

Jeans' Gravitational Instability of a Thermally Conducting, Unbounded, Partially Ionized Plasma

Shaista Shaikh^a, Aiyub Khan^a, and Prem Kumar Bhatia^b

^a Department of Mathematics and Statistics, J. N. Vyas University, Jodhpur, India

^b Department of Mathematics, M. B. M. Engineering College, J. N. Vyas University, Jodhpur, India

Reprint requests to S. S.; E-mail: shaikh_shaista@yahoo.com

Z. Naturforsch. **61a**, 275 – 280 (2006); received March 6, 2006

The gravitational instability of an infinitely extending homogenous, partially ionized plasma, permeated by an oblique magnetic field, has been studied to investigate the effects of Hall currents, finite conductivity, viscosity, collision with neutrals and thermal conductivity on the growth rate of the disturbance. The dispersion relation obtained has been solved numerically, and it has been found that Hall currents and collision with neutrals have a destabilizing influence on the growth rate while the other mechanisms reinforce the gravitational instability. Jeans' criterion, derived within a purely hydrodynamic framework, has been rediscussed along a nonextensive kinetic theory. A new Jeans' criterion was deduced, which depends on the nonextensive parameter q and the standard Jeans' wave number is recovered in the limiting case $q = 1$.

Key words: Partially Ionized; Jeans' Criterion; Thermal Conductivity.

1. Introduction

The gravitational instability problem of an infinite homogenous medium was first considered by Jeans [1]. According to Jeans' criterion, an infinite homogenous self-gravitating atmosphere is unstable for all wave numbers k smaller than the Jeans' wave number $k_J = \sqrt{\frac{G\rho}{S}}$, where ρ is the density, G the gravitational constant and $S = \sqrt{\frac{kT}{m}}$ the velocity of sound in the gas, k the Boltzmann's constant, T the temperature and m the mass of the particle.

Since then, several researchers have studied this problem under varying assumptions of hydrodynamics and hydromagnetics. A comprehensive account of these studies has been given by Chandrasekhar [2], who showed that Jeans' criterion remains unaffected by the separate or simultaneous presence of a uniform rotation and a uniform magnetic field. Chhajlani and Vaghela [3] have studied the stability of a self-gravitating plasma with thermal conduction and Finite Larmor Radius (FLR) through porous medium. The combined effects of viscosity, FLR, finite conductivity and Hall currents on the gravitational instability of a homogenous, rotating, thermally conducting plasma have been investigated by Mehta and Bhatia [4].

In cosmic physics, there are several situations, such as chromospheres, the solar photosphere and cool interstellar clouds, where the plasmas are frequently not

fully ionized but may instead be partially ionized, so that the interaction between the ionized fluid and the neutral gas becomes important. Research considering the case of partially ionized plasmas has been carried out by Alfven [5], Lehnert [6], Kumar and Srivastava [7], Cadez [8], Ali and Bhatia [9] and Hazarika and Bhatia [10].

In recent years, Jeans' stability problems have been analyzed in the framework of nonextensive Tsallis [11] statistics and its associated kinetic theory. Noteworthy work in this field has been done by Lima, Silva and Santos [12] and Du [13, 14].

In the present paper we discuss Jeans' gravitational instability for a plasma model endowed with several mechanisms, namely ion viscosity, collision with neutrals, Hall currents, finite conductivity and thermal conductivity. The main result is that these mechanisms play different physical roles in the perturbation. Jeans' criterion is derived in the framework of fluid theory. Moving along Du's line of approach [14], a possible kinetic treatment of Jeans' criterion is also discussed. It is shown that Jeans' criterion is modified due to the nonextensive effect by a factor that depends on the nonextensive parameter q .

