A Note on Rotation Matrices

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Rotation matrices $d^{(j)}_{m,m'}(\Theta)$ find wide applications in molecular and nuclear theory. They are known since the famous work of WIGNER [1] on the rotation group. As the derivation is complicated and needs group theory, a simple algebraic method will be valuable. The present note gives a simplification of Schwinger's solution of $d^{(j)}_{m,m'}(\Theta)$ avoiding the unnecessary complicated generating function trick [2]. We follow the customary notation and repeat for completeness the defining equations.

The problem is the evaluation of the matrixelement

$$d^{(j)}_{m,m'}(\Theta) = \langle j\, m \,|\, \exp(-i\,\Theta\,\hat{J}_y)\,|\, j\, m'\rangle. \qquad (1)$$

SCHWINGER [2] introduced a boson representation of angular momentum in terms of the twodimensional isotropic harmonic oscillator creation and annihilation operators $\hat{a}^+_\pm$ and $\hat{a}_\pm$:

$$\hat{J}_+ = \hat{J}_x + i\,\hat{J}_y = \hat{a}^+_+\,\hat{a}_-\,, \qquad (2)$$

$$\hat{J}_- = \hat{J}_x - i\,\hat{J}_y = \hat{a}^+_-\,\hat{a}_+\,, \qquad (3)$$

$$\hat{J}_z = \tfrac{1}{2}\,(\hat{a}^+_+\,\hat{a}_+ - \hat{a}^+_-\,\hat{a}_-), \qquad (4)$$

and

$$|\, j\, m'\rangle = [(j+m')!\,(j-m')!]^{-1/2}\,(\hat{a}^+_+)^{j+m'}\,(\hat{a}^+_-)^{j-m'}\,|\,00\rangle. \qquad (5)$$

By substitution, $\hat{J}_y$ is

$$\hat{J}_y = \frac{1}{2\,i}\,(\hat{a}^+_+\,\hat{a}_- - \hat{a}^+_-\,\hat{a}_+). \qquad (6)$$

If the identity operator $\hat{I}$

$$\hat{I} = \exp(i\,\Theta\,\hat{J}_y)\,\exp(-i\,\Theta\,\hat{J}_y) \qquad (7)$$

is introduced the problem of calculating $d^{(j)}_{m,m'}(\Theta)$ is reduced to a canonical transformation

$$d^{(j)}_{m,m'}(\Theta) = [(j+m')!\,(j-m')!]^{-1/2}\,\langle j\, m\,|\,\exp(-i\,\Theta\,\hat{J}_y)\,(\hat{a}^+_+)^{j+m'}\,(\hat{a}^+_-)^{j-m'}\,\exp(i\,\Theta\,\hat{J}_y)\exp(-i\,\Theta\,\hat{J}_y)\,|\,00\rangle \qquad (8)$$

which will be performed by means of the well known expansion theorem:

$$\exp(\hat{B})\,\hat{A}^k\exp(-\hat{B}) = [\exp(\hat{B})\,\hat{A}\exp(-\hat{B})]^k = \left(\hat{A} + [\hat{B},\hat{A}]_- + \frac{1}{2!}[\hat{B},[\hat{B},\hat{A}]_-]_- + \dots\right)^k = \left(\sum_{S=0}^{\infty}\frac{1}{S!}\{\hat{B}^S,\hat{A}\}_-\right)^k. \qquad (9)$$

In the present case, the $\hat{B}$ operator is realized by $(-i\,\Theta\,\hat{J}_y)$ and the $\hat{A}$ operator by $\hat{a}^+_+$ and $\hat{a}^+_-$, respectively. The commutator $[\hat{B},\hat{A}]_-$ is $(\Theta/2)\,\hat{a}^+_-$ and $-(\Theta/2)\,\hat{a}^+_+$ in this special case. Therefore one obtains

$$\exp(-i\,\Theta\,\hat{J}_y)\,\hat{a}^+_+\,\exp(i\,\Theta\,\hat{J}_y) = \hat{a}^+_+\,\cos(\Theta/2) + \hat{a}^+_-\,\sin(\Theta/2), \qquad (10)$$

$$\exp(-i\,\Theta\,\hat{J}_y)\,\hat{a}^+_-\,\exp(i\,\Theta\,\hat{J}_y) = \hat{a}^+_-\,\cos(\Theta/2) - \hat{a}^+_+\,\sin(\Theta/2). \qquad (11)$$

As the operator $\exp(-i\,\Theta\,\hat{J}_y)$ contains normalordered annihilation operators, all terms except the first one vanish in the series expansion. The final result for $d^{(j)}_{m,m'}(\Theta)$ is written with the binomial theorem:

$$d^{(j)}_{m,m'}(\Theta) = [(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!]^{-1/2}\sum_{k=0}^{j+m'}\sum_{l=0}^{j-m'}(-1)^l\binom{j+m'}{k}\binom{j-m'}{l}\cos^{(2j-k-l)}(\Theta/2)\,\sin^{(k+l)}(\Theta/2)\,\langle 00\,|\,(\hat{a}_+)^{j+m}\,(\hat{a}_-)^{j-m}\,(\hat{a}^+_+)^{j+m'-k+l}\,(\hat{a}^+_-)^{j-m'-l+k}\,|\,00\rangle. \qquad (12)$$

The matrix element vanishes for all cases

$$j+m \neq j+m'-k+l \quad \text{and/or} \quad j-m \neq j-m'-l+k\,. \qquad (13)$$

If the expressions are equal, the matrixelement is $(j+m)!\,(j-m)!$. By easy trigonometric manipulations the $d^{(j)}_{m,m'}(\Theta)$ can be brought to the customary form of Jacobi-polynomials.

[1] E. P. WIGNER, Group Theory and its Application to the Quantum Mechanics of Atomic Spectra, Academic Press, New York 1959.

[2] J. SCHWINGER, On Angular Momentum, reprinted in: Quantum Theory of Angular Momentum, L. C. BIEDENHARN and H. VAN DAM, eds., Academic Press, New York 1965.