Existence and Uniqueness of Solutions of Piecewise-Defined Continuous Dynamical Systems

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The question of existence and uniqueness of trajectories in piecewise-defined autonomous dynamical systems is investigated. Examples of prototype equations violating existence and/or uniqueness are presented as well as criteria testing these properties.

Recently a number of applied mathematical papers have been published investigating the properties of piecewise-defined autonomous dynamical systems, specifically piecewise-linear ones. Most of these papers do not discuss existence and uniqueness of the solutions, because this seems to be trivial task. Nevertheless there are pitfalls even in apparently simple systems, as will be shown.

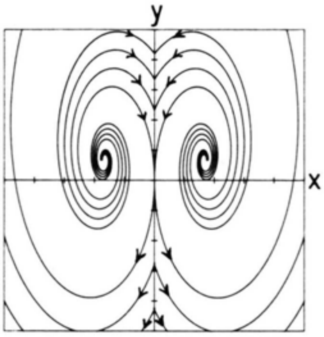
Let (τ, Φ_t) denote a dynamical system with the state space τ being a differentiable manifold embedded in a Banach space, and Φ_t the flow of the system. The system is called piecewise-defined, if a finite or even a countable family of dynamical systems (τ^i, Φ_t^i) exists such that all the (so-called regions) τ^i are simply connected disjoint open sets with the property $\bigcup_i \bar{\tau}^i = \tau$. The trajectories of the partial systems are solutions of the differential equations $dx/dt = F^i(x)$.

Let us now assume that each of the functions F^i is defined and locally Lipschitz on an open set $\tilde{\tau}^i$ containing $\bar{\tau}^i$ (the closure of τ^i) as a proper subset, i.e., the F^i fulfill the prerequisites of the existence and uniqueness theorem (cf. e.g. [1], pp. 162) on the $\tilde{\tau}^i$. Trouble can then appear only at boundaries $\partial\tau^i$ of regions. (The boundary of τ itself, if there is any, is not to be considered here.) There the F's are switching, so the possibility arises that a boundary point may not lie on a single (unique) trajectory only, or that there exists no trajectory at all using this point (see Fig. 1).

The concept of exit and entry points is considerably helpful to see how a solution inside a region is extended to its boundary. A point s is called an exit point of the region τ^i, subject to the dynamics F^i if a $y \in \tau^i$ and a $0 < T < \infty$ exist, such that both, for all $0 < t < T$ the $\Phi_t^i(y)$ are from τ^i, and $s = \lim_{t \to T} \Phi_t^i(y)$ hold true. In other words, a trajectory starting at point y, and leaving τ^i in s, reaches the boundary at time T for the first time. Entry points are defined analogously for negative times.

The other idea to be introduced here is the concept of shadow solutions. Let $I(s)$ be the set of all indices with $s \in \partial\tau^i$; in the following we shall imply $i \in I(s)$ for a „general i". Since all F^i are locally Lipschitz, for a boundary point s a unique solution of the initial value problem exists inside each $\tilde{\tau}$. These solutions are called shadow solutions. They are said to become manifest if a $\delta > 0$ exists, such that for all $t \in (0, \delta)$ the points $\Phi_t^i(s)$ are from τ^i. Now it is easy to see that a system possesses a unique trajectory using s, for the case of exactly one shadow solution becoming manifest in positive time, and exactly one in negative time, i.e., if one incoming and one outgoing solution exists uniquely. A point possessing this property is said to be regular.

A boundary point s is called C^1, if it possesses an open neighbourhood $\mathcal{N}(s)$, such that for all boundary points $s' \in \mathcal{N}(s)$ and for all $i \in I(s')$ the condition $F^i(s') = \lambda(s')$ is fulfilled, with $\lambda(s')$ an element of the tangential space to τ in s'. This means that the F^i match at C^1 boundary points, and in the neighborhood of such points yielding a con-



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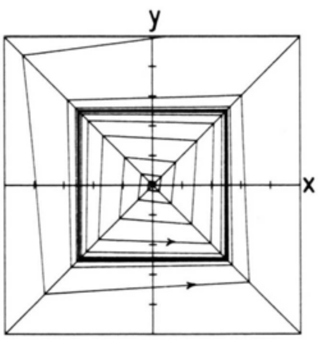

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Fig. 1. Simple examples violating existence and uniqueness. (a) All points of the boundary ($x = 0$) are nonregular. For $y > 0$ two incoming solutions are found while for $y < 0$ two trajectories are outgoing. At $y = 0$ two tangential trajectories simultaneously use this boundary point. (b) A dissipative oscillator possessing a limit cycle. Here
$F^1(x, y) = (\varrho(1 - x), 1)^T$, $\quad F^2(x, y) = (-1, \varrho(1 - y))^T$,
$F^3(x, y) = (-\varrho(1 + x), -1)^T$, and
$F^4(x, y) = (1, -\varrho(1 + y))^T$.
For $\varrho < 1$ (0.15 in the example shown) at the origin, being not a fixed point of the flow, no solution exists; while for $\varrho > 1$ at this point four outgoing solutions become manifest, with no trajectory coming in.