2. Perturbation Equations

Consider the motion of an infinite homogenous, thermally conducting hydromagnetic fluid of density ρ

and finite conductivity permeated by neutrals of density ρ_d ($\rho \gg \rho_d$) in the presence of Hall currents. We assume that the two components of the partially ionized plasma (the ionized fluid and the neutral gas) behave as a continuum, and that their steady state velocities are equal. We assume that the magnetic field interacts only with the ionized component of the plasma, and that the frictional force of the neutral gas on the ionized fluid is of the same order of magnitude as that of the pressure gradient of the ionized fluid. The force due to the thermal conductivity, viscosity and the pressure gradient of the neutral gas is much smaller than that of the ionized fluid. Thus, we are considering here only the mutual frictional effects between the neutral gas and the ionized plasma in the presence of an oblique magnetic field. The problem is tractable mathematically, and it is hoped that the model considered reveals the essential features of the problem.

Under the foregoing assumptions, the linearized perturbation equations appropriate to the flow of the partially ionized plasma are

$$\begin{aligned} \rho \frac{\partial \vec{u}}{\partial t} = & -\nabla \delta p + (\nabla \times \vec{h}) \times \vec{H} + \rho (\nabla \delta \phi) \\ & + \mu \nabla^2 \vec{u} + \frac{1}{3} \mu \nabla (\nabla \cdot \vec{u}) + \rho_d v_c (\vec{u}_d - \vec{u}), \end{aligned} \quad (2.1)$$

$$\frac{\partial \vec{u}_d}{\partial t} = -v_c (\vec{u}_d - \vec{u}) + \nabla \delta \phi, \quad (2.2)$$

$$\begin{aligned} \frac{\partial \vec{h}}{\partial t} = & \nabla \times (\vec{u} \times \vec{h}) - \frac{1}{N_e} \nabla \times [(\nabla \times \vec{h}) \times \vec{H}] \\ & + \eta \nabla^2 \vec{h}, \end{aligned} \quad (2.3)$$

$$\frac{\partial}{\partial t} \delta \rho + (\vec{u} \nabla) \rho = 0, \quad (2.4)$$

$$\nabla^2 \delta \phi = -G \delta \rho, \quad (2.5)$$

$$\frac{\partial}{\partial t} (\delta p - \gamma S^2 \delta \rho) = \gamma \chi \nabla^2 (\delta p - S^2 \delta \rho), \quad (2.6)$$

where $u(u, v, w)$, $h(h_x, h_y, h_z)$, $\delta \rho$, δp and $\delta \phi$ are, respectively, the perturbations in velocity u , magnetic field H , density ρ , pressure p and gravitational potential ϕ . The corresponding quantities for the neutral gas are denoted by u_d and ρ_d . The collision frequency between the two components is represented by v_c . Here μ is the viscosity, G the gravitational constant, η the magnetic resistivity, N the electron number density and e the charge of the electron. Also, γ denotes an adiabatic exponent and χ is the thermal conductivity.

The magnetic field is assumed to be uniform and oblique, i.e. $H = (H_x, 0, H_z)$. We seek the solutions of (2.1)–(2.6), whose dependence on the space coordinates (x, y, z) and time t is of the form

$$\exp(i k \sin \theta x + i k \cos \theta z + int), \quad (2.7)$$

where $\vec{k} = (k \sin \theta, 0, k \cos \theta)$ is the wave number of perturbation making an angle θ with the x -axis, and n is the frequency of perturbation. Eliminating $\delta \rho$, $\delta \phi$, and δp from the above equations and writing $\alpha = \rho_d/\rho$, we get six equations governing the perturbation of the velocity and magnetic field, which can be written in the matrix form

$$[A][B] = 0, \quad (2.8)$$

where $[A]$ is a sixth-order square matrix and $[B]$ is a single column matrix in which the elements are $(u, v, w, h_x, h_y, h_z)^T$. The elements of $[A]$ are

