Connection Aspects of the Non-linear Schrödinger Equation

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Some geometrical features of the non-linear Schrödinger equation are studied and it is described how the Schrödinger equation can be obtained from the non-linear scattering equations. The SL(2, ℝ)-valued elements of the matrix of the non-linear scattering problem are interpreted as matrix-valued forms. We calculate the curvature form with respect to a basis of the Lie algebra. If the curvature form equals zero then we obtain the non-linear Schrödinger equation.

Wahlquist and Estabrook have given a geometric approach [1] and they find a prolongation structure for non-linear partial differential equations in two independent variables. They have applied their method to the non-l near Schrödinger equation and they discussed the relationship between the pseudopotentials and the inverse scattering method (ISM) of Zakharov and Shabat [2], and Ablowitz, Kaup, Newell and Segur [3]. Sasaki [4] has shown that the ISM is given by a completely integrable pfaffian system and that the equations of Ablowitz, Kaup, Newell and Segur can be described by a pseudospherical surface. Furthermore, Sasaki [5] has given pseudopotentials for equations of Ablowitz, Kaup, Newell und Segur.

We give a further example of the Lie algebraic method to study the geometrical aspects of the non-linear Schrödinger equation. First we cite the scattering equations (ISM) and some formulae [6]. We use the (ISM) in the form

$$\frac{\partial}{\partial x}\begin{pmatrix}\varphi^1\\ \varphi^2\end{pmatrix}=\begin{pmatrix}\eta & q(x,t)\\ r(x,t) & -\eta\end{pmatrix}\begin{pmatrix}\varphi^1\\ \varphi^2\end{pmatrix}, \tag{1}$$

where the time evolution of the functions $\varphi^1(x,t)$ and $\varphi^2(x,t)$ are given by

$$\frac{\partial}{\partial t}\begin{pmatrix}\varphi^1\\ \varphi^2\end{pmatrix}=\begin{pmatrix}A(x,t;\eta) & B(x,t;\eta)\\ C(s,t;\eta) & -A(x,t;\eta)\end{pmatrix}\begin{pmatrix}\varphi^1\\ \varphi^2\end{pmatrix}. \tag{2}$$

The coefficients A, B and C are functions of x and time t. The quantity η is the eigenvalue of the scattering problem and we assume $\eta \neq \eta(t)$. The system (1) and (2) can be rewritten in matrix notation. We have

$$\frac{\partial\varphi^k}{\partial x^j}+\sum_{p=1}^{2}\Omega^k_{pj}\cdot\varphi^p=0, \tag{3}$$

where $j, k = 1, 2$ and $x^1 = x$, $x^2 = t$. The $\varphi^k(x,t)$ are the components of a two-component field on the principal bundle $P(M, G)$. The Ω^k_{pj} are given by the components of the matrix in (1) and (2). The theory of Cartan-Ehresmann connections (see [7]) generalizes the Gaussian curvature of a Riemannian 2-dimensional manifold. The curvature form is given by the exterior covariant derivative of the 1-form ω on $P(M, G)$ with the values in a finite-dimensional vector space V in the form

$$\Theta=\nabla\omega=\mathrm{d}\omega\circ h, \tag{4}$$

where Θ is a $\boldsymbol{g}$-valued 2-form. The $(p+1)$-form $\nabla\omega$ is given by

$$\nabla\omega(X_1,\ldots,X_{p+1}) = \mathrm{d}\omega(hX_1,\ldots,hX_{p+1}), \tag{5}$$

and h: $T_p(P(M,G))\to S_p$ the projection of T_p (tangential space) onto its horizontal subspace S_p. The space of vertical vectors $V_p = T_p \ominus S_p$ lies tangential to the fibre.

In a previous paper [6] we have expressed the curvature form Θ on bundles $P(M, G)$ in the form

$$\Theta=\sum_{i=1}^{3}\mathrm{d}\omega^i\otimes X_i+\tfrac{1}{2}\sum_{i,j=1}^{3}(\omega^i\wedge\omega^j)\otimes[X_i,X_j], \tag{6}$$

where ω^k $(k=1,2,3)$ are arbitrary one-forms and $[X_p, X_q]$ is the commutator of the quantities X_k. The two-form Θ is called the curvature of the connection. The $\{X_k\}_{k=1}^3$ are a basis of the Lie algebra $\boldsymbol{g} = \mathrm{SL}(2, \mathbb{R})$, and we choose

$$X_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad X_2=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad X_3=\begin{pmatrix}0&0\\1&0\end{pmatrix}. \tag{7}$$

In view of (3) we can write for the 1-forms

$$\omega^1=-(\eta\,\mathrm{d}x+A\,\mathrm{d}t),\quad \omega^2=-(q\,\mathrm{d}x+B\,\mathrm{d}t),\quad \omega^3=-(r\,\mathrm{d}x+C\,\mathrm{d}t). \tag{8}$$

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If we rewrite the curvature form (6) with (7) and (8) then we obtain the expression

$$\begin{aligned}\Theta = & (qC - rB - A_x) \cdot \mathrm{d}x \wedge \mathrm{d}t \otimes X_1 \\ & + (2\eta B - 2qA + q_t - B_x) \\ & \quad \cdot \mathrm{d}x \wedge \mathrm{d}t \otimes X_2 \\ & + (-2\eta C + 2rA + r_t - C_x) \\ & \quad \cdot \mathrm{d}x \wedge \mathrm{d}t \otimes X_3 . \end{aligned} \tag{9}$$

If we choose

$$\begin{aligned} r = -q^* = -u^*, \quad & A = 2i\eta^2 + i|u|^2, \\ B = iu_x + 2i\eta u, \quad & C = iu_x^* - 2i\eta u^*, \end{aligned} \tag{10}$$

then we obtain

$$\begin{aligned}\Theta = & (-2ui|u|^2 + u_t - iu_{xx})\, \mathrm{d}x \wedge \mathrm{d}t \otimes X_2 \\ & + (-2u^* i|u|^2 - u_t^* - iu_{xx}^*) \\ & \quad \cdot \mathrm{d}x \wedge \mathrm{d}t \otimes X_3 . \end{aligned} \tag{11}$$

If $\Theta = 0$ we obtain

$$\begin{aligned} iu_t + u_{xx} + 2|u|^2 u &= 0, \\ -iu_t^* + u_{xx}^* + 2|u|^2 u^* &= 0, \end{aligned} \tag{12}$$

for the real and imaginary part of the non-linear Schrödinger equation. Moreover, from the condition $\Theta = 0$ we conclude that

i) ω satisfies the Maurer-Cartan structural equation

$$\mathrm{d}\omega + \tfrac{1}{2}[\omega, \omega] = 0,$$

ii) the connection in $P(M, G)$ is flat.

To summarize: We have used the Lie algebraic method to give a geometrical interpretation of the non-linear Schrödinger equation. We see that ω satisfies the structure equation of Maurer-Cartan, which implies that the canonical flat connection has zero curvature. Therefore we can say that the non-linear Schrödinger equation can be deduced from the fact that the SL(2, ℝ) connection associated with the scattering equation has zero curvature. The Bäcklund transformation for the non-linear Schrödinger equation is given by Lamb [8] in a different way by using the method of pseudopotentials [1].

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