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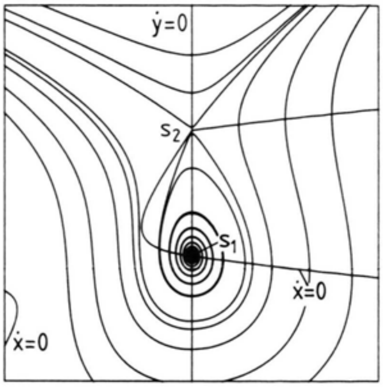

Fig. 2. Phase portrait of a system possessing a C^1 type boundary containing two critical points. Here $F^1(x, y) = (y^2 - y - xy - \varrho x, x)^T$ and $F^2(x, y) = (y^2 - y - \varrho x, x)^T$, with $\varrho = 0.1$. At the stable focus $s_1 = (0, 0)^T$ both F^1 and F^2 simultaneously vanish and their linearisations for the two regions coincide; while at the saddlepoint $s_2 = (0, 1)^T$ the F's become zero but their linearisations are different.

tinuously differentiable flow there. All other boundary points are said to be C^0.

The existence of a solution, containing a point of C^1 type as initial value, is guaranteed by Peano's theorem [2]. To show uniqueness we have to consider three cases. (1) If s is a fixed point of the flow it solves the initial value problem for all times, and there exists no trajectory approaching it in finite time. (2) A solution running for an interval of time inside a boundary is determined uniquely as long as all its points are C^1. This is because all the F^i involved, and hence all shadow solutions, coincide for C^1 boundary points. (3) For a transversal boundary point the solution found may exit τ^i in s and enter τ^j. Since the $F^k(s)$ $(k \in I(s))$ coincide, a $\Delta > 0$ exists such that for all $\delta \in (0, \Delta)$ the points $s + \delta F^k(s)$ are from τ^j and hence all shadow solutions are running inside τ^j for a finite interval of time. The same situation is found for negative times, leaving only $\Phi_t^j(s)$ as a manifest trajectory.

The preceding argument yields the idea of how to generalize the uniqueness criterion for C^0 boundary points: A trajectory that exits τ^i in s possesses a unique continuation, if a $\Delta > 0$ exists such that for all $j \in I(s) \setminus \{i\}$ and for all $\delta \in (0, \Delta)$ the points $s + \delta F^j(s)$ are from the same τ^k, i.e., all derivatives of the potentially outgoing trajectories (shadow solutions) are pointing towards the same region. Note that for $k = i$ (a case that may appear at a corner of a boundary) of course the above condition has to be fulfilled for all $j \in I(s)$. This criterion is sufficient to guarantee that only the trajectory $\Phi_t^k(s)$ becomes manifest. Applying the same procedure for negative times allows to decide whether there is also a unique incoming trajectory converging towards s.

A negative criterion would be the existence of at least two different τ^k where the $F^j(s)$ are pointing to. If, however, one or more of the $F^j(s)$ are tangential to the boundary (provided it is smooth), the derivatives of the (shadow) flows yield no decision. So the solutions (belonging to the tangential $F^j(s)$) themselves have to be investigated.

To conclude, existence and uniqueness of solutions using a boundary point can be expressed in terms of the number of manifest shadow solutions. If there is neither an incoming nor an outgoing manifest flow, no trajectory exists. For exactly one shadow solution becoming manifest both in positive and negative time, the trajectory is uniquely defined, while in all other cases uniqueness is violated or the trajectory possesses no continuation, respectively.

The criterion presented in this note shows two things: (1) The existence and uniqueness of the solutions can be guaranteed at C^0 boundary points, where no Lipschitz condition is fulfilled, too; and (2) the conditions for existence and uniqueness in piecewise-defined autonomous dynamical systems are mild. Still, as the examples illustrate, even rather simple systems may behave irregularly. If not a whole boundary, but only singular points are non-regular, they can easily be overlooked, specifically in higher dimensions. Therefore, in every concrete case it is recommendable to check the regularity of all boundary points.

Acknowledgements

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[1] M. W. Hirsch and S. Smale, Differential Equations, Dynamical Systems, and Linear Algebra. Academic Press, New York 1974.

[2] G. Peano, Démonstration de l'integrabilité des équations différentielles ordinaires. Math. Ann. **76,** 182–228 (1890).