$$\begin{aligned} A_{11} = & in + \frac{i n \alpha v_c}{in + v_c} - i D \sin^2 \theta + v_0 k^2 \left(1 + \frac{1}{3} \sin^2 \theta \right), \\ A_{12} = 0, \quad A_{13} = & \left(\frac{1}{3} v_0 k^2 - i D \right) \sin \theta \cos \theta, \\ A_{14} = & -\frac{H_z}{\rho} i k \cos \theta, \quad A_{15} = 0, \quad A_{16} = \frac{H_z}{\rho} i k \sin \theta, \\ A_{21} = 0, \quad A_{22} = & in + \frac{i n \alpha v_c}{in + v_c} + v_0 k^2, \quad A_{23} = 0, \\ A_{24} = 0, \quad A_{25} = & -\frac{ik}{\rho} (H_x \sin \theta + H_z \cos \theta), \quad A_{26} = 0, \\ A_{31} = & \left(\frac{1}{3} v_0 k^2 - i D \right) \sin \theta \cos \theta, \quad A_{32} = 0, \\ A_{33} = & in + \frac{i n \alpha v_c}{in + v_c} - i D \cos^2 \theta + v_0 k^2 \left(1 + \frac{1}{3} \cos^2 \theta \right), \\ A_{34} = & \frac{H_x}{\rho} i k \cos \theta, \quad A_{35} = 0, \quad A_{36} = -\frac{H_x}{\rho} i k \sin \theta, \\ A_{41} = & -H_z i k \cos \theta, \quad A_{42} = 0, \quad A_{43} = H_x i k \cos \theta, \\ A_{44} = & in + \eta k^2, \quad A_{45} = \frac{k^2 \cos \theta}{N_e} (H_x \sin \theta + H_z \cos \theta), \\ A_{46} = 0, \quad A_{51} = 0, \quad A_{52} = & -ik (H_x \sin \theta + H_z \cos \theta), \\ A_{53} = 0, \quad A_{54} = & -\frac{k^2 \cos \theta}{N_e} (H_x \sin \theta + H_z \cos \theta), \\ A_{55} = & in + \eta k^2, \quad A_{56} = \frac{k^2 \sin \theta}{N_e} (H_x \sin \theta + H_z \cos \theta), \\ A_{61} = & H_z i k \sin \theta, \quad A_{62} = 0, \quad A_{63} = -H_x i k \sin \theta, \\ A_{64} = 0, \quad A_{65} = & -\frac{k^2 \sin \theta}{N_e} (H_x \sin \theta + H_z \cos \theta), \\ A_{66} = & in + \eta k^2, \end{aligned} \quad (2.9)$$

where we have written

$$D = \frac{S^2 k^2 \gamma (in + \chi k^2) (in + v_c) - G\rho \{in + (1 + \alpha)v_c\} (in + \gamma \chi k^2)}{n(in + v_c)(in + \gamma \chi k^2)}. \quad (2.10)$$

3. Dispersion Relation

The vanishing of $|A|$ gives the dispersion relation as the product:

$$(in + \eta k^2) \left[\left\{ \left(in + \frac{in\alpha v_c}{in + v_c} + v_0 k^2 \right) \frac{k^4}{N_e^2} (H_x \sin \theta + H_z \cos \theta)^2 + (in + \eta k^2) \left\{ \left(in + \frac{in\alpha v_c}{in + v_c} + v_0 k^2 \right) (in + \eta k^2) + \frac{k^2}{\rho} (H_x \sin \theta + H_z \cos \theta)^2 \right\} \right\} \left\{ \left(in + \frac{in\alpha v_c}{in + v_c} - iD \sin^2 \theta + v_0 k^2 \left(1 + \frac{1}{3} \sin^2 \theta \right) \right) \cdot \left(in + \frac{in\alpha v_c}{in + v_c} - iD \cos^2 \theta + v_0 k^2 \left(1 + \frac{1}{3} \cos^2 \theta \right) \right) - \sin^2 \theta \cos^2 \theta \left(\frac{1}{3} v_0 k^2 - iD \right)^2 \right\} - \left\{ \left(in + \frac{in\alpha v_c}{in + v_c} + v_0 k^2 \right) (in + \eta k^2) + \frac{k^2}{\rho} (H_x \sin \theta + H_z \cos \theta)^2 \right\} \cdot \left\{ H_x ik \left\{ H_x \frac{ik}{\rho} \left(in + \frac{in\alpha v_c}{in + v_c} - iD \sin^2 \theta + v_0 k^2 \left(1 + \frac{1}{3} \sin^2 \theta \right) \right) + H_z \frac{ik}{\rho} \sin \theta \cos \theta \left(\frac{1}{3} v_0 k^2 - iD \right) \right\} + H_z ik \left\{ H_x \frac{ik}{\rho} \sin \theta \cos \theta \left(\frac{1}{3} v_0 k^2 - iD \right) + H_z \frac{ik}{\rho} \left(in + \frac{in\alpha v_c}{in + v_c} - iD \cos^2 \theta + v_0 k^2 \left(1 + \frac{1}{3} \cos^2 \theta \right) \right) \right\} \right\} \right] = 0. \quad (3.1)$$

