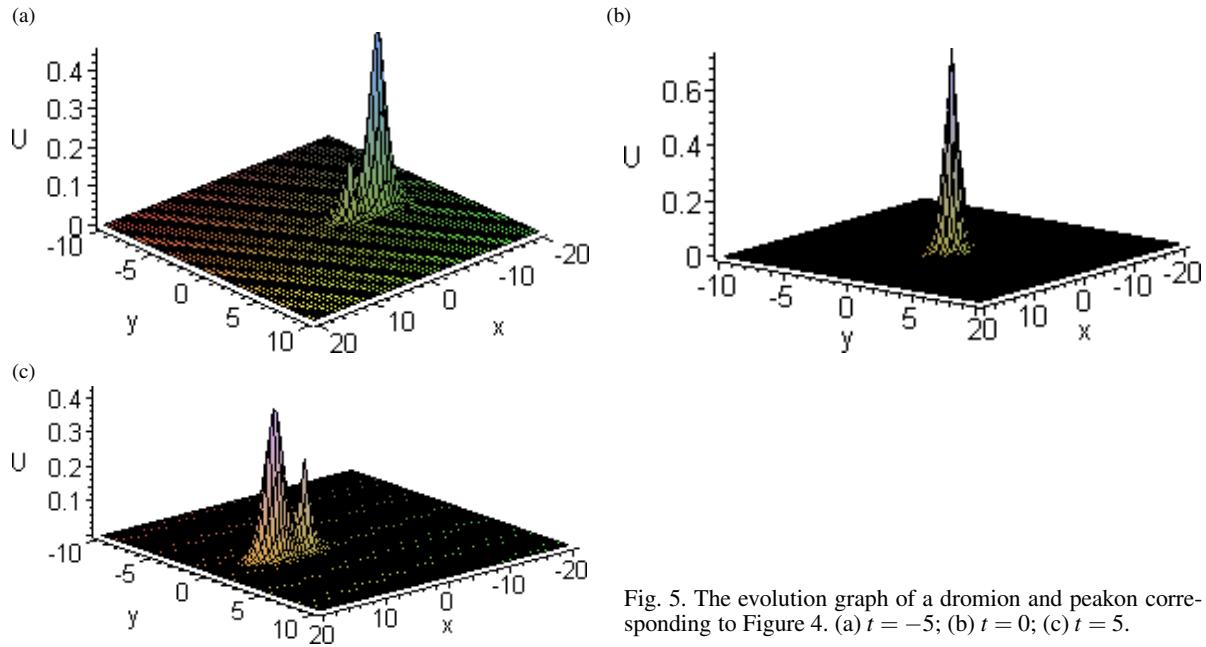


Fig. 5. The evolution graph of a dromion and peakon corresponding to Figure 4. (a) $t = -5$; (b) $t = 0$; (c) $t = 5$.

They are introduced to consider so-called symmetrical representations of the hyperbolic Lucas functions, and they may present a certain interest for modern theoretical physics taking into consideration the great role played by the Golden Section, Golden Proportion, Golden Ratio, Golden Mean in modern physical researches [22]. The symmetrical Lucas cotangent function ($\cot Ls$) is $\cot Ls(x) = \frac{1}{tLs(x)}$, the symmetrical Lucas secant function ($\sec Ls$) is $\sec Ls(x) = \frac{1}{cLs(x)}$, the symmetrical Lucas cosecant function ($\csc Ls$) is $\csc Ls(x) = \frac{1}{sLs(x)}$. These functions satisfy the following relations [23]:

$$\begin{aligned} cLs^2(x) - sLs^2(x) &= 4, \\ 1 - tLs^2(x) &= 4 \sec Ls^2(x), \\ \cot Ls^2(x) - 1 &= 4 \csc Ls^2(x). \end{aligned} \quad (33)$$

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Also, from the above definition, we give the derivative formulas of the symmetrical Lucas functions as follows:

$$\begin{aligned} \frac{d sLs(x)}{dx} &= cLs(x) \ln \alpha, \\ \frac{d cLs(x)}{dx} &= sLs(x) \ln \alpha, \\ \frac{d tLs(x)}{dx} &= 4 \sec Ls^2(x) \ln \alpha. \end{aligned} \quad (34)$$

The above symmetrical hyperbolic Lucas functions are connected with the classical hyperbolic functions by the simple correlations

$$\begin{aligned} sLs(x) &= 2 \sinh(x \ln \alpha), \\ cLs(x) &= 2 \cosh(x \ln \alpha), \\ tLs(x) &= \tanh(x \ln \alpha). \end{aligned} \quad (35)$$

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