The first factor gives

$$n = i\eta k^2, \quad (3.2)$$

which corresponds to a viscous type of damped mode modified by a finite conductivity. By writing $n = iW$ and using the value of D in the second factor of (3.1), we obtain the resulting dispersion relation, which is an equation of twelfth degree in W of the form

$$\begin{aligned} W^{12} - c_{11}W^{11} + c_{10}W^{10} - c_9W^9 + c_8W^8 \\ - c_7W^7 + c_6W^6 - c_5W^5 + c_4W^4 \\ - c_3W^3 + c_2W^2 - c_1W + c_0 = 0. \end{aligned} \quad (3.3)$$

The coefficients c_0 to c_{11} in (3.3) can obviously be obtained from (3.1). We use here explicitly only the coefficient c_0 since we discuss the nature of the roots of W with its help:

$$\begin{aligned} c_0 = \gamma^2 \chi^2 k^6 v_c^4 (S^2 k^2 - G\rho(1 + \alpha)) \\ \cdot [v_0 k^2 (M^2 v_0 + M V^2 + v_0 k^4 L^2) \\ + M v_0 + V^2], \end{aligned} \quad (3.4)$$

where

$$\begin{aligned} M &= \eta k^2, & V^2 &= \frac{(H_x \sin \theta + H_z \cos \theta)^2}{\rho}, \\ L &= \frac{(H_x \sin \theta + H_z \cos \theta)}{N_e}, & v_0 &= \frac{\mu}{\rho}. \end{aligned} \quad (3.5)$$

4. Discussion

4.1. Analysis of Jeans' criterion

This section deals with the analysis of Jeans' criterion from two different approaches – the fluid theory approach and the kinetic theory approach.

(i) Jeans' criterion in a purely hydrodynamic framework:

The solution of the dispersion relation (3.3) leads to a critical wave number for the instability

$$k_C = \frac{\sqrt{G\rho(1 + \alpha)}}{S} = k_J \sqrt{(1 + \alpha)}, \quad (4.1)$$

where k_J is Jeans' wave number. Clearly from (3.4), the nature of the roots of (3.3) depends on the nature of the factor $(S^2 k^2 - G\rho(1 + \alpha))$. Two possible cases have been discussed.

When $(S^2 k^2 - G\rho(1 + \alpha)) < 0$, the product of the roots is negative. Therefore, (3.3) has always one negative real root. The considered plasma is, therefore, unstable when $(S^2 k^2 - G\rho(1 + \alpha)) < 0$, which is precisely Jeans' criterion for a partially ionized plasma.

However, when $(S^2 k^2 - G\rho(1 + \alpha)) > 0$, the product of the roots is positive. Equation (3.3) has therefore either all positive roots or pairs of complex conjugate roots. The real roots correspond to a stable mode. The complex roots also correspond to a stable mode since $\text{Re}W$ is always positive, as the equation satisfied by $\text{Re}W$ (equation (3.3)) turns out to be one that has its coefficients alternately positive and negative. Hence, Jeans' criterion for gravitational instability remains unchanged in this framework for the considered plasma model.

(ii) Jeans' criterion in a kinetic framework:

Studies involving the statistical description of self-gravitating systems with long range interactions revealed some peculiar features exhibited by these systems, such as negative specific heats [15, 16] and the so-called gravothermal instability [17], which could not be accounted for by the conventional theory. Hence, the need for an extension to the standard Boltzmann-Gibbs approach was felt. In 1988, Tsallis provided a solution to such problems by proposing the nonextensive generalization of Boltzmann-Gibbs statistical mechanics that came to be known as "Tsallis' statistics". Tsallis' statistics provided the appropriate framework for the analysis of nonextensive effects that modify Jeans' gravitational instability criterion. Lima, Silva, Santos and J. Du have efficiently tackled this problem, considering a q -nonextensive velocity distribution function of the kinetic theory in accordance with Tsallis' thermostatistics. The difference in the two approaches lies in the choice of the definition of the mean square velocity $\langle v^2 \rangle$ of the particle. The former has worked with the standard definition of the expectation value, while the latter has used the new expectation value as defined in Tsallis' statistics. We wish to move along the line of Du's approach, as the second choice seems more appropriate.

According to Du, when the nonextensive effect is considered in the framework of Tsallis' statistics, the equation of state of an ideal gas can be written as

$P_q = nkT_q$ with the physical temperature T_q , a variable that depends on the nonextensive parameter q as $T_q = \frac{2T}{(5-3q)}$. Consequently, the speed of sound can be written as $S_q = \sqrt{\frac{k_B T_q}{m}}$, significantly different from the one in Boltzmann-Gibbs' statistics ($q = 1$, $T_q = T$). Equation (2.6) is thus modified as

$$\frac{\partial}{\partial t}(\delta p - \gamma S_q^2 \delta \rho) = \gamma \chi \nabla^2(\delta p - S_q^2 \delta \rho), \quad (4.2)$$

and on solving Jeans' problem determined by (2.1)–(2.5) and (4.2), we get the critical wave number for the instability:

$$k_C = \frac{\sqrt{G\rho(1+\alpha)}}{S_q} = k_J \sqrt{\frac{(5-3q)}{2}} \sqrt{(1+\alpha)}, \quad (4.3)$$

where k_J is Jeans' wave number. Hence, Jeans' criterion is modified due to the nonextensive effects by the factor $\sqrt{\frac{(5-3q)}{2}}$ where q lies in the range $0 < q < 5/3$. The new critical wave number k_C depends on the nonextensive parameter q as follows:

- (a) If $q = 1$, Jeans' criterion is recovered perfectly; $k < k_C = k_J$.
- (b) If $0 < q < 1$, Jeans' criterion is modified as $k < k_C > k_J$.
- (c) If $1 < q < 5/3$, Jeans' criterion is modified as $k > k_C < k_J$.
- (d) If $q \rightarrow 5/3$, $k_C \rightarrow 0$, i.e. $\lambda_C \rightarrow \infty$, the self-gravitating system is always stable.

The results (b) and (c) are worth to be analyzed.

Considering (b), for a gas with $0 < q < 1$, the critical wave number is increased by $\sqrt{\frac{(5-3q)}{2}}$ and the system may exhibit gravitational instability even if the wave number of the density fluctuation is greater than Jeans' wave number k_J . Hence, the system with $0 < q < 1$, which was believed to be gravitationally stable on the basis of the standard Boltzmann-Gibbs' statistics, may now be unstable according to Tsallis' statistics. However, from result (c) it is seen that for a gas with $1 < q < 5/3$, the critical wave number is decreased by $\sqrt{\frac{(5-3q)}{2}}$ and the system may remain stable even if the wave number of the density fluctuation is smaller than k_J . Hence, such a system, which was believed to be gravitationally unstable under the standard Boltzmann-Gibbs' statistics may now remain stable according to Tsallis' statistics.

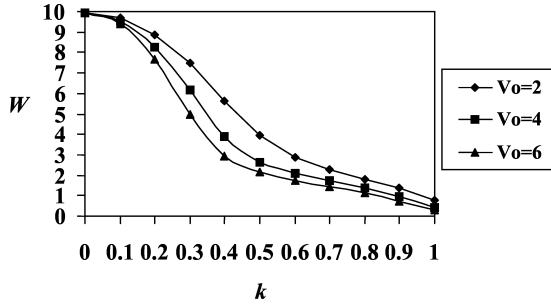


Fig. 1. Variation of the growth rate (negative real W) with the wave number (k) for $L = 1.0$, $v_c = 0.1$, $\chi = 1.0$ and $\eta = 1.0$.

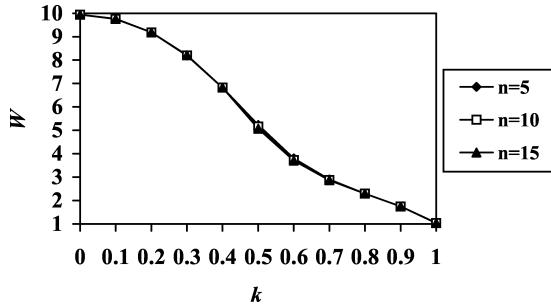


Fig. 2. Variation of the growth rate (negative real W) with the wave number (k) for $v_0 = 1.0$, $L = 1.0$, $v_c = 0.1$ and $\chi = 1.0$.

These results are quite different from the fluid theory approach discussed in part (i) of this section. However, the basic instability criterion is maintained: perturbations with $k > k_C$ do not grow while instability takes place for $k < k_C$.

4.2. Numerical Calculations

In order to study the influence of various physical parameters on the growth rate of an unstable mode, we have performed numerical calculations of the dispersion relation (3.3) to locate the roots of W against k for several values of parameters. For these calculations we took numerical values for conditions prevailing in magnetic collapsing clouds:

$$\begin{aligned} \rho &= 1.7 \times 10^{-21} \text{ kg m}^{-3}, \\ G &= 6.658 \times 10^{-11} \text{ kg}^{-1} \text{ m}^3 \text{ s}^{-2}, \\ S^2 &= 2.5 \times 10^8 \text{ m}^2 \text{ s}^{-2}, \\ V^2 &= 5 \times 10^8 \text{ m}^2 \text{ s}^{-2}. \end{aligned}$$

The critical wavelength for the gravitational instability of a homogenous infinitely extending plasma is of the order 10^{20} m. We have, therefore, calculated the

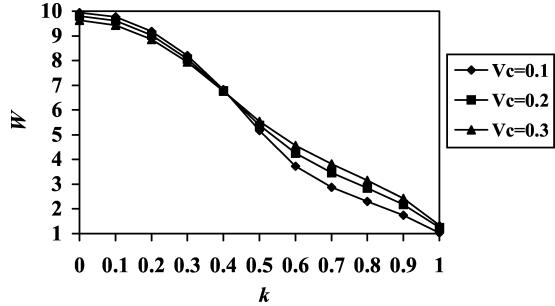


Fig. 3. Variation of the growth rate (negative real W) with the wave number (k) for $v_0 = 1.0$, $L = 1.0$, $\chi = 1.0$ and $\eta = 10.0$.

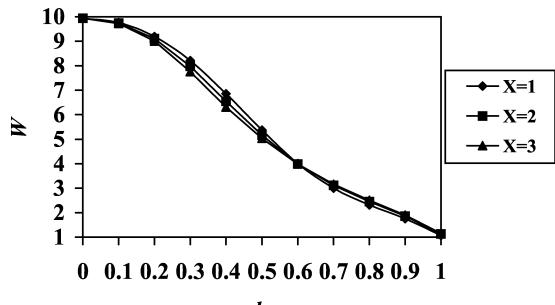


Fig. 4. Variation of the growth rate (negative real W) with the wave number (k) for $v_0 = 1.0$, $L = 1.0$, $v_c = 0.1$ and $\eta = 1.0$.

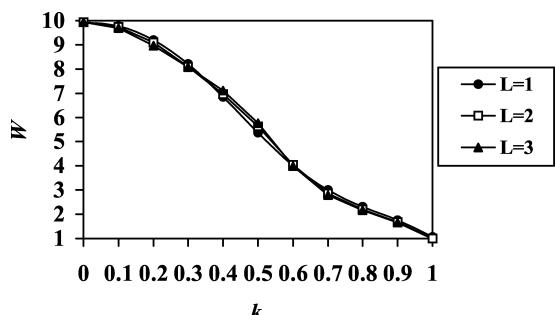


Fig. 5. Variation of the growth rate (negative real W) with the wave number (k) for $v_0 = 1.0$, $v_c = 0.1$, $\chi = 1.0$ and $\eta = 1.0$.

roots of the dispersion relation (3.3) for different values of the parameters v_0 , η , v_c , L , χ characterizing, respectively, the viscosity, finite conductivity, collision with neutrals, Hall currents and thermal conductivity, taking multiples of 10^{-20} as the values for the wave number.

The numerical calculations are presented as graphs in Figs. 1–5 where we have taken the growth rate (negative real root of W after multiplying by 10^4) against the wave number k (after multiplying by 10^{20}) for $v_0 = 2, 4, 6$ (after multiplying by 10^4); for $\eta = 5, 10, 15$ (after multiplying by 10^4); for $v_c = 0.1, 0.2, 0.3$

(after multiplying by 10^4); for $L = 1, 2, 3$ (after multiplying by 10^4) and for $\chi = 1, 2, 3$ (after multiplying by 10^4).

It is seen from Figs. 1, 2 and 4 that for a constant wave number k , the growth rate W decreases with increase of the ion viscosity (v_0), finite conductivity (η) and thermal conductivity (χ), respectively, thereby proving the stabilizing character of these physical parameters.

Similarly, by studying Figs. 3 and 5 it is clear that for a constant wave number k , the growth rate W predominantly increases with increasing collisions with neutrals (v_c) and Hall currents (L), respectively, thereby emphasizing the destabilizing influence of these parameters on the growth rate.

We may thus conclude that the viscosity, finite conductivity and thermal conductivity have a stabilizing influence, while collision with neutrals and Hall currents have a destabilizing influence on the growth rate of an unstable mode of disturbance.

Conclusion

We have studied Jeans' gravitational instability of an infinitely extending, homogenous, partially ionized plasma under the combined effects of several physical parameters. Particularly, Hall currents and collision with neutrals are found to have a destabilizing influence on the growth rate of unstable modes of disturbance. The effect of nonextensivity on Jeans' criterion has been discussed in the framework of the kinetic theory and, although, the Jeans' criterion is modified by the presence of the nonextensive q parameter, the basic instability criterion is maintained.

Acknowledgement

P. K. B. is thankful to AICTE for the award of an Emeritus Fellowship to him during the tenure of which this work has been done.

- [1] J. H. Jeans, Phil. Trans. R. Soc. (London) **A 199**, 1 (1902).
- [2] S. Chandrasekhar, Hydrodynamics und Hydromagnetic Stability, Clarendon Press, Oxford 1961.
- [3] R. H. Chhajlani and D. S. Vaghela, Astrophys. Space Sci. (Historical Archive) **134**, 301 (1987).
- [4] V. Mehta and P. K. Bhatia, Contr. Plasma Phys. **29**, 617 (1989).
- [5] H. Alfven, On the Origin of the Solar System, Clarendon Press, Oxford 1954.
- [6] B. Lehnert, Cosmic Electrodyn. **1**, 397 (1970).
- [7] N. Kumar and K. M. Srivastava, Astrophys. Space Sci. **174**, 211 (1990).
- [8] V. M. Cadez, Astron. Astrophys. **235**, 1 (1990).
- [9] A. Ali and P. K. Bhatia, Astrophys. Space Sci. **195**, 38 (1992).
- [10] A. B. R. Hazarika and P. K. Bhatia, Phys. Scr. **51**, 775 (1995).
- [11] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).
- [12] J. A. S. Lima, R. Silva, and J. Santos, Astron. Astrophys. **396**, 309 (2002).
- [13] J. L. Du, Europhys. Lett. **67**, 893 (2004).
- [14] J. L. Du, Phys. Lett. A **320**, 347 (2004).
- [15] S. W. Hawking, Nature **248**, 30 (1974).
- [16] T. Padmanabhan, Phys. Rep. **188**, 285 (1990).
- [17] A. Taruya and M. Sakagami, Physica A **318**, 387 (